A LYAPUNOV APPROACH TO H_2 ITERATIVE ADJUSTMENT FOR FIXED STRUCTURE CONTROLLERS

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Abstract

The adjustment of fixed structure controllers becomes a major issue in the development of complex control systems. Indeed, for long term project, the controller structure is often fixed at an early stage while the tuning of controller parameters remains possible all along the project. It is also a key point whenever multi-objective design problems are tackled.

The objective of the fixed structure controllers H_2 adjustment is to adjust selected gains of a given control law in order to reduce the H_2 norm of a transfer function representative of some performance criterium while minimizing all other changes in the closed loop behavior.

This paper deals with methodological aspects of H_2 adjustment. We present the development of a Lyapunov approach based on the classical Lyapunov computation of H_2 norms. An analytic sensitivity analysis is carried out that leads to an efficient gradient like search. Furthermore we show that fast and accurate numerical procedures may be derived allowing to work with large scale models often encountered in adjustment aeronautical applications.

1 Introduction

1.1 Motivation

Our main concern is in flight control system applications for large passenger transportation aircraft. The aircraft control law design is seldom a direct operation. Generally multiple stages are necessary to obtain a satisfactory result. When the design conditions are modified, to start again the work since the beginning is not necessarily the most effective way. Thus the need for a retuning of the control laws may arise all along the development of an aircraft, each time the design model or the specifications evolve.

Obviously the availability of efficient adjustment tools is of particular importance in the end of the project where delays may have highly negative economic consequences on the overall project. At this stage the structure of the controller is frozen, and the adjustment procedures only intend to adapt some controller parameter so that every control law requirements are satisfied. Thus the development of adjustment tools dedicated to fixed structure controllers becomes a major issue in the development of flight control systems.

Among various objectives, adjustment of H_2 norms is often required because a lot of design objectives express as H_2 criteria. Adjustment tools allow to compensate for differences between the aircraft design model and the actual aircraft model that includes all the more recent knowledge of the aircraft behavior. This later model often includes a lot of modes neglected at the design step, but may also differ because of new available flight test identification results.

Another need for H_2 adjustment tools results from the fact that a lot of H_2 constraints are just roughly taken into account for when designing the control law, or even not at all. For example an accurate evaluation of the aircraft behavior in the presence of turbulence wind in terms of servomechanisms fatigue, mechanical loads undergone by the aircraft structure or passengers comfort may leads to the adjustment of the control law gains.

1.2 Methodological aspects

We begin to show in section 2 that the adjustment problem may be seen has a decentralized static output feedback design problem. Fixed order and fixed structure design problems are known to be generally nonconvex. For example existence and uniqueness of stabilizing controllers of a given order or structure is still an open question [1].

Several heuristic procedures have been proposed to solve fixed order design problems. They mostly rely on numerical optimization [2] and thus may be used in the adjustment context. Indeed, even if our objective is not to find an H_2 optimal solution, iterative techniques dedicated to the minimization of H_2 norms provide a good basis for the development of H_2 adjustment procedures.

Proposed approaches are often based on a two-stage optimization process: $V-K$ iterations [2], alternating convex projection methods [3] [4], dual iterations [5]. Each stage is a (quasi) convex optimization problem set up within the linear matrix inequality (LMI) framework. However optimization problems that involve large scale models are not currently tractable by LMI solvers.

We deal here with such large scale problems. The question of stabilization is bypassed assuming that an initial feasible point has been yet found. We just have to preserve stability. Thus our objective is to develop efficient numerical procedures allowing

to adjust selected gains of a given control law in order to
- reduce the H_2 norm of a transfer function representative of some performance criterium or constraint,
- preserve stability,
- and also minimize all other changes in the closed loop behavior.
The satisfaction of this two later points simply results here from the fact that we develop iterative small gain corrections methods. At each step the adjustment can be stopped if unexpected changes appear. A more accurate processing of these constraints would be interesting but is out of the scope of the present paper.

