A PATH-FOLLOWING PROBLEM FOR A CLASS OF NON-LINEAR UNCERTAIN SYSTEMS

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Abstract

This paper focuses on a path-following problem for a suitable class of Chaplygin-like non linear uncertain systems. A feedforward/feedback strategy is proposed where the feedforward action is based on an exact dynamic inversion of the nominal system and the feedback correction depends on the error of the actual output with respect to the path to be followed. A convergence analysis of the resulting dynamic inversion based controller is presented and a simulation for a three dimensional path following problem with a simple aircraft model is included.

1 Introduction

Chaplygin-type nonholonomic systems are known to be an important class of mechanical systems (see [1, 4]) that includes, for example, the kinematic models of various vehicles for planar or aerial navigation; related motion planning techniques have been reported in [5, 6, 7] mainly exploiting the geometric phase idea.

In paper [3] a special motion planning is addressed for a class of Chaplygin-like nonholonomic systems that exhibit a time-varying drift term in the so-called "fiber" equation: namely, given a n-dimensional path on the fiber space the problem is to find the feedforward input that force the system to follow exactly the given path. In this paper we deal with an uncertain model(see (10)) and propose a controller (see (12) that makes use of the error between the real position and the path location. We give an estimate of such error in terms of the norm of the initial error and the bounds of the uncertainties (see (19)) that shows the robustness of the control action. We generalize an idea already used in [2] in a similar two-dimensional problem for a car like-vehicle. The developed strategy can be applied to the maneuver regulation problem of autonomous aerial vehicle where a displaced point has to follow a given path.

2 Problem formulation

The general problem may be introduced by an example. Let the motion of the center of gravity $P(t)$ of an aircraft be given by the following equations:

$$\dot{P}(t) = v(t) \begin{pmatrix} \cos\theta_1(t)\cos\theta_2(t) \\ \sin\theta_1(t)\cos\theta_2(t) \\ \sin\theta_2(t) \end{pmatrix}$$

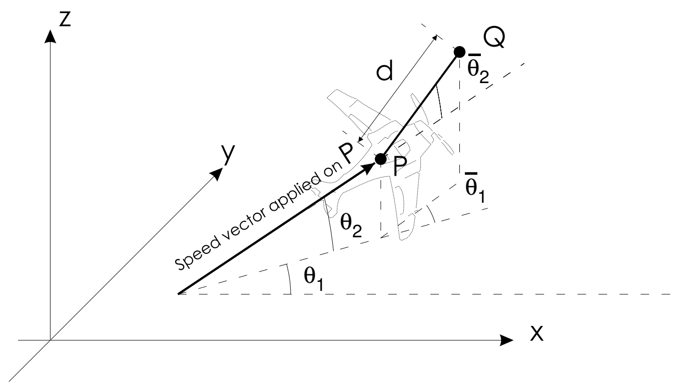

Figure 1: Aircraft simplified model with the point **Q**.

where $v(t)$ is a given smooth positive function defined in $[0, +\infty[$ and $\theta_1(t)$, $\theta_2(t)$ are the angular polar coordinates of $P(t)$. Choose a point $Q(t)$ rigidly linked to $P(t)$, that is $\|Q(t) - P(t)\| = d > 0$ and the coordinates of the vector $Q(t) - P(t)$ are given by $\theta_1(t) + \bar{\theta}_1$, $\theta_2(t) + \bar{\theta}_2$ where $\bar{\theta}_1$ and $\bar{\theta}_2$ are two constants such that $-\frac{\pi}{2} < \bar{\theta}_1 < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \bar{\theta}_2 < \frac{\pi}{2}$. Therefore $Q(t)$ is given by:

$$Q(t) = P(t) + d \begin{pmatrix} \cos(\theta_1(t) + \bar{\theta}_1)\cos(\theta_2(t) + \bar{\theta}_2) \\ \sin(\theta_1(t) + \bar{\theta}_1)\cos(\theta_2(t) + \bar{\theta}_2) \\ \sin(\theta_2(t) + \bar{\theta}_2) \end{pmatrix}$$

