ROBUSTNESS VERSUS UNMATCHED UNCERTAINTIES OF A HYBRID VARIABLE STRUCTURE CONTROL STRATEGY

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Abstract

The robustness features versus unmatched uncertainties of a hybrid variable structure control strategy for a class of second order systems are analyzed in this paper. The hybrid control relies on a subdivision of the system state space into nested regions, and on an event-driven switching among the control laws associated with each region. By componing globally stabilizing variable structure laws, and avoiding the generation of limit cycles, the proposed strategy proves to globally stabilize the origin of the system state space. Moreover, the equivalent system state is proved to be ultimately bounded with respect to an arbitrarily small set.

1 Introduction

The use of variable structure control (VSC), [12]-[5], within a framework typical of hybrid systems (see, for instance, [10]-[8]), has been first discussed in [2]-[7]. The motivation for using VSC to design a hybrid strategy directly comes from applications: in various contexts (aerospace systems, underwater mobile robotics, traffic controllers, etc.) there are cases in which different types of on-off or relay actuators need to be used depending on some criteria (usually, internal energy or distance from the equilibrium) or to comply with conflicting requirements. Even in the case of VSC systems, the control design relies on a state space subdivision through a border, the so–called sliding manifold, which is a linear or nonlinear function of the full system state, so that the control law is switched on crossing it.

The hybrid variable structure control (*hybrid-VSC*) system presented in [6] relies on a peculiar system state decomposition into countable regions by means of a grid of conventional sliding manifolds, and a set of nested switching boundaries. Each region is a "block" in the sense used in [4], and a block–invariant control law is associated with it. On the whole, the choice of the control laws corresponding to the blocks included between two switching boundaries (note that also infinity and the origin of the state space can be interpreted in this way) concurs to the attainment of the objective of either reaching a particular sliding manifold, or crossing the switching boundary closer to the origin.

In the present paper, the robustness features versus unmatched uncertainties of a hybrid variable structure control strategy for a class of second order systems inspired by [6] are analyzed. As it well-known, VSC systems, during sliding mode, are insensitive to bounded matched uncertainties [12]. In contrast, the influence of unmatched uncertainties on the equivalent system cannot be eliminated by suitably choosing the control gain. Then, the stability of the origin of the closed-loop system, as well as the convergence of the state trajectory need to be investigated, especially in view of the hybrid nature of the proposed controller, which, in principle, could determine a loss of the stabilizing properties and of the robustness characteristics of the single feedback law associated with the state space regions [8].

In this paper, it is proved that the effect of unmatched uncertainties on the equivalent system dynamics can be reduced by suitable selecting the switching boundary and the design parameter of the sliding manifold associated with the inner region (that containing the origin). The proposed *hybrid-VSC* strategy proves to globally stabilize the origin of the system state space. By avoiding the enforcement of limit cycles, a sliding mode in the inner region is ultimately reached. Moreover, the equivalent system state during that sliding mode results in being ultimately bounded with respect to an arbitrarily small set containing the origin.

2 Problem formulation

Consider the second-order continuous-time dynamical system

$$\dot{x}(t) = f(x(t),t) + g_1 u_1(t) + g_2 u_2(t), \quad x(0) = \overline{x} \tag{1}$$

where $x = [x_1 \ x_2]' \in \Re^2$ is the state vector, and $f(x(t),t) = [x_2(t) \quad 0]' + \bar{f}(x(t),t) : \Re^2 \times \Re_+ \to \Re^2$, with $\bar{f}(x(t),t) = [\bar{f}_1(x(t),t) \ \bar{f}_2(x(t),t)]'$, is a Lipschitz vector function. Moreover, the scalar functions $\bar{f}_i(x(t),t) : \Re^2 \times \Re_+ \to \Re$ are uncertain but such that

$$\left|\bar{f}_i(x(t),t)\right| < k_i, \quad i = 1,2, \quad \forall\, t \geq 0 \tag{2}$$

The control variables $u_1 \in \Re$, $u_2 \in \Re$ influence the state vector linearly through the constant vectors $g_1 = [0 \ \bar{g}_1]'$, $g_2 = [0 \ \bar{g}_2]'$, where, without loss of generality, $\bar{g}_i$, $i = 1,2$, are supposed to be positive constants. As it is apparent, the uncertain term $\bar{f}(x(t),t)$, which describes the system nonlinearities and model uncertainties, does not satisfy the matching condition, that is $f(x(t),t) \neq B\xi(x(t),t)$, for any bounded function $\xi(x(t),t) : \Re^2 \times \Re_+ \to \Re^2$ and $B := [g_1 \ g_2]$, [5].

