AN ITERATIVE ALGORITHM FOR THE MIXED $\mathrm{H}_2/\mathrm{H}_\infty$ CONTROL PROBLEM USING H_2 NORM DECREASING CONTROLLER SETS

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Abstract

This paper is concerned with the mixed H_2/H_∞ control problem. The purpose of this paper is to give an iterative algorithm for finding a sub-optimal static state feedback controller for the mixed H_2/H_∞ control problem. The key idea of our algorithm is to construct two "controller sets": one is a set of controllers that improve the H_2 norm of the closed loop map for a given controller and the other is a set of controllers whose elements satisfy the H_∞ norm constraint. Using two controller sets, we propose an iterative algorithm. The obtained controller is either the global optimal solution if the H_∞ norm constraint is satisfied until the H_2 norm of the objective closed loop map converges to the H_2 optimal value or a sub-optimal solution on the boundary of the H_∞ norm constraint.

1 Introduction

Recently, multiobjective control problems have received a great deal of attention [1, 2], [4]-[7], [9]-[14]. In particular, the so-called mixed H_2/H_∞ control problem for linear time invariant (LTI) systems has been studied by many researchers. In this problem, the H_2 and H_∞ norms are measures for optimal performance and robustness, respectively. The purpose of the mixed H_2/H_∞ control problem is to find a controller which minimizes the H_2 norm of one closed-loop map with an H_∞ norm constraint of another closed-loop map. That is, this problem is to find the best performance controller among the robustly stabilizing controllers. Both the H_2 and H_∞ control theories have almost been established. However the mixed H_2/H_∞ control problem have not completely been solved. This is because the mixed H_2/H_∞ control problem is quite difficult to be solved theoretically, and it is known that the order of the optimal mixed H_2/H_∞ controller is not finite in some cases. Even for a fixed order controller the problem is still very difficult, because it is a non-convex problem. For this non-convex problem, various approaches to find a sub-optimal solution have been explored. However, there is no method to obtain the global optimal solution except some special cases. Standard technique to get a sub-optimal solution is to use a common LMI solution at the expense of conservatism [2, 4]. Recently, new methods using uncommon LMI solutions have been proposed [1], [9]-[14]. However, all of them except that in [14] do not show what kind of solutions are obtained, although it is important for sub-optimal methods to show the properties of the obtained solutions.

The purpose of this paper is to give a new iterative algorithm for finding a sub-optimal static state feedback controller for the mixed H_2/H_∞ control problem. The key idea of our algorithm is to introduce two "controller sets": one is a set of controllers that improve the H_2 norm of the closed loop map for a given controller and the other is a set of controllers whose elements satisfy the H_∞ norm constraint. Using these sets, we propose an iterative algorithm which produces a controller that satisfies a necessary condition for global optimality. That is, the obtained controller is either the global optimal solution of the unconstraint objective function or a solution on the boundary of the H_∞ norm constraint. Numerical examples show the effectiveness of our algorithm.

2 Problem Formulation

In this paper, consider the following LTI system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_1 w_1(t) + B_2 w_2(t), &(1)\\ z_1(t) &= C_1 x(t) + D_1 u(t), &(2)\\ z_2(t) &= C_2 x(t) + D_2 u(t), &(3)\\ y(t) &= x(t), &(4) \end{aligned}$$

where x is the plant state, $w_i (i = 1, 2)$ are any exogenous inputs, u is the control input and $z_i (i = 1, 2)$ are the performance outputs. Throughout this paper, the following assumptions are made:

1. (A, B) is controllable.
2. $(A, B_i, C_i)(i = 1, 2)$ are controllable and observable.
3. $D_i^T D_i = I (i = 1, 2)$.
4. B_2 has full column rank.
5. $\begin{bmatrix} A - j\omega & B \\ C_2 & D_2 \end{bmatrix}$ has full column rank for all $\omega \in R$.