and its motion is governed by the following equations:

$$\begin{gathered} \dot{Q} = v(t) \begin{pmatrix} \cos\theta_1(t)\cos\theta_2(t) \\ \sin\theta_1(t)\cos\theta_2(t) \\ \sin\theta_2(t) \end{pmatrix} + \\ + d \begin{pmatrix} -\sin(\theta_1(t) + \bar{\theta}_1)\cos(\theta_2(t) + \bar{\theta}_2) & -\cos(\theta_1(t) + \bar{\theta}_1)\sin(\theta_2(t) + \bar{\theta}_2) \\ \cos(\theta_1(t) + \bar{\theta}_1)\cos(\theta_2(t) + \bar{\theta}_2) & -\sin(\theta_1(t) + \bar{\theta}_1)\sin(\theta_2(t) + \bar{\theta}_2) \\ 0 & \cos(\theta_2(t) + \bar{\theta}_2) \end{pmatrix} \cdot \\ \cdot \begin{pmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{pmatrix}. \end{gathered}$$

Now if we set $z(t) = (\theta_1(t) + \bar{\theta}_1, \theta_2(t) + \bar{\theta}_2)^T$ and make the change of variable $u = A(z)\dot{z}$, where $A(z) = \begin{pmatrix} d\cos z_2 & 0 \\ 0 & d \end{pmatrix}$, then the previous system becomes:

$$\begin{cases} \dot{Q} = v(t) \begin{pmatrix} \cos(z_1(t) - \bar{\theta}_1)\cos(z_2(t) - \bar{\theta}_2) \\ \sin(z_1(t) - \bar{\theta}_1)\cos(z_2(t) - \bar{\theta}_2) \\ \sin(z_2(t) - \bar{\theta}_2) \end{pmatrix} + \\ + \begin{pmatrix} -\sin z_1(t) & -\cos z_1(t)\sin z_2(t) \\ \cos z_1(t) & -\sin z_1(t)\sin z_2(t) \\ 0 & \cos z_2(t) \end{pmatrix} u(t) \\ \dot{z} = A^{-1}(z)u(t). \end{cases} \tag{1}$$

Therefore if Ω is an open subset of $\mathbb{R}^{n-1}$, the previous system fits in the following general framework:

$$\left\{\begin{array}{l}\dot{x}(t)=F(t,z(t))+G(z(t))u(t)\\ \dot{z}(t)=H(z(t))u(t)\\ x(0)=x_0,\quad z(0)=z_0\end{array}\right. \tag{2}$$

where $F:[0,+\infty[\times\Omega\to\mathbb{R}^n$, $G:\Omega\to\mathbb{R}^{n\times n-1}$, $H:\Omega\to\mathbb{R}^{(n-1)\times(n-1)}$ are continuous maps having the following properties:

- F, G, H are bounded lipschitz map with lipschitz constants respectively L_F, L_G, L_H, $G(z)$ is a $\mathcal{C}^1$ $n\times(n-1)$ orthonormal matrix and
$$-\infty<\inf_{t\geq 0,\, z\in\Omega}\{\|F(t,z)\|\}\leq\sup_{t\geq 0, z\in\Omega}\{|F(t,z)\|\}<+\infty,$$
- $\det(F(t,z)),G(z))\neq 0, \quad \det H(z)\neq 0, \forall t\geq 0, \forall z\in\Omega$ and $\sup_{z\in\Omega}\{\|H^{-1}(z)\|\}\triangleq C_{H^{-1}}<+\infty$
- $\frac{F(t,z)^T}{\|F(t,z)\|}G^{\perp}(z)\geq\cos\beta, \quad \forall t\geq 0,\forall z\in\Omega$

where $0<\beta<\frac{\pi}{2}$ and $G^{\perp}(z)$ is an unitary vector orthogonal to the $(n-1)$ columns of $G(z)$, in other words the angle between the vectors $F(t,z)$ and $G^{\perp}(z)$ is always less than a fixed angle β.