Assume that the control variables u_1 and u_2 can be alternatively used in two different regions $\Omega^1(x)$ and $\Omega^2(x)$ of the state space, bounded by a switching boundary $\bar{\varphi}$ defined by $\varphi(x) = 0$, where

$$\varphi(x) = x'Px - c, \quad P = P' = diag\{p_1, p_2\} > 0 \qquad (3)$$

Specifically

$$\Omega^1(x) = \{x : \varphi(x) > 0\}, \quad \Omega^2(x) = \{x : \varphi(x) < 0\}$$

Then, the problem in question is to find a hybrid control strategy capable of globally stabilizing the origin of the state space in spite of the presence of a non completely matched uncertainty vector term acting on the controlled system.

3 The hybrid control strategy

With reference to Ω^1 and Ω^2 introduce the linear functions

$$\sigma_i(x) = x_2 + \alpha_i x_1, \quad \alpha_i > 0, \quad i = 1, 2 \qquad (4)$$

with $\alpha_1 < \alpha_2$, and the corresponding sliding manifolds $\sigma_i(x) = 0$, $i = 1, 2$ respectively. Then, define the hybrid control law as

$$\begin{cases} u_1 = -K_1 sign(\sigma_1(x)), & K_1 > 0 \\ u_2 = 0 \end{cases} \quad when \;\; x \in \Omega^1(x) \qquad (5)$$

and

$$\begin{cases} u_1 = 0 \\ u_2 = -K_2 sign(\sigma_2(x)), & K_2 > 0 \end{cases} \quad when \;\; x \in \Omega^2(x) \qquad (6)$$

where $K_1 \geq \bar{K}_1$ and $\bar{K}_1 > 0$ is such that the reaching condition $\sigma_1(x)\dot{\sigma}_1(x) \leq -\eta_1 |\sigma_1(x)|$ is fulfilled in $\Omega^1(x)$, while $K_2 \geq \bar{K}_2$ and $\bar{K}_2 > 0$ satisfies the analogous reaching condition $\sigma_2(x)\dot{\sigma}_2(x) \leq -\eta_2 |\sigma_2(x)|$ in $\Omega^2(x)$, η_1 and η_2 being strictly positive constants. This choice ensures that, in case $\Omega^1(x) = \Re$ and $\Omega^2(x) = \emptyset$ ($\emptyset$ being the empty set), or, alternatively, $\Omega^1(x) = \emptyset$ and $\Omega^2(x) = \Re$, the sliding manifold $\sigma_1(x) = 0$ or $\sigma_2(x) = 0$, respectively, would be reached in a finite time [11]. As usual in *VSC* control, (see, e.g. [12], [5]), the switching functions $\sigma_i(x)$, $i = 1, 2$, are selected so that when the state of system (1) is restricted to lay on the sliding manifolds, the system dynamics exhibits the desired behaviour.

The proposed *hybrid-VSC* strategy does not guarantee by itself the global stability of the origin of the controlled system state space, but some further conditions on the gains K_1 and K_2 must be imposed. To this end, first observe that each region $\Omega^i(x)$, $i = 1, 2$, can be partitioned into eight different regions $\Omega^i_{a,b,c}(x)$ where

$$a = \begin{cases} 1 & if \; x_2 > 0 \\ -1 & if \; x_2 < 0 \end{cases}, \quad b = \begin{cases} 1 & if \; sign(\sigma_1(x)) > 0 \\ -1 & if \; sign(\sigma_1(x)) < 0 \end{cases}$$

$$c = \begin{cases} 1 & if \; sign(\sigma_2(x)) > 0 \\ -1 & if \; sign(\sigma_2(x)) < 0 \end{cases}$$

Depending on the relative value of the coefficients α_1 and α_2 in (4), some $\Omega^i_{a,b,c}(x)$, $i = 1, 2$, are empty. Specifically, if $\alpha_2 > \alpha_1$, as in the considered case, the empty regions are $\Omega^i_{-1,1,-1}(x)$, $i = 1, 2$, and $\Omega^i_{1,-1,1}(x)$, $i = 1, 2$; while if it were $\alpha_1 > \alpha_2$, the empty regions would be $\Omega^i_{-1,-1,1}(x)$, $i = 1, 2$, and $\Omega^i_{1,1,-1}(x)$, $i = 1, 2$.