One approach to controller H_2 adjustment for such high dimensional systems is through the use of Lyapunov solvers. This is the way followed in this paper.

In the first section we develop a sensitivity analysis of H_2 criterium based on Lyapunov theory. We show that the sensitivity with respect to controller gains expresses as a function of the matrix solutions of a set of coupled Lyapunov equations. This analytic expression of the sensitivity is then used in section 3 where an H_2 adjustment procedure is proposed. Such ideas may be seen as a reminiscence of [6] [7] where more complex mixed H_2/H_∞ design problems are studied. The main advantage of this Lyapunov approach is that numerically efficient Lyapunov solvers exist. This lead us to show by numerical experiments that, for each adjustment iteration, a linearly increasing CPU time with respect to the model order may be achieved (section 4).

It is worth to be pointed out that another Lyapunov based approach may be followed. As a matter of fact the H_2 optimal static output feedback may be characterized by a set of bilinear matrix equations. Alternating projection methods may be then applied. This yields to an iterated resolution of Lyapunov equation sets. In this spirit, the Kleinman algorithm [8] dedicated to the H_2 optimal state feedback case is known to converge. The extension to the static output feedback case is under study.

As an alternative strategy for addressing H_2 optimal problems subject to architecture constraints, homotopic technique have been also proposed. Related works may be found in [9] [10] [11] where homotopic and descent algorithms are discussed. Such techniques might be also very interesting within the adjustment context.

Finally, note that direct synthesis approaches is out off the scope of the present paper, since we follow here a multiple steps design approach.

2 H_2 norm sensitivity analysis

2.1 Modeling dedicated to adjustment

We consider an $M-\Delta$ representation of a closed system model as depicted on figure (1). The matrix $\Delta = K$ involves the controller gains. Some of them are to be adjust in order to decrease the H_2 norm of the closed-loop transfer G from w to e. Such a structured gain is often called *decentralized static output feedback* and captures a large class of controller architectures [12].

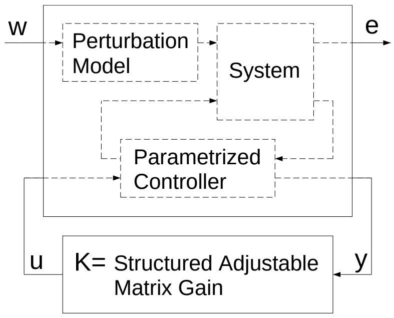

Figure 1: $M-\Delta$ representation of a closed system

The closed-loop state space representation writes as:

$$\begin{aligned} \dot{x} &= (A+BKC_y)x + B_w w \\ e &= (C_e + D_{eu}KC_y)x \end{aligned} \tag{1}$$

The A matrix includes the open loop system dynamic, the controller dynamic and the perturbation model dynamic. Thus u and y on figure (1) are not the true system input and output, but fictitious signals that are equivalent to control and measurement signals for the $M-\Delta$ representation.

We assume that the H_2 norm between w and e is finite whatever is the feedback gain K. This implies that D_{ew}, D_{yw} and D_{yu} are all zero matrices.

2.2 Computation of the H_2 sensitivity

We denote by J the squared H_2 norm of G: $J = \|G\|_2^2$. The classical computation of J within the Lyapunov approach is first recalled.

$$J = trace\left\{(C_e + D_{eu}KC_y)X(C_e + D_{eu}KC_y)^T\right\} \tag{2}$$

where X satisfies the Lyapunov equation

$$(A+BKC_y)X + X(A+BKC_y)^T + B_w B_w^T = 0 \tag{3}$$

Within the stochastic framework, if w is a white noise process with power spectral density equal to 1, then the definite non negative matrix X is the covariance of the state variable x.

Since all closed loop matrices are linear functions of the matrix gain K it is always possible to shift them towards zero. Thus we can assume that the initial value of K is zero.