Remark that in the case of the aircraft, H is a bounded lipschitz map if we take $\Omega=\mathbb{R}\times]-\frac{\pi}{2}+\epsilon,\frac{\pi}{2}-\epsilon[$, where ϵ is any real number such that $0<\epsilon<\frac{\pi}{2}$. The systems of type (2) are called Chaplygin systems and are an important class of mechanical systems (see for instance [1, 4]) which includes the kinematic models of various vehicles for planar and aerial navigation.

To introduce the problem, let $\gamma:[0,+\infty[\to\mathbb{R}^n$ be a given $\mathcal{C}^2$ arc length parametrized curve ($\|\dot{\gamma}\|=1,\forall\lambda\geq 0$) and set $\bar{\kappa}=\sup_{\lambda\geq 0}\{\kappa(\lambda)\}$, where $\kappa(\lambda)\triangleq\|\ddot{\gamma}(\lambda)\|$.

In the paper [3] we found sufficient conditions on x_0, z_0, and $\bar{\kappa}$ which guarantee that there exist a control u and a maximal interval $[0,t_M[$ such that the problem (2) is solved on $[0,t_M[$, $x(t)\in\Gamma=\gamma([0,+\infty[)$, $\forall t\in[0,t_M[$ (with $\|\dot{x}(t)\|>0$) furthermore $t_M=+\infty$ and $x([0,+\infty[)=\Gamma$ otherwise $\lim_{t\to t_M}d(z(t),\partial\Omega)=0$, that is $x(t)$ follows the path Γ with a positive speed and covers all Γ unless $z(t)$ does converge to the boundary $\partial\Omega$ of Ω, ($d(z(t),\partial\Omega)=\inf_{w\in\partial\Omega}\{\|z(t)-w\|\}$ with the convention that $d(z(t),\partial\Omega)=+\infty$ if $\partial\Omega=\emptyset$).

To recall precisely this result we have to introduce the following matrixes, set $\forall z\in\Omega$:

$$X(z)=-(G^{\perp}(z))^T\frac{dG}{dz}(z)H(z),\quad S_x(z)=\frac{X(z)+X^T(z)}{2} \tag{3}$$

(S_z is the symmetric part of X). Since G is a Lipschitz map and H is bounded, there exists a positive constant M such that

$$\|X(z)w\|\leq M\|w\|,\quad\forall w\in\mathbb{R}^{n-1},\quad\forall z\in\Omega. \tag{4}$$

Furthermore we will suppose for the following that there exists a constant $m>0$ such that

$$w^TS_x(z)w\geq m\|w\|^2,\quad\forall w\in\mathbb{R}^{n-1},\quad\forall z\in\Omega. \tag{5}$$

Then by the result exposed in ([3]), we can deduce the following theorem.

Theorem 1 *(The open loop case)*

In the previous hypotheses and notations, suppose that

$$\gamma(0)=x_0,\quad\dot{\gamma}^T(0)G^{\perp}(z_0)>0,\quad\bar{\kappa}<m. \tag{6}$$

Then there exists a maximal interval $[0,t_M[$ and a control $u(t)$ such that the problem (2) is solvable on $[0,t_M[$, ($\|\dot{x}(t)\|>0$), $x(t)\in\Gamma=\gamma([0,+\infty[)$, $\forall t\in[0,t_M[$; furthermore if $t_M<+\infty$ then $\lim_{t\to t_M}d(z(t),\partial\Omega)=0$ and $x([0,t_M[)\subsetneq\Gamma$; if $t_M=+\infty$ then $x([0,+\infty[)=\Gamma$; therefore $t_M=+\infty$ and $x([0,+\infty[)=\Gamma$ if $\Omega=\mathbb{R}^{n-1}$.