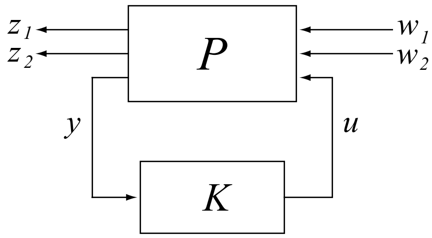

Figure 1: System model for the mixed H_2/H_∞ control

Let us consider the static feedback controller:

$$u(t) = Kx(t). \tag{5}$$

Via the static feedback controller the closed loop system is described by

$$\begin{aligned} \dot{x}(t) &= A_{cl}x(t) + B_1 w_1(t) + B_2 w_2(t), & (6)\\ z_1(t) &= C_{cl1}x(t), & (7)\\ z_2(t) &= C_{cl2}x(t), & (8) \end{aligned}$$

where

$$A_{cl} = A + BK, C_{cli} = C_i + D_i K (i = 1, 2). \tag{9}$$

Let $T_{z_i w_i}(K)$ denote the closed-loop transfer function from w_i to z_i. For this system the mixed H_2/H_∞ control problem is defined as follows.

The mixed H_2/H_∞ control problem (OP): *Given an achievable H_∞ norm bound γ, find a controller that satisfies*

$$\min_K \|T_{z_2 w_2}(K)\|_2 \,\text{subject to}\, \|T_{z_1 w_1}(K)\|_\infty < \gamma, \tag{10}$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the H_2 and H_∞ norms, respectively.

3 The global optimal solution of the mixed H_2/H_∞ control problem

In this section, we show the property of the global optimal solution of the mixed H_2/H_∞ control problem. In general, a global optimal solution of an optimization problem is either a stationary point of the objective function or a feasible solution on the boundary of a constraint. However, such property of the problem (OP) cannot be discussed, because the H_∞ norm constraint in (10) has no boundary. Hence, we modify the problem (OP) into the following problem.

The modified mixed H_2/H_∞ control problem (MP): *Given an achievable H_∞ norm bound γ and sufficiently small ε, find a controller that satisfies*

$$\min_K \|T_{z_2 w_2}(K)\|_2 \,\text{subject to}\, \|T_{z_1 w_1}(K)\|_\infty \le \gamma - \varepsilon. \tag{11}$$

To state the property of the stationary point of the objective function let

$$J(K) := \|T_{z_2 w_2}(K)\|_2^2 = \text{trace} B_2^T G B_2 \tag{12}$$

where G is the observability Gramian that is the solution of the following Lyapnov equation:

$$GA_{cl} + A_{cl}^T G + C_{cl2}^T C_{cl2} = 0. \tag{13}$$

The global optimal H_2 state feedback controller without the H_∞ norm constraint is given by

$$K = K_2^* := -B^T Z_2 - D_2^T C_2 \tag{14}$$

where Z_2 is the stabilizing solution of the Riccati equation

$$\begin{aligned} &Z_2(A - BD_2^T C_2) + (A - BD_2^T C_2)^T Z_2\\ &-Z_2 BB^T Z_2 + C_2^T (I - D_2 D_2^T) C_2 = 0. \end{aligned} \tag{15}$$

Lemma 1 *[14] $J(K)$ has the unique stationary point at $K = K_2^*$ over all internally stabilizing controllers.*

Now, we can state the property of the global solution of the modified mixed H_2/H_∞ control problem.

Proposition 1 *Let K_m^* be the global optimal solution of the modified mixed H_2/H_∞ control problem (MP). If $\|T_{z_1 w_1}(K_2^*)\|_\infty \le \gamma - \varepsilon$ then $K_m^* = K_2^*$. Otherwise K_m^* exists on the boundary of the H_∞ norm constraint, i.e., $\|T_{z_1 w_1}(K_m^*)\|_\infty = \gamma - \varepsilon$.*

Remark 1 *From Proposition 1 it is a necessary condition for a controller K to be the global optimal controller of the problem (MP) that $K = K_2^*$ or K is on the boundary of the H_∞ norm constraint.*

4 Controller Sets

In this section, we define a "controller set" $S_2(K_i)$ whose element achieves the better H_2 norm than K_i. After then, we show that a "controller sequence" chosen from $S_2(K_i)$ achieves a monotonically non-increasing H_2 norm which converges to the unconstraint H_2 optimal value. Similarly, we define a "controller set" $S_\infty(K_i)$ via whose element the closed loop satisfies the H_∞ norm constraint.