The sensitivity matrix of J with respect to K is a matrix denoted $S = \partial J/\partial K$ with the same dimensions as K. It depends on K. Its value at the initial point $K = K_0 = 0$ is given by the following lemma (2.1). However it must be pointed out that this lemma only applies to non zero terms of S. Since we consider structured feedback, if $K_{i,j}$ is fixed then obviously $S_{i,j} = 0$.

Lemma 2.1 *The general term of the sensitivity matrix S is given by:*

$$S_{i,j} = tr\left\{C_e W_{i,j} C_e^T + D_i C_j X C_e^T + C_e X (D_i C_j)^T\right\}$$

where $X \geq 0$ and $W_{i,j}$ satisfy the set of Lyapunov equations:

$$\begin{array}{l} AX + XA^T + B_w B_w^T = 0 \\ AW_{i,j} + W_{i,j} A^T + B_i C_j X + X (B_i C_j)^T = 0 \end{array} \tag{4}$$

and where B_i (resp. D_i) stands for the ith column of B (resp. D_{eu}) and C_j stands for the jth row of C.

Proof 2.2 *The dimension of K is $m \times p$. We denote by $e_{i,m}$ the ith vector of the natural basis of $\mathbb{R}^m$. We consider a variation of K with norm ρ and which is zero everywhere except at row i and column j. This means that K writes:*

$$K = \rho\, e_{i,m} e_{j,p}^T \tag{5}$$

We have:

$$\begin{array}{rcl} B_i &=& B_u e_{i,m} \\ C_j &=& e_{j,p}^T C_y \\ D_i &=& D_{eu} e_{i,m} \end{array}$$

Then, from equations (2) and (3), J expresses as a function of ρ:

$$\begin{array}{l} (A + \rho B_i C_j) X + X (A + \rho B_i C_j)^T + B_w B_w^T = 0 \\ J(\rho) = tr\left\{(C_e + \rho D_i C_j) X (C_e + \rho D_i C_j)^T\right\} \end{array}$$

The derivative of J with respect to this parameter $\rho = K_{i,j}$ is the general term of the sensitivity matrix:

$$\begin{array}{rcl} S_{i,j} &=& dJ/d\rho \\ &=& tr\left\{D_i C_j X (C_e + \rho D_i C_j)^T \right. \\ && +(C_e + \rho D_i C_j) W_{i,j} (C_e + \rho D_i C_j)^T \\ && \left. +(C_e + \rho D_i C_j) X (D_i C_j)^T \right\} \end{array}$$

where $W_{i,j}$ is the matrix defined by $W_{i,j} = dX/d\rho$.

In order to evaluate this later matrix we compute the derivative of the Lyapunov equation satisfied par X (equation 3). Its derivative with respect to ρ writes as:

$$\begin{array}{l} B_i C_j X + (A + \rho B_i C_j) W_{i,j} \\ +W_{i,j} (A + \rho B_i C_j)^T + X (B_i C_j)^T = 0 \end{array}$$

Finally taking $\rho = 0$ yields to the expected lemma result.

3 An H_2 adjustment procedure

3.1 Maximal sensitivity descent

The behavior of J about the initial point $K_0 = 0$ is described at the first order by:

$$J \approx J_0 + \mathrm{tr}\left\{S^T (K - K_0)\right\} \tag{6}$$

where J_0 is the value when $K = K_0 = 0$. If a gradient algorithm were used to decrease J, with length ρ, then on should take :

$$K = K_0 - \rho \frac{S}{\mathrm{tr}\left\{S^T S\right\}} \tag{7}$$

However our objective is to reduce J but without any other important change. This constraint led us not to use a gradient search algorithm but a descent algorithm along the direction of maximal sensitivity.