Furthermore the control u is given by the following dynamic inversion based controller:

$$\left\{\begin{array}{ll}\dot{\mu}=F_{\dot{\gamma}}(\mu,\zeta) & ,\mu(0)=0\\ \dot{\zeta}=-H(\zeta)F_G(\mu,\zeta) & ,\zeta(0)=z_0\\ u=-F_G(\mu,\zeta)\end{array}\right. \tag{7}$$

where we have set, $\forall\lambda\geq 0$, $\forall z\in\Omega$ with $\dot{\gamma}^T(\lambda)G^{\perp}(z)\neq 0$:

$$\begin{array}{c}F_{\dot{\gamma}}(\lambda,z)=\dfrac{F^T(t,z)G^{\perp}(z)}{\dot{\gamma}^T(\lambda)G^{\perp}(z)},\\ F_G(\lambda,z)=G^T(z)(F(t,z)-F_{\dot{\gamma}}(\lambda,z)\dot{\gamma}(\lambda)).\end{array} \tag{8}$$

Remark that $F(t,z)=\dot{\gamma}(\lambda)F_{\dot{\gamma}}(\lambda,z)+G(z)F_G(\lambda,z)$, since

$$\begin{array}{c}\forall w\in\mathbb{R}^n, w=\dot{\gamma}(\lambda)\dfrac{w^TG^{\perp}(z)}{\dot{\gamma}^T(\lambda)G^{\perp}(z)}+\\ +G(z)G^T(z)(w-\dfrac{w^TG^{\perp}(z)}{\dot{\gamma}^T(\lambda)G^{\perp}(z)}\dot{\gamma}(\lambda)).\end{array} \tag{9}$$

For istance in the case of the aircraft, if at the initial time $x_0=\gamma(0)$, the speed $\dot{\gamma}(0)$ is not orthogonal to $Q(0)-P(0)$ (remark that $G^{\perp}(z_0)=\frac{Q(0)-P(0)}{\|Q(0)-P(0)\|}$) and the curvature of γ is not too high, namely less than $\frac{1}{d}$,(remark that $X(z)=\begin{pmatrix}\frac{1}{d}&0\\0&\frac{1}{d}\end{pmatrix}$) then there exists a control u such that point $Q(t)$ will follow all the path Γ unless $z(t)$ does converge to the boundary of $\Omega=\mathbb{R}\times]-\frac{\pi}{2}+\epsilon,\frac{\pi}{2}-\epsilon[$.

In general the real motion of the system is not equal to the ideal one described by system (2), to simulate the real behavior we suppose that it is given by the following perturbed system

$$\left\{\begin{array}{l}\dot{x}(t)=F(t,z(t))+G(z(t))u(t)+e_x(t)\\ \dot{z}(t)=H(z(t))u(t)+e_z(t)\\ x(0)=x_0,\quad z(0)=z_0\end{array}\right. \tag{10}$$

where the uncertainties are performed by the continuous vector maps e_x and e_z satisfying the condition:

$$\|e_x(t)\|\leq B_x,\quad\|e_z(t)\|\leq B_z,\quad\forall t\geq 0. \tag{11}$$

Furthermore we allow that at the initial time $x(0)$ may be different from $\gamma(0)$. Therefore if we use the control given by the generator (7), $x(t)$ may not belong anymore to Γ, that is, $d(x(t),\Gamma)=\inf_{\lambda\geq 0}\{\|x(t)-\gamma(\lambda)\|\}>0$.