Associated with $\Omega^i_{a,b,c}(x)$, $i = 1, 2$, it is also possible to define the $\Delta^i_{a,b,c}(x)$, $i = 1, 2$, vicinity of the switching boundary $\bar{\varphi}$ inside $\Omega^i_{a,b,c}(x)$ as follows

$$\Delta^i_{a,b,c}(x) = \left\{x \in \Omega^i_{a,b,c}(x) : (|x_2| > \delta_1) \cap (\|x - \bar{\varphi}\| < \delta_2)\right\}$$

$i = 1, 2$, where δ_1 and δ_2 are arbitrarily small positive constants. Moreover, denote by

$$\Delta^i(x) = \bigcup \Delta^i_{a,b,c}(x), \quad a = \pm 1, \; b = \pm 1, \; c = \pm 1, \; i = 1, 2$$

the $\Delta^i(x)$, $i = 1, 2$, vicinity of the switching boundary inside $\Omega^1(x)$ and $\Omega^2(x)$.

Now let $\tilde{K}_i$, $i = 1, 2$, be two values such that

$$\tilde{K}_i > \frac{|x'Pf(x(t), t)|}{|x'Pg_i|}, \quad \forall x \in \Delta^i(x), \quad i = 1, 2 \qquad (7)$$

and finally assume that the control gains K_i, $i = 1, 2$, of the hybrid control law (5), (6) are chosen as follows

$$K_i \geq \max\left\{\bar{K}_i, \tilde{K}_i\right\}, \quad i = 1, 2 \qquad (8)$$

4 Robustness, stability and convergence analysis

In this section, the robustness features versus unmatched uncertainties of the proposed *hybrid-VSC* strategy will be analyzed. The stability of the origin of the closed-loop system, as well as the convergence of the state trajectory will be also discussed. For the sake of clarity, two cases will be considered separately: the case of $\bar{f}_1(x(t), t) = 0$, $\forall\, t \geq 0$, (matched uncertainty case), and that of $\bar{f}_1(x(t), t) \neq 0$, satysfying the assumption of Section 2.

4.1 The matched uncertainty case

Now assume that $\bar{f}_1(x(t), t) = 0, \forall\, t \geq 0$. Since $\bar{f}_2(x(t), t)$ affects the system by acting in the input channel, such an uncertainty is matched. This case has been analyzed in datail in [6]. Hereafter, the main results are briefly recalled for the reader's convenience, they being the basis for the treatment of the unmatched uncertainty case.

In [6], the stability of the origin of the closed-loop system (1), (5), (6), (8) has been investigated by analyzing the behaviour of the state trajectories in the vicinities $\Delta^i(x)$, $i = 1, 2$, of the switching boundary $\bar{\varphi}$. In particular, a sort of "reaching condition" for $\bar{\varphi}$ is studied, in order to establish which parts of the switching boundary exerts an attractive or repulsive action on the controlled state trajectories. Indeed, note that in view of definition (3), system (1) and the hybrid control law (5), (6), in $\Delta^i(x)$, $i = 1, 2$, it results that

$$\dot{\varphi}(x) = 2x'Pf(x(t), t) - 2x'Pg_i K_i sign(\sigma_i(x)) \qquad (9)$$

Moreover, in $\Delta^1_{a,b,c}(x)$ one has $\varphi(x) > 0$ and, in view of (7), (8),

$$sign(x'Pg_1K_1sign(\sigma_1(x))) = sign(ab)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = -sign(ab) \tag{10}$$

On the contrary, in $\Delta^2_{a,b,c}(x)$ one has $\varphi(x) < 0$,

$$sign(x'Pg_2K_2sign(\sigma_2(x))) = sign(ac)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = sign(ac) \tag{11}$$