Let us consider a Singular Value Decomposition of S : $S = U\Sigma V^T$. We can write :

$$K = K_0 + URV^T$$

Then we have :

$$\begin{array}{rcl} J - J_0 &=& \mathrm{tr}\left\{V\Sigma^T U^T URV^T\right\} = \mathrm{tr}\left\{V\Sigma^T RV^T\right\} \\ &=& \mathrm{tr}\left\{V^T V\Sigma^T R\right\} = \mathrm{tr}\left\{\Sigma^T R\right\} \\ &=& \mathrm{tr}\left\{R\Sigma^T\right\} = \sum_i R_{ii}\sigma_i \end{array}$$

Let us remark that $\|K - K_0\|_2 = \|URV^T\|_2 = \|R\|_2$. Thus if this norm is fixed and equal to ρ, the maximal decrease of J is then achieved with

$$K = K_0 - \rho \frac{u_1 v_1^T}{\sigma_1} \tag{8}$$

where u_1 and v_1 are the left and right singular vectors associated to the largest singular value σ_1 of S.

The first order variation of J is $J = J_0 - \rho$. This suggests the H_2 adjustment procedure described hereafter:

Procedure

(i) Compute the current closed loop model in order to shift K towards $K_0 = 0$.

(ii) Compute $dK^* = u_1 v_1^T/\sigma_1$, the direction of variation for K maximizing the sensitivity of S about K_0.

(iii) Search for an optimal ρ so that the gain variation $-\rho\, dK^*$ yields to a minimal J value, while keeping the closed loop stable.

(iv) Iterate if required.

Before each step of the algorithm, the previously computed K matrix is included in the system closed loop model. Then the sensitivity is estimated, and a K-variation is proposed along the maximal sensitivity direction. A priori a one dimensional search algorithm may be used to solve step *(iii)*. Furthermore, in order to ensure that the iterations stay inside the stability domain, a logarithmic barrier functional may be used in the spirit of interior point methods. However this is a theoretical solution. Simpler and more intuitive processes, as proposed below, may be as efficient and far less time consuming.

3.2 Optimization over an adaptive mesh

The above procedure involves a constraint one dimensional optimization that may be tackled from different angles. We first note that with : $\rho = \alpha J_0$, the new algorithm step α is just equal to the expected relative decrease of J. Within the adjustment context a typical maximal value for the relative criterium decrease is $\alpha_{max} = 10\%$. And a typical minimal value is $\alpha_{min} = 1\%$, which corresponds to the minimal criterium decrease that is worth to do. Thus it seems to be reasonable to constraint the one dimensional ρ-optimization to the range $J_0 \times [\alpha_{min}, \alpha_{max}]$.

In order to speed up the procedure we propose to just test several ρ values over the previously defined range and to jump to the best stable point. We currently use the following mesh for α:

$$\alpha \in \alpha_{max} \times \{0.1\,,0.3\,,0.7\,,1\} \tag{9}$$

Such a very simple algorithm works quite well when applied to our physical aeronautical problems. When iterated, the adjustment steps produce a sequence of decreasing criterium values. It stops when the relative variation of the criterium is less than α_{min}.

However in some cases it stops prematurely. As a matter of fact, since no continuous penalty function is used to prevent from instability, the current point may converge towards the stability domain barrier. Then it may happens that none of the tested points of the mesh satisfy the stability constraint, and the procedure stops even if the current value of K is far from being locally optimal.

To prevent from such a behavior, the range of the mesh (parameter α_{max}) is adapted. At the beginning of each iteration the value of α_{max} is initialized at 10%. Then it is reduced (multiplication by 0.9) until there exists at least one stable point in the mesh.

One can thinks that the procedure may be used to minimize a H_2 criterium under fixed structure constraints and stability constraint. However it is worth to be pointed out that even if the proposed procedure always converges, it may not always lead to a local optimum. As a matter of fact, even with an adaptive mesh, the descent direction K^* may not be admissible with respect to the stability constraint. Thus we do not really solve the fixed structure H_2 optimization problem under stability constraint.