To overcome this problem, the idea is to build a new inversion based generator of the control function, correcting the controller (7) by means of the error $E(t) = x(t) - \gamma(\mu(t))$ between the actual position $x(t)$ and the estimated one $\gamma(u(t))$ and the error $(z(t) - \zeta(t))$ in the following way:

$$\begin{cases} \dot{\mu} = F_{\dot{\gamma}}(\mu, \zeta) + \chi E_{\dot{\gamma}}, \quad \mu(0) = 0 \\ \dot{\zeta} = -H(\zeta)(F_G(\mu, \zeta) + \chi E_G) + \theta(z - \zeta), \quad \zeta(0) = z_0 \\ u = -(F_G(\mu, \zeta) + \chi E_G) \end{cases} \tag{12}$$

where χ and θ are positive gain parameters and $E_{\dot{\gamma}}$ and E_G are the decomposition of E with respect to the orthogonal system $(\dot{\gamma}, G)$ namely

$$E_{\dot{\gamma}} = \frac{E^T(t) G^{\perp}(\zeta(t))}{\dot{\gamma}(\mu(t)) G^{\perp}(\zeta(t))},$$
$$E_G(t) = G^{\perp}(\zeta(t))(E(t) - \dot{\gamma}(\mu(t)) E_{\dot{\gamma}}(t))$$

(remark that $E(t) = \dot{\gamma}(\mu(t)) E_{\dot{\gamma}}(t) + G(\zeta(t)) E_G(t)$). Therefore the controlled motion of $x(t)$ is given by the following closed loop system

$$\begin{cases} \dot{x} = F(t, z) + G(z)u + e_x \quad , x(0) = x_0 \\ \dot{z} = H(z)u + e_z \quad , z(0) = z_0 \\ \dot{\mu} = F_{\dot{\gamma}}(\mu, \zeta) + \chi E_{\dot{\gamma}}, \quad \mu(0) = 0 \\ \dot{\zeta} = -H(\zeta)(F_G(\mu, \zeta) + \chi E_G) + \theta(z - \zeta), \quad \zeta(0) = z_0 \\ u = -(F_G(\mu, \zeta) + \chi E_G). \end{cases} \tag{13}$$

The closed loop system (13) provides a feedforward/feedback

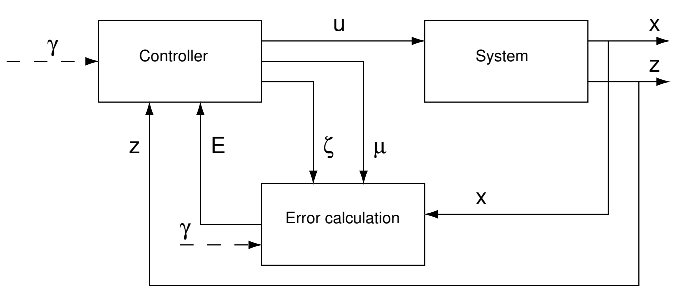

Figure 2: Robust path-following scheme.

strategy where the feedforward term is determined by a dynamic generator based on an exact dynamic inversion over the nominal system and the feedback is mainly achieved by correcting the generator with the components $E_{\dot{\gamma}}$ and E_G of the error E (with respect to the system $(\dot{\gamma}(\mu(t)), \ G(\zeta(t)))$ between the real position $x(t)$ and the estimated one $\gamma(\mu(t))$. A similar technique has been used also in a two-dimensional path-following problem for a car-like vehicle (see paper [2]).

3 The main results

We can state now the following convergence theorem which is a straight consequence of the more technical (but more general) robustness theorem 3.