For a given controller K_i let $G_i = G_i^T > 0$ be the observability Gramian, i.e., the solution of

$$G_i A_i + A_i^T G_i + C_{2i}^T C_{2i} = 0 \tag{16}$$

where

$$A_i = A + BK_i, C_{2i} = C_2 + D_2 K_i, \tag{17}$$

and define a controller set $S_2(K_i)$ as

$$S_2(K_i) := \{K | L_2^{G_i}(K) \le 0\} - \{K_i\} \tag{18}$$

where

$$\begin{aligned} L_2^{G_i}(K) := &\, G_i(A + BK) + (A + BK)^T G_i\\ &+ (C_2 + D_2 K)^T (C_2 + D_2 K). \end{aligned} \tag{19}$$

This controller set $S_2(K_i)$ has the next property.

Lemma 2 *If $K_i \neq K_2^*$ then every $K \in S_2(K_i)$ is an internally stabilizing controller.*

Using the controller set $S_2(K_i)$ a controller sequence $\Pi = \{K_i, i = 0, 1, 2, \cdots\}$ is defined as follows:

Algorithm 1: Construction of a controller sequence Π.

STEP 1 Give a stabilizing controller $K_0 (\neq K_2^*)$ and let $i := 0$.

STEP 2 Get $G_i > 0$ which is the solution of (16).

STEP 3 Choose any controller from $S_2(K_i)$ and let it be K_{i+1}. If $K_{i+1} = K_2^*$ then exit. Otherwise $i := i + 1$ and go to STEP 2.

This controller sequence has the next properties.

Lemma 3 *Suppose $K_i \neq K_2^*$ then the following (i)-(ii) hold:*

(i) The inequality $G_i \geq G_{i+1}$ holds.

(ii) The H_2 norm of the closed loop via the controller K_i is monotonically non-increasing, i.e., $J(K_i) \geq J(K_{i+1})$.

Proof: From the definition of Π we have

$$G_i A_{i+1} + A_{i+1}^T G_i + C_{2i+1}^T C_{2i+1} \leq 0, \quad (20)$$

$$G_{i+1} A_{i+1} + A_{i+1}^T G_{i+1} + C_{2i+1}^T C_{2i+1} = 0. \quad (21)$$

Subtracting (21) from (20) to get

$$(G_i - G_{i+1}) A_{i+1} + A_{i+1}^T (G_i - G_{i+1}) \leq 0. \quad (22)$$

Since A_{i+1} is stable it follows that $G_i - G_{i+1} \geq 0$, which implies (i) and hence

$$\mathrm{trace} B_2^T G_i B_2 \geq \mathrm{trace} B_2^T G_{i+1} B_2. \quad (23)$$

Thus $J(K_i) \geq J(K_{i+1})$. □

To state that the controller sequence Π converges to the unconstraint H_2 optimal controller K_2^* we need the next lemma.

Lemma 4 *The set $S_2(K_i)$ is empty ($S_2(K_i) = \phi$) if and only if $K_i = K_2^*$.*

Proof: (only if): Suppose $S_2(K_i) = \phi$ but $K_i \neq K_2^*$. Then let $\tilde{K} = -B^T G_i - D_2^T C_2$ and using (16) to get

$$L_2^{G_i}(\tilde{K}) = -(K_i + B^T G_i + D_2^T C_2)^T (K_i + B^T G_i + D_2^T C_2). \quad (24)$$