4 Numerical aspects

4.1 Accuracy

From a numerical point of view, our procedure mainly relies on the resolution of a set of Lyapunov equation, and a singular value decomposition. At each step the number of Lyapunov equations to be solved is equal to the number of free parameters in K plus the number of points of the mesh. Since an important characteristic of the state space models involved in our aeronautical applications is their high dimension, then a lot of care must be paid to the choice of the Lyapunov solver. Indeed even the criterium calculus may be ill-conditioned. And accurate solutions of all the involved Lyapunov equations are not easy to found.

Within such large systems it is recommended to use efficient numerical routines such as the ones of the SLICOT package (*http://www.win.tue.nl/niconet/niconet.html*). That is what we do. The adjustment procedure was thus first developed under MATLAB, with an embedded SLICOT Lyapunov solver. This led us to verify the good behavior of the proposed adjustment procedure.

As regards the SVD, it must be pointed out that it only applies to a matrix whose dimension is equal to that of K. An accurate solution is thus easy to found.

4.2 Computation cost

As regards the computation cost, the first adjustment routine was developed without any special attention dedicated to the computation cost. The average step duration was less than 1.4 s on a SPARC station IV, when a 25 dimensional model was used. In order to increase the size of the models to which the procedure applies a more adapted MATLAB routine has then been developed.

It relies on the fact that at each step, all the Lyapunov equations to be solved involve the same A matrix. Classical Lyapunov solvers use a schur factorization followed by a Gauss pivoting method applied to solve a set of linear triangular equations. Consequently the computational cost of the schur factorization needs not to be repeated. Such a routine yields to an average step duration which is equal to 0.3 s.

With this new algorithm, numerical experiments show that the average CPU time duration of each adjustment iteration increases linearly when the size of the aircraft state space model is greater than about 130 (figures 2). Above this limit each iteration increases of about 0.1s per state. These results were

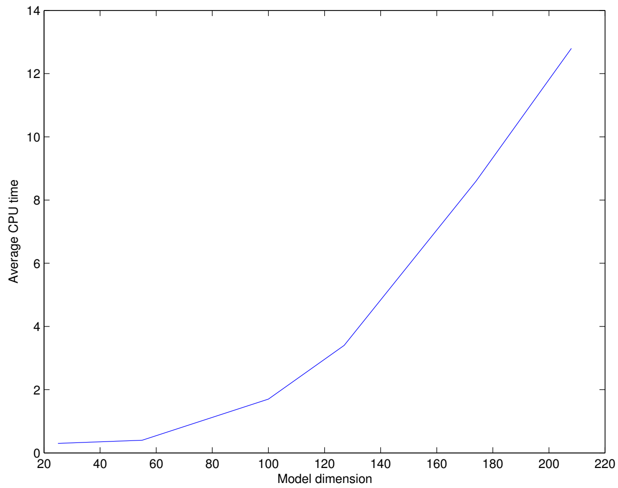

Figure 2: One iteration average CPU time v.s model order

obtained with reduced state space models computed from a unique initial aircraft structural model. For that purpose a reduction procedure dedicated to aircraft structural model reduction has been developed.

5 Application

5.1 An academic example

A first order system has been used to check the behavior of the adjustment algorithm. It is described on figure (3). The corresponding system matrices are the following:

$$A = -1 \qquad B_w = 1 \qquad B_u = 1$$
$$C_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad D_{ew} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad D_{eu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$C_y = 1 \qquad D_{yw} = 0 \qquad D_{yu} = 0$$

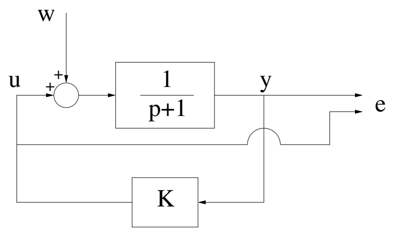

Figure 3: Academic example

For such a simple system, it is quite easy to found the optimal value of the gain K by solving the Riccati equation associated the quadratic criterium. One find $K = 1 - \sqrt{2} = -0.4142$ and the optimal criterium value is $J_{opt} = \sqrt{2} - 1 = 0.4142$.