Theorem 2 *(The closed loop case) In the previous hypotheses and notations, suppose that:*

$$\gamma(0) = x_0, \quad \dot{\gamma}^T(0) G^{\perp}(z_0) > 0, \quad \bar{\kappa} < m.$$

If the uncertainty bounds B_x and B_z are such that:

$$0 \leq B_x < \inf_{t \geq 0, z \in \Omega} \{\|F(t, z)\|\} \cos\beta, \quad 0 \leq B_z (< +\infty) \tag{14}$$

then there exists $\bar{\theta} > 0$, such that $\forall \theta > \bar{\theta}, \forall \chi > 0$ there exists a maximal interval $[0, t_M[$ ($0 < t_M \leq +\infty$) where the open-loop system (13) is solvable on $[0, t_M[$ and

$$d(x(t), \Gamma) \leq \frac{2}{\chi}(B_x + B_z), \quad \forall t \in [0, t_M[.$$

Therefore $\forall \epsilon > 0$, the closed-loop system (13) with $\theta = \bar{\theta}$ and $\chi = \frac{2(B_x + B_z)}{\epsilon}$ is solvable on $[0, t_M[$ and

$$d(x(t), \Gamma) \leq \epsilon, \forall t \geq 0.$$

Furthermore $t_M < +\infty$ only if $\lim_{t \to t_M} d(z(t), \partial\Omega) = 0$ or $\lim_{t \to t_M} d(\zeta(t), \partial\Omega) = 0$, then $t_M = +\infty$ if for instance $\Omega = \mathbb{R}^{n-1}$.

Proof.

It comes out directly from Theorem 3 since $E(0) = x(0) - \gamma(0) = 0.\square$

The main result of the paper is the following theorem of robustness in terms of the initial error $E(0) = x(0) - \gamma(0)$ and the bounds B_x, B_z of the uncertainties that gives a deep insight to the behaviour of the output $x(t)$ of the closed-loop system (13), with respect to the trajectory $\Gamma = \gamma([0, +\infty[)$.

Theorem 3 *In the previous notations and hypotheses, suppose that*

$$\dot{\gamma}^T(0) G^{\perp}(z_0) > 0 \tag{15}$$

$$\bar{\kappa} < m. \tag{16}$$

$$B_x < \inf_{t \geq 0, z \in \Omega} \{\|F(t, z)\|\} \cos\beta, \quad 0 \leq B_z (< +\infty) \tag{17}$$

Let $\bar{\chi}$ be such that

$$\bar{\chi}\|E(0)\| + B_x < \inf_{t \geq 0, z \in \Omega} \{\|F(t, z)\|\} \cos\beta \tag{18}$$

then there exists $\bar{\theta} > 0$ (independent on $\bar{\chi}$ if E(0)=0) such that $\forall \theta \geq \bar{\theta}$, $\forall \chi$ with $0 < \chi \leq \bar{\chi}$, there exists a maximal interval $[0, t_M[$ ($0 < t_M \leq +\infty$) where the closed-loop system is solvable and the following estimate holds:

$$d(x(t), \Gamma) \leq \|E(t)\| \leq \|E(0)\| e^{-\frac{\chi}{2} t} + \frac{2}{\chi}(B_x + B_z), \forall t \in [0, t_M[\tag{19}$$

(for a sharper estimate see (26)), then (18) holds for any $\chi > 0$ if $x(0) = \gamma(0)$, that is $E(0) = 0$. Furthermore $t_M < +\infty$ only if $\lim_{t \to t_M} d(z(t), \partial\Omega) = 0$ or $\lim_{t \to t_M} d(\zeta(t), \partial\Omega) = 0$, therefore $t_M = +\infty$ if, for instance, $\Omega = \mathbb{R}^{n-1}$.

Proof.

Let us define the following constants:

$$\begin{cases} i_F = \inf_{t\geq 0, z\in\Omega}\{\|F(t,z)\|\} \\ A = \frac{2M(\bar{C}_F+\bar{\chi}\|E(0)\|+2(B_x+B_z(1+C_{H^{-1}})))}{i_F\cos\beta-(\bar{\chi}\|E(0)\|+B_x)} \\ R = \min\{\cos(\arcsin((m^2+A^2)^{-\frac{1}{2}}\bar{\kappa})+ \\ +\arctan\frac{A}{m}), \dot{\gamma}^T(0)G^\perp(0)\}. \end{cases} \quad (20)$$