Since the RHS is negative semi-definite it follows that $\tilde{K} \in S_2(K_i)$. This contradicts $S_2(K_i) = \phi$.
(if): Suppose $K_i = K_2^*$ and let $\hat{K}$ be any controller such that $L_2^{G_i}(\hat{K}) \leq 0$ and $\hat{G}$ be the solution of (13) where $K = \hat{K}$. Then $G_i \geq \hat{G}$ from Lemma 3-(i). On the other hand, $G \geq Z_2 (= G_i)$ for any G and K that satisfy (13) (see [14]). Hence, we have $G_i = \hat{G}$, which implies $J(K_2^*) = J(K_i) = J(\hat{K})$. From Lemma 1 $\hat{K} = K_2^*$ and it follows $S_2(K_i) = \phi$. □

Theorem 1 *The controller sequence Π converges to the unconstraint H_2 optimal controller K_2^*, i.e.,*

$$\lim_{i \to \infty} K_i = K_2^*, (K_i \in \Pi). \quad (25)$$

Proof: If $K_i = K_2^*$ for some $i > 0$ (25) is obvious. Hence, suppose $K_i \neq K_2^*$ for all $i \geq 0$. Since G_i is monotonically non-increasing and bounded below ($G_i \geq G_{i+1} \geq Z_2 > 0$) G_i converges as $i \to \infty$. Hence, from the definition K_i also converges and let $K_\infty := \lim_{i \to \infty} K_i$. If $S_2(K_\infty)$ is not empty we can choose a new controller $\tilde{K}_\infty \in S_2(K_\infty)$ in STEP 3 of Algorithm 1, which contradicts the assumption that K_∞ is the limit of K_i. Hence $S_2(K_\infty)$ is empty and $K_\infty = K_2^*$ from Lemma 4. □

Next, we construct a controller set for a given controller such that any controller in the set satisfies the H_∞ norm constraint. Suppose a given controller K_i satisfies the H_∞ norm constraint. Then there exists $X_i = X_i^T (> 0)$ which satisfies

$$A_i X_i + X_i A_i^T + \gamma^{-2} X_i C_{1i}^T C_{1i} X_i + B_1 B_1^T < 0 \quad (26)$$

where

$$A_i = A + BK_i, \; C_{1i} = C_1 + D_1 K_i, \quad (27)$$

and a controller set $S_\infty(K_i)$ is defined as

$$S_\infty(K_i) = \{K | L_\infty^{X_i}(K) < 0\} \quad (28)$$

where

$$L_\infty^{X_i}(K) := (A + BK) X_i + X_i (A + BK)^T + \gamma^{-2} X_i (C_1 + D_1 K)^T (C_1 + D_1 K) X_i + B_1 B_1^T. \quad (29)$$

This controller set $S_\infty(K_i)$ has the next property.

Lemma 5 *Every $K \in S_\infty(K_i)$ satisfies the H_∞ norm constraint, i.e., $\|T_{z_1 w_1}(K)\|_\infty < \gamma$ for $K \in S_\infty(K_i)$.*

Proof: Obvious from the definition of $S_\infty(K_i)$. □

5 Iterative Algorithm

In this section, we propose an iterative algorithm for the modified mixed H_2/H_∞ control problem. For a given controller K_i any controller in $S_2(K_i) \cap S_\infty(K_i)$ achieves the better H_2 norm of the closed loop $T_{z_2 w_2}(K)$ than K_i while it satisfies the H_∞ norm constraint. Hence the controller chosen in $S_2(K_i) \cap S_\infty(K_i)$ is a better mixed H_2/H_∞ controller than K_i.

An iterative algorithm we propose for the modified mixed H_2/H_∞ control problem (MP) is described as follows:

Algorithm 2 : Iterative algorithm for the modified mixed H_2/H_∞ control problem (MP).

STEP 1 Take an initial stabilizing controller K_0 which satisfies the H_∞ norm constraint and let $i := 0$.

STEP 2 Get G_i and X_i which satisfy (16) and (26), respectively.

STEP 3 Choose any controller from $S_2(K_i) \cap S_\infty(K_i)$ and let it be K_{i+1}.

STEP 4 If the H_2 norm is not improved (i.e., $J(K_i) = J(K_{i+1})$) or $\gamma - \|T_{z_1w_1}(K_{i+1})\|_\infty < \varepsilon$ then let $K^* = K_i$ and exit. Otherwise let $i := i + 1$ and go to STEP 2.