Four different cases can occur:

case 1 : when $ab = -1$ and $ac = -1$, $\varphi(x)\dot{\varphi}(x) > 0$ in $\Delta^1_{a,b,c}(x)$, while $\varphi(x)\dot{\varphi}(x) < 0$ in $\Delta^2_{a,b,c}(x)$, so that the state trajectories move from $\Delta^2_{a,b,c}(x)$ to $\Delta^1_{a,b,c}(x)$;

case 2 : when $ab = 1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) < 0$ in $\Delta^1_{a,b,c}(x)$, while $\varphi(x)\dot{\varphi}(x) > 0$ in $\Delta^2_{a,b,c}(x)$, so that the state trajectories move from $\Delta^1_{a,b,c}(x)$ to $\Delta^2_{a,b,c}(x)$;

case 3 : when $ab = -1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) > 0$ both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$. Hence, the state trajectories cannot go through the switching boundary $\bar{\varphi}$, which is "repulsive" both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$, note that if $\alpha_2 > \alpha_1$ this case never applies;

case 4 : when $ab = 1$ and $ac = -1$, $\varphi(x)\dot{\varphi}(x) < 0$ both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$. Hence, the state trajectories point to the switching boundary $\bar{\varphi}$ both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$. In this case some parts of $\bar{\varphi}$ turn out to be a sliding mode domain of the controlled system. The direction of the sliding movement is from σ_1 to σ_2, being determined by the sign of x_2 (i.e., the sign of a), which is equal to the sign of $\dot{x}_1$ (see, eq.(1)). Note also that when $\alpha_1 > \alpha_2$ this case never applies (see, for instance, [7]).

Figure 1 illustrates a sketch of the state space together with the switching boundary $\bar{\varphi}$ (the ellipse in continuous line) and the sliding manifolds σ_1 and σ_2 when $\alpha_2 > \alpha_1$, as assumed in this paper. The figure also shows the labels a, b, c associated with the different parts of $\bar{\varphi}$.

Relying on a limiting procedure, the previous analysis can be directly extended to the points of $\varphi(x)$ such that $|x_2| \le \delta_1$, $\delta_1 \to 0$. In [6], the following results, here reported without proofs, have been proved.

Lemma 1:
Given system (1) controlled by the *hybrid-VSC* strategy (5), (6), (8), in the matched uncertainty case ($\bar{f}_1(x(t), t) = 0$, $\forall\, t \ge 0$), the trajectories of the hybrid closed-loop system do not have any limit cycle.

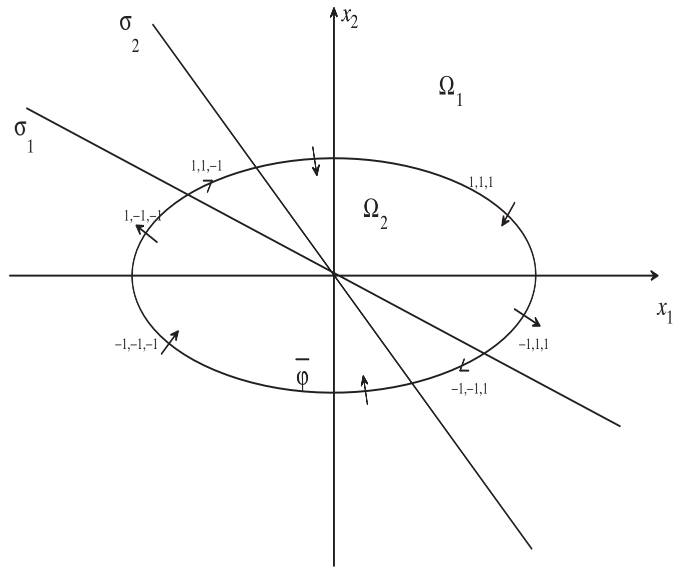

Figure 1: The state space partition.

Corollary 1:
Given system (1) controlled by the *hybrid-VSC* strategy (5), (6), (8), in the matched uncertainty case ($\bar{f}_1(x(t), t) = 0$, $\forall\, t \ge 0$), any trajectory moving from Ω^2 to Ω^1, reaches in Ω^1 the sliding manifold $\sigma_1(x) = 0$.

Theorem 1:
Given system (1) controlled by the *hybrid-VSC* strategy (5), (6), (8), in the matched uncertainty case ($\bar{f}_1(x(t), t) = 0, \forall\, t \ge 0$), the origin of the state space is a globally asymptotically stable equilibrium point of the controlled system.