By (15), the definition of R and the local existence theory for ordinary differential systems, $\forall\theta > 0, \forall\chi : 0 < \chi \leq \bar{\chi}$ there exists $\epsilon > 0$, $x \in \mathcal{C}^1([0,\epsilon],\mathbb{R}^n)$, $z \in \mathcal{C}^1([0,\epsilon],\Omega)$, $\zeta \in \mathcal{C}^1([0,\epsilon],\Omega)$, $\mu \in \mathcal{C}^1([0,\epsilon],\mathbb{R})$ which solve (13) on $[0,\epsilon]$ and $\dot{\gamma}^T(\mu)G^\perp(\zeta) \geq \frac{R}{2}$, $F(t,\zeta)^T G^\perp(\zeta) + \chi E^T G^\perp(\zeta) > 0$, $\|E(t)\| \leq \|E(0)\| + 2(B_x+B_z)+1$, remark that $F(0,z_0)^T G^\perp(z_0) + \bar{\chi}E^T(0)G^\perp(z_0) \geq i_F\cos\beta - \bar{\chi}\|E(0)\| > 0$, by (17).

Set

$$\begin{aligned} t_M = \sup\{\epsilon | (13)\text{is solvable}, \dot{\gamma}(\mu)G^\perp(\zeta) \geq \frac{R}{2}, \\ F(t,\zeta)^T G^\perp(\zeta) + E^T G^\perp(\zeta) > 0, \\ \|E(t)\| \leq \|E(0)\| + \frac{2}{\chi}(B_x+B_z)+1 \text{ on } [0,\epsilon]\}, \end{aligned} \quad (21)$$

therefore $t_M > 0$, in particular we get that μ and ζ are lipschitz maps and μ is a monotone strictly increasing map (since $\dot{\mu}(t) > 0$) on $[0,t_M[$. we will show that we can find a $\bar{\theta}$ such that $\forall\theta \geq \bar{\theta}, \forall\chi : 0 < \chi \leq \bar{\chi}$

$$\dot{\gamma}^T(\mu)G^\perp(\zeta) > R, \quad \text{on } [0,t_M[, \quad (22)$$

$$F(t,\zeta)^T G^\perp(\zeta) + \chi E^T G^\perp(\zeta) > C \quad \text{on } [0,t_M[, \quad (23)$$

$$\|E(t)\| \leq \|E(0)\|e^{-\frac{\chi}{2}t} + \frac{2}{\chi}(B_x+B_z) \quad \text{on } [0,t_M[. \quad (24)$$

This will imply, by the local existence theorem for ordinary differential systems and a maximality argument, that $t_M = +\infty$ otherwise $\lim_{t\to t_M} d(z(t),\partial\Omega) = 0$ or $\lim_{t\to t_M} d(\zeta(t),\partial\Omega) = 0$ and the theorem has been proved. To this goal we need the following Lemma whose proof will be omitted for sake of brevity.

Lemma 1 *In the previous hypotheses there exists $\bar{\theta} > 0$ such that $\forall\theta \geq \bar{\theta}, \forall\chi : 0 < \chi \leq \bar{\chi}$*

$$\|z(t)-\zeta(t)\| \leq \frac{B_z}{\varphi(\theta,\chi)}, \quad \textit{on } [0,t_M[\quad (25)$$

$$\begin{aligned} \|E(t)\| \leq \|E(0)\|e^{-\chi(1-(1+\frac{2}{R})\frac{L_G B_z}{\varphi(\theta,\chi)})t} + \\ +[\chi(1-(1+\frac{2}{R})\frac{L_G B_z}{\varphi(\theta,\chi)})]^{-1}\cdot \\ \cdot(B_x + B_z\frac{L_F+(1+\frac{2}{R})C_F L_G}{\varphi(\theta,\chi)}) \leq \\ \leq \|E(0)\|e^{-\frac{\chi}{2}t} + \frac{2}{\chi}(B_x+B_z), \quad \textit{on } [0,t_M[\end{aligned} \quad (26)$$