Table (1) shows that, when iterated, the adjustment procedure converge to the optimal gain. N is the number of iterations required for the algorithm to converge. Obviously this number increases when the initialization moves away from the optimum, because at each step the criterium decrease is restricted to a maximal value of 10%.

K_{init}	J_{init}	K_{final}	J_{final}	N
0	0.50	-0.414	0.414	4
-10	4.59	-0.438	0.414	25
-100	49.51	-0.438	0.414	48

Table 1: H_2 optimization for a first order system

This example shows that the proposed adjustment procedure, when iterated, is able to deliver results that are coherent with the known optimal value.

5.2 Adjustment dedicated to load alleviation

We now consider a realistic model that is well representative of the vertical behavior of a large four engine civil aircraft, with conventional control surfaces. It is a linear aero-elastic model based on a modal description. The overall aircraft model with its control law involves more than 250 states. Such a high dimension modeling is the price to be paid for an accurate representation of the structural dynamic behavior of the aircraft in turbulent wind and over a rather large frequency range.

The mechanical load outputs we dealt with here are:
1. the vertical bending moment at the wing root,
2. the vertical bending moment at the tail root,
3. the torsion moment at a forward fuselage point,
4. the torsion moment at a rear fuselage point.
This preliminary study uses a Dryden turbulent wind model. The Lyapunov analysis introduced above allows us to compute the mechanical load power without carrying on any simulation.

The current control law is based on five measurement outputs: the pitch rate q, and the normal acceleration at four points (crew station, forward and rear fuselage, left and right external engines). The five components of the input vector correspond to rudder and aileron deflections. The controller involves five selected gains to be adjust in order to lower the mechanical loads. Gains 1 to 3 correspond to the classical control of pitch rate and normal acceleration responses. Gains 4 and 5 are intended to reduce the vibration level by aileron and elevator actuation respectively.

For each of the four mechanical load outputs an iterated adjustment of the current control law has been carried out that is intended to minimize the power of the output under consideration (cases 1 to 4). The Lyapunov analysis of these results are summarized in the following table (2):

Moment	Adjustment case			
	1	2	3	4
Bending Wing root	-10	34	46	10
Bending Tail root	20	-60	500	1398
Torsion Forward fuselage	9	78	-40	48
Torsion Rear fuselage	-2	66	-4	-64

Table 2: Load power variations (%)

This table (2) brings to the fore the balance between the two bending moments. If the bending moment at the wing root is reduced by 10 percent (first column), the bending moment at the tail root is increased by 20 %. On the other hand (second column) if this later moment is minimized down to 60 %, then the former increases by 34 %. As regards the torsion fuselage moments, the table depicts the fact that the minimization of these moments may result in a very large increase of the bending moments (columns 3 and 4). Thus, in order to achieve a reasonable adjustment, these two criteria must be completed with wing and tail bending terms.

The adjustment algorithm is obviously not aimed to minimize an H_2 criterium. It is a tool that may help an engineer to tune a given control law, in a multi-objective and highly constraint framework. This example demonstrates that the proposed procedure is able to deal with high dimensional models as far as adjustment of H_2 norms is concerned.

6 Conclusion

In this paper, we propose to use a gradient-like algorithm to solve the iterative H_2 adjustment problem. An analytical expression of the sensitivity function is first derived, and an adjustment algorithm based on a maximal sensitivity descent is then described. Numerical experiments show that the procedure is well suited to solve H_2 adjustment for large scale systems.

Many other tools may be used to tackle the adjustment problem. We must emphasis on the LMI approach, which provides us with a very interesting framework for the iterative design, as illustrated in the references hereafter. For example a method based on LMI computation of the H_2 norm is developed in [13]. However, at the time being, none of these approaches is numerically accurate and efficient enough to be applied within the context of high dimension models.

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