4.2 The unmatched uncertainty case

Now we consider the original formulation of the control problem, with $\bar{f}_1(x(t), t)$, generally different from zero, satisfying the assumptions of Section 2. In this case, the origin of the state space cannot be a globally asymptotically stable equilibrium point of the controlled system, since, as mentioned before, the effect on the equivalent system dynamics of the unmatched uncertainty term cannot be eliminated by simply varying the control amplitude. The aim of the present section is to prove, however, that the origin of the state space is a globally stable equilibrium point of the system controlled via the proposed *hybrid-VSC* strategy, and that the scalar state of the equivalent system is ultimately bounded with respect to an arbitrarily small set containing the origin. To this end, the following preliminary result can be proved.

Lemma 2:
Given system (1), assume that $\Omega^1(x) = \Re$, $\Omega^2(x) = \emptyset$, and that the control signals are chosen according to the following

strategy

$$\left\{\begin{array}{l} u_1 = -K_1 sign(\sigma_1(x)), \quad K_1 > 0 \\ u_2 = 0 \end{array}\right. \qquad when \;\; x \in \Omega^1(x) \tag{12}$$

where $\sigma_1(x)$ is defined in (4), and, by assumption, $K_1 \geq \bar{K}_1$ and $\bar{K}_1 > 0$ is such that the reaching condition $\sigma_1(x)\dot{\sigma}_1(x) \leq -\eta_1 |\sigma_1(x)|$ is fulfilled in $\Omega^1(x)$, then, the state $x_1(t)$ is ultimately bounded with respect to the set $\Theta_1 = \left\{x_1 : |x_1| < \frac{k_1}{\alpha_1}\right\}$, where α_1 and k_1 are known strictly positive constants defined in (4) and (2), respectively.

Proof: In this particular case the control strategy is a conventional sliding mode control. Then, the sliding manifold $\sigma_1(x) = 0$ is reached in a finite time $\bar{t}_1 \leq |\sigma_1(x(0))| / \eta_1$ [11]. From $\bar{t}_1$ on, the system dynamics can be described by the reduced order equivalent system

$$\dot{x}_1(t) = -\alpha_1 x_1(t) + \bar{f}_1(x(t), t) \tag{13}$$

$t \geq \bar{t}_1, x_1(\bar{t}_1) \;\; s.t. \;\; \sigma_1(x(\bar{t}_1)) = 0.$

Consider the Lyapunov function

$$V(t) = \frac{1}{2\alpha_1} x_1^2(t) \tag{14}$$

Differentiating (14) along the trajectory of (13), it yields

$$\begin{aligned} \dot{V}(t) &= \frac{1}{\alpha_1} x_1(t)(-\alpha_1 x_1(t) + \bar{f}_1(x(t), t)) && (15) \\ &= -x_1^2(t) + \frac{1}{\alpha_1} x_1(t) \, \bar{f}_1(x(t), t) && (16) \\ &\leq -x_1^2(t) + \frac{1}{\alpha_1} |x_1(t)| \cdot \left| \bar{f}_1(x(t), t) \right| && (17) \\ &\leq -|x_1(t)| \cdot \left(|x_1(t)| - \frac{k_1}{\alpha_1} \right) && (18) \end{aligned}$$

From (18), it appears that for $x_1(t) \notin \Theta_1$, $\dot{V}(t) < 0$. Then, there is a time instant $t_{\Theta_1} \geq \bar{t}_1$ such that for $t > t_{\Theta_1}$ the equivalent system state $x_1(t) \in \Theta_1$, that is $x_1(t)$ is ultimately bounded with respect to the set Θ_1.

Relying on Lemma 2 and Corollary 1, the following result can be proved.

Theorem 2:
Given system (1) controlled by the *hybrid-VSC* strategy (5), (6), (8), assume that the switching boundary $\bar{\varphi}$ is chosen so that

$$c > k_1^2 \left(\frac{p_1}{\alpha_1^2} + p_2 \right) \tag{19}$$

where c, p_1, p_2, k_1 and α_1 are known strictly positive constants defined in (3), (2), and (4), respectively, then, the origin of the state space is a globally stable equilibrium point of the controlled system. Moreover, the controlled system state is ultimately bounded with respect to a region centered at origin of the state space and internal to $\Omega^2(x)$.

Proof: Assume first that $x(0) \in \Omega^2(x)$. Then, three cases are possible.