Remark 2 *The problem to find a new controller K_{i+1} in STEP 3 of Algorithm 2 is described as an LMI feasible problem that can efficiently be solved numerically.*

Remark 3 *The controller sequence $K_i (i = 0, 1, \cdots)$ produced by Algorithm 2 approaches to the unconstraint H_2 optimal controller K_2^* until it encounters the boundary of the H_∞ norm constraint.*

Theorem 2 *Let*

$$\tilde{K} := \lim_{i \to \infty} K_i \tag{30}$$

where $K_i (i = 0, 1, \cdots)$ is the controller sequence produced by Algorithm 2. Then the following (i) and (ii) hold:

(i) If $\|T_{z_1w_1}(\tilde{K})\|_\infty < \gamma - \varepsilon$ then $\tilde{K} = K_2^*$. In this case, $\tilde{K}$ is the global optimal solution of the modified mixed H_2/H_∞ control problem.

(ii) Otherwise $\tilde{K}$ exists on the boundary of the H_∞ norm constraint of the modified mixed H_2/H_∞ control problem, i.e., $\|T_{z_1w_1}(\tilde{K})\|_\infty = \gamma - \varepsilon$.

Proof: This follows immediately from construction of K_i.□

6 Numerical Examples

Consider the following state-space matrices:

$$A = \begin{bmatrix} -0.40 & -0.04 & 0.59 \\ -0.11 & 0.37 & -0.23 \\ 1.21 & 0.39 & -0.35 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.29 & -1.10 \\ -0.02 & -1.04 \\ 1.05 & -0.91 \end{bmatrix}, B_1 = \begin{bmatrix} -0.98 & -0.90 \\ -0.68 & -0.41 \\ 1.33 & -0.50 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.80 & -0.08 \\ 0.04 & -2.00 \\ -0.75 & 1.08 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.24 & 1.36 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2.51 & -0.67 & 0 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Figure 2 shows the behavior of $\|T_{z_2w_2}(K_i)\|_2$ on the controller sequence Π produced by Algorithm 1 as a function of the iteration number i. The common LMI solutions for $\gamma = 5$ are taken as an initial controller K_0. Figure 2 shows that $\|T_{z_2w_2}(K_i)\|_2$ is monotonically non-increasing on the controller sequence Π and converges to the H_2 optimal value.

Figure 3 and Figure 4 show the behaviors of $\|T_{z_2w_2}(K_i)\|_2$ and $\|T_{z_1w_1}(K_i)\|_\infty$ on the controller sequence produced by Algorithm 2 as a function of the iteration number i for $\gamma = 5$. The common LMI solution for $\gamma = 5$ is taken as an initial controller K_0. Figure 3 shows that $\|T_{z_2w_2}(K_i)\|_2$ is monotonically non-increasing. Figure 4 shows that the controller obtained by Algorithm 2 satisfies $\|T_{z_1w_1}(K^*)\|_\infty = 5 - \varepsilon$, i.e., the obtained controller exists on the boundary of the H_∞ norm constraint of the modified H_2/H_∞ control problem.

Figure 5 and Figure 6 show the behaviors of $\|T_{z_2w_2}(K^*)\|_2$ and $\|T_{z_1w_1}(K^*)\|_\infty$ as a function of the H_∞ norm bound γ. For each γ the common LMI solutions are taken as an initial controller K_0. Figure 5 and Figure 6 also show $\|T_{z_2w_2}(K_c)\|_2$ and $\|T_{z_1w_1}(K_c)\|_\infty$, where K_c is a controller obtained by common LMI solutions. Figure 5 shows the controllers obtained by Algorithm 2 achieve lower H_2 norms than the controllers obtained by common LMI solutions for all γ. Furthermore, $\|T_{z_2w_2}(K^*)\|_2$ goes to the unconstraint H_2 optimal value as γ increases. Let γ_2^* be the H_∞ norm of $T_{z_1w_1}(K)$ via the unconstraint H_2 optimal controller K_2^*, i.e., $\gamma_2^* := \|T_{z_1w_1}(K_2^*)\|_\infty = 6.8858$. Figure 5 shows the controllers obtained by Algorithm 2 are the global optimal solutions for $\gamma \geq \gamma_2^*$, and Figure 6 shows they are on the boundary of the H_∞ norm constraint for $\gamma < \gamma_2^*$.