A1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_2(x) = 0$ in a finite time $\bar{t}_2 \leq |\sigma_2(x(0))| / \eta_2$ [11], and, from $\bar{t}_2$ on, the system dynamics can be described by the reduced order equivalent system

$$\dot{x}_1(t) = -\alpha_2 x_1(t) + \bar{f}_1(x(t), t) \tag{20}$$

$t \geq \bar{t}_2$, $x_1(\bar{t}_2) \;\; s.t. \;\; \sigma_2(x(\bar{t}_2)) = 0$. Then, according to the proof of Lemma 2, there exists a time instant $t_{\Theta_2} \geq \bar{t}_2$ such that for $t > t_{\Theta_2}$ the equivalent system state $x_1(t) \in \Theta_2 = \left\{x_1 : |x_1| < \frac{k_1}{\alpha_2}\right\}$, that is $x_1(t)$ is ultimately bounded with respect to the set Θ_2.

A2 The trajectory reaches a domain of $\bar{\varphi}$ of sliding mode (when $\alpha_2 > \alpha_1$, such domains exist as discussed in [6]), then it goes to the sliding manifold $\sigma_2(x) = 0$ and case A1 applies.

A3 The trajectory leaves $\Omega^2(x)$ and enters $\Omega^1(x)$. In view of Corollary 1, which still holds in this case, it reaches the sliding manifold $\sigma_1(x) = 0$ in a finite time, then, the case B1 below holds.

When $x(0) \in \Omega^1(x)$, one of the following three cases holds.

B1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_1(x) = 0$ in a finite time $\bar{t}_1 \leq |\sigma_1(x(0))| / \eta_1$ [11], and, from $\bar{t}_1$ on, the system dynamics can be described by the reduced order equivalent system

$$\dot{x}_1(t) = -\alpha_1 x_1(t) + \bar{f}_1(x(t), t) \tag{21}$$

$t \geq \bar{t}_1$, $x_1(\bar{t}_1) \;\; s.t. \;\; \sigma_1(x(\bar{t}_1)) = 0$. Then, according to the proof of Lemma 2, the state $x_1(t)$ is ultimately bounded with respect to the set $\Theta_1 = \left\{x_1 : |x_1| < \frac{k_1}{\alpha_1}\right\}$. Yet, by virtue of condition (19), the choice of $\bar{\varphi}$ satisfies the inequality

$$p_1 x_1^2 + p_2 x_2^2 > k_1^2 \left(\frac{p_1}{\alpha_1^2} + p_2 \right) \tag{22}$$

that is, all the $x \in \Omega^1(x)$ are such that

$$|x_1| > \sqrt{\frac{k_1^2}{\alpha_1^2} + \frac{k_1^2 p_2}{p_1}} > \frac{k_1}{\alpha_1} \tag{23}$$

As a consequence, $\Theta_1 \subset \Omega^2(x)$. Then, the switching boundary $\bar{\varphi}$ is reached in finite time, and the controlled system behaviour is equivalent to the one considered in case A2.

B2 The trajectory reaches a sliding mode domain of $\bar{\varphi}$ (case A2).

B3 The trajectory enters $\Omega^2(x)$ and one of the cases A1-A3 applies.

Thus, in any case, the controlled system state reaches, in finite time, the region centered at the origin

$$\Theta = \{x \in \Re^2 : x_1 \in \Theta_2,\ |x_2| < k_1\} \cap \{x \in \Re^2 : \sigma_2(x) = 0\} \tag{24}$$

with $\Theta \subset \Omega^2(x)$.

Remark: Note that, on the basis of Theorem 2, the state $x_1(t)$ is ultimately bounded with respect to a set, centered at the origin, which can be made arbitrarily small by increasing the design parameter α_2. In contrast, the other state variable, is ultimately bounded with respect to a set, centered at the origin, the dimension of which only depends on the unmatched uncertainty bound k_1.

4.2.1 Example 1

Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + 10\sin(x_1(t)) - 2\cos(x_1(t)) + 4\sin(10t) \\ \dot{x}_2(t) &= -x_2(t) - 3\sin(x_1(t)) + u_1(t) + u_2(t) \\ x(0) &= \overline{x} \end{aligned} \tag{25}$$

where $\bar{x} = [-40\ \ 16]'$. The switching boundary is $\varphi(x) = x'Px - 9$, $P = diag(1,4)$, and, according to the hybrid control strategy described in Section 2, $u_1(t)$ and $u_2(t)$ are designed as in (5), (6), with $K_1 = 150$, $K_2 = 100$, the corresponding sliding manifolds being, respectively,

$$\begin{aligned} \sigma_1(x) &= x_2(t) + x_1(t) \\ \sigma_2(x) &= x_2(t) + 4x_1(t) \end{aligned} \tag{26}$$

Note that in this case, the choice $c = 9$ does not respect condition (19) in Theorem 2, since a reasonable choice for k_1 is 20, while $p_1 = 1$, $p_2 = 4$, and $\alpha_1 = 1$.

The simulation results of the application of the described *hybrid-VSC* strategy are shown in Figure 2. In this case, the controlled system state does not reach the switching boundary $\bar{\varphi}$, and results in being ultimately bounded with respect to a set external to $\Omega^2(x)$.

4.2.2 Example 2

Consider again system (25) where $\bar{x} = [-40\ 16]'$. The switching boundary is $\varphi(x) = x'Px - 2500$, $P = diag(1,4)$, and, according to the hybrid control strategy described in Section 2, $u_1(t)$ and $u_2(t)$ are designed as in (5), (6), with $K_1 = 150$, $K_2 = 100$, the corresponding sliding manifolds being the same as in (26). Note that in this case, the choice $c = 2500$ satisfies condition (19) in Theorem 2.

The simulation results of the application of the described *hybrid-VSC* strategy are shown in Figure 3. As expected, in this case, the controlled system state reaches in finite time the switching boundary $\bar{\varphi}$, and results in being ultimately bounded with respect to a set included in $\Omega^2(x)$. Note that, the equivalent system state $x_1(t)$ is ultimately bounded with respect to a set $\Theta_2 \subset \Theta_1$, denoting with Θ_1 the convergence set for $x_1(t)$ in Example 1. Moreover, the amplitude of the set Θ_2 can be reduced by increasing the design parameter α_2, as shown in Figure 4, where the case with $\alpha_2 = 10$ is reported (all the other parameters are the same as in Example 2).

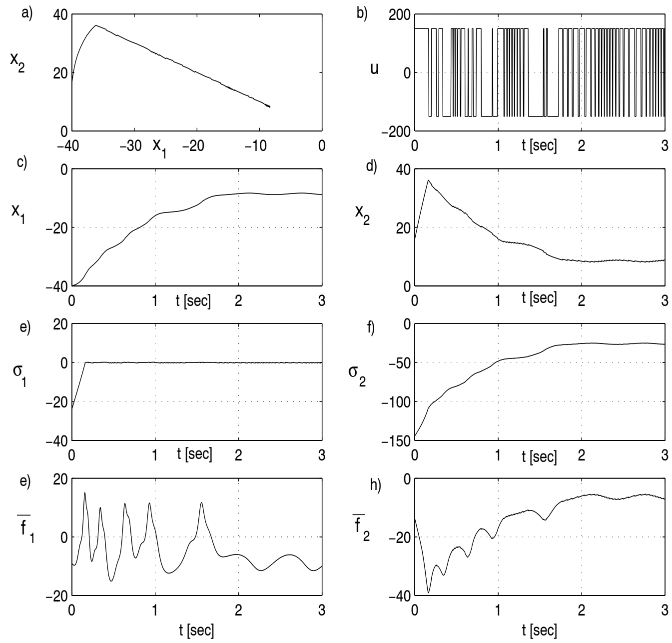

Figure 2: a) $x_2(t)$ versus $x_1(t)$, b) the control signal $u(t)$, c) $x_1(t)$ versus time, d) $x_2(t)$ versus time, e) $\sigma_1(x)$ versus time, f) $\sigma_2(x)$ versus time, g) $\bar{f}_1(x(t),t)$ versus time, h) $\bar{f}_2(x(t),t)$ versus time, in Example 1.