7 Conclusions

In this paper, we introduced two controller sets and showed that a controller sequence derived by the controller sets achieves the monotonically non-increasing H_2 norm that converges to the unconstraint H_2 optimal value. Using the two controller sets we proposed an iterative algorithm for the modified mixed H_2/H_∞ control problem and show the effectiveness of our algorithm by numerical examples.

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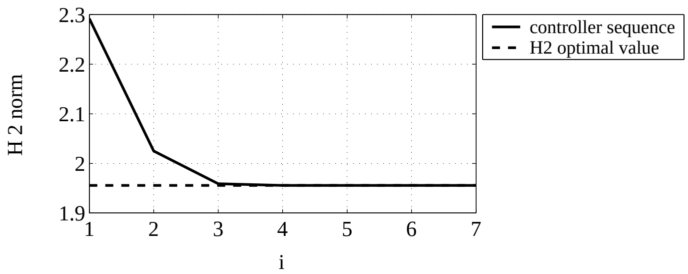

Figure 2: H_2 norm behavior on the controller sequence Π

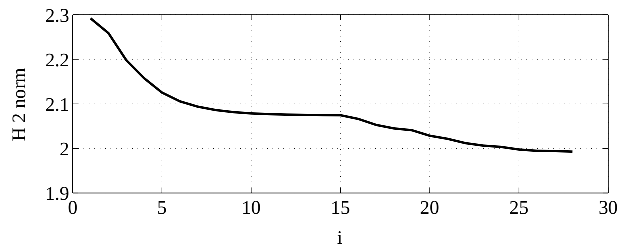

Figure 3: H_2 norm vs. iteration number i $(\gamma = 5)$

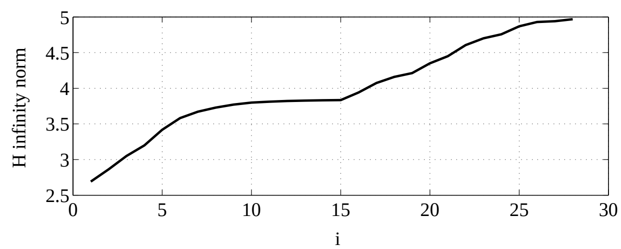

Figure 4: H_∞ norm vs. iteration number i $(\gamma = 5)$

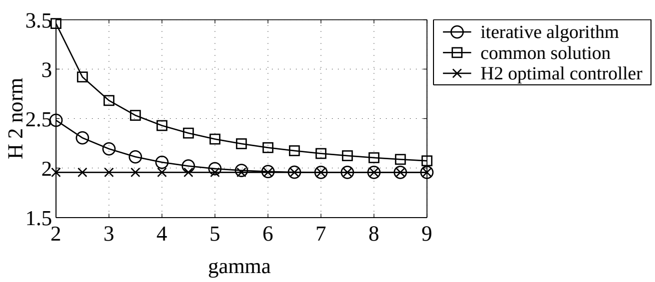

Figure 5: H_2 norm as a function of γ

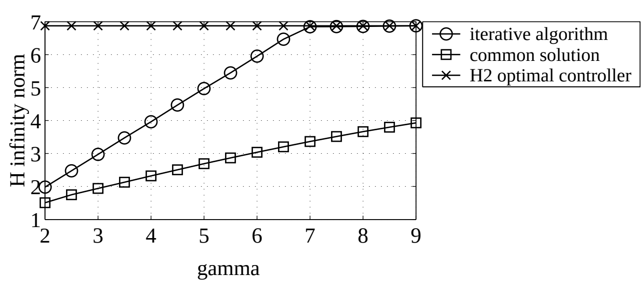

Figure 6: H_∞ norm as a function of γ

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