5 Conclusions

In this paper, the robustness features versus unmatched uncertainties of a hybrid variable structure control strategy for a class of second order systems are analyzed and discussed. The hybrid strategy is based on a system state subdivision into regions with which different control laws are associated, so that the selection among the various laws is event-driven. Sliding mode behaviors are generated on the sliding manifolds and on a suitably defined switching boundary. As conventional sliding mode control, the proposed *hybrid-VSC* strategy is robust with respect to matched bounded uncertainties. The effect of unmatched uncertainties on the equivalent system dynamics can be reduced by suitable selecting the switching boundary and the design parameter of the sliding manifold associated with the inner region. The proposed *hybrid-VSC* strategy proves to globally stabilize the origin of the system state space. Moreover, the equivalent system scalar state during sliding mode in the inner region, which is a system mode reached in finite time, is ultimately bounded with respect to an arbitrarily small set containing the origin.

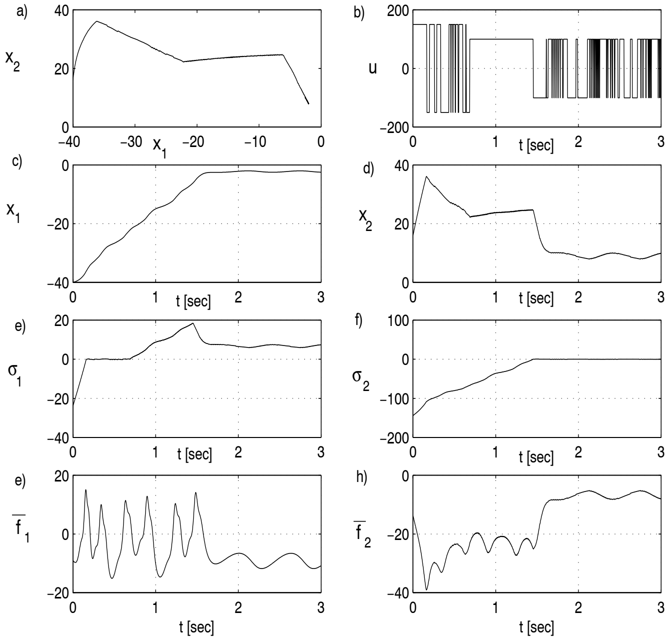

Figure 3: a) $x_2(t)$ versus $x_1(t)$, b) the control signal $u(t)$, c) $x_1(t)$ versus time, d) $x_2(t)$ versus time, e) $\sigma_1(x)$ versus time, f) $\sigma_2(x)$ versus time, g) $\bar{f}_1(x(t),t)$ versus time, h) $\bar{f}_2(x(t),t)$ versus time, in Example 2 with $\alpha_2 = 4$.

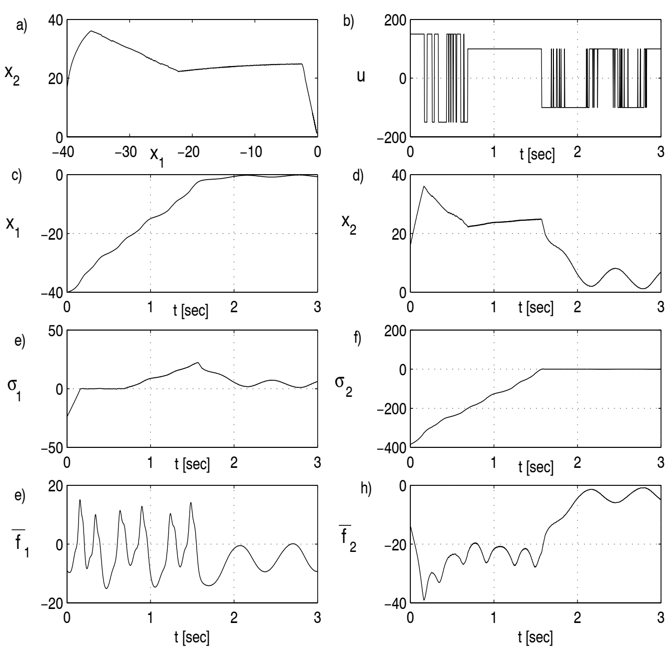

Figure 4: a) $x_2(t)$ versus $x_1(t)$, b) the control signal $u(t)$, c) $x_1(t)$ versus time, d) $x_2(t)$ versus time, e) $\sigma_1(x)$ versus time, f) $\sigma_2(x)$ versus time, g) $\bar{f}_1(x(t),t)$ versus time, h) $\bar{f}_2(x(t),t)$ versus time, in Example 2 with $\alpha_2 = 10$.

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