where $\varphi(\theta,\chi) = \theta-(1+\frac{2}{R})L_H(C_F+\chi\|E(0)\|+2(B_x+B_z))$; namely it suffices that $\bar{\theta}$ verifies the following inequality:

$$\varphi(\bar{\theta},\bar{\chi}) > \max\{2(1+\frac{2}{R})L_G B_z, L_F+(1+\frac{2}{R})C_F L_G\}, \quad (27)$$

therefore if $x(0) = \gamma(0)$, that is $\|E(0)\| = 0$, then the choice of $\bar{\theta}$ is independent of $\bar{\chi}$; for instance (27) holds if:

$$\bar{\theta} > 2(1+\frac{2}{R})[(L_F+L_G)(C_F+B_z)+L_H B_z+L_F+\bar{\chi}L_H\|E(0)\|]. \quad (28)$$

Now, continuing the proof of Theorem 3, (26) implies directly (24), to verify (23) we can suppose, unless of increasing $\bar{\theta}$, that

$$\begin{aligned} \psi(\bar{\theta},\bar{\chi}) \triangleq [(1-(1+\frac{2}{R}))\frac{L_G B_z}{\varphi(\bar{\theta},\bar{\chi})}]^{-1}(B_x+ \\ +B_z\frac{L_F+(1+\frac{2}{R})C_F L_G}{\varphi(\bar{\theta},\bar{\chi})}) < \\ < i_F\cos\beta - \bar{\chi}\|E(0)\|; \end{aligned} \quad (29)$$

for instance, if $\bar{\chi}\|E(0)\| + 2(B_x+B_z) < i_F\cos\beta$, then it suffices that $\bar{\theta}$ verifies (28). Therefore, by (26), $\forall\theta \geq \bar{\theta}$, $\forall\chi : 0 < \chi \leq \bar{\chi}$

$$\|E(t)\| \leq \|E(0)\| + \frac{\psi(\bar{\theta},\bar{\chi})}{\chi}, \quad \text{on } [0,t_M[\quad (30)$$

and

$$\begin{aligned} F(t,\zeta)^T G^\perp(\zeta) + \chi E^T G^\perp(\zeta) \geq i_F\cos\beta - \chi\|E\| \geq \\ \geq i_F\cos\beta - \bar{\chi}\|E(0)\| - \psi(\bar{\theta},\bar{\chi}) \triangleq C > 0 \end{aligned}$$

therefore (23) holds. To prove (22), we remark first of all that it is equivalent to show that

$$\dot{\gamma}^T(\lambda)G^\perp(\zeta(\mu^{-1}(\lambda))) > R, \quad \forall\lambda \in [0,\lambda_M[\quad (31)$$

where $\lambda_M = \sup_{t\in[0,t_M[}\{\mu(t)\}$. By differentiating we get:

$$\begin{aligned} \frac{d}{d\lambda}(\dot{\gamma}^T(\lambda)G^\perp(\zeta(\mu^{-1}(\lambda)))) = \ddot{\gamma}G^\perp + \dot{\gamma}^T\frac{dG^\perp}{dz}\frac{d\zeta}{dt}\frac{d\mu^{-1}}{d\lambda} = \\ = \ddot{\gamma}^T G^\perp + \dot{\gamma}^T(-G(G^\perp)^T\frac{dG}{dz})\cdot \\ \cdot(Hu+\theta(z-\zeta))\frac{\dot{\gamma}G^\perp}{F^T G^\perp + \chi E^T G^\perp} = \\ = \ddot{\gamma}^T G^\perp + (\dot{\gamma}^T G^\perp)(G^T\dot{\gamma})\cdot \\ \cdot(\frac{Xu}{F^T G^\perp + \chi E^T G^\perp} + \theta\frac{XH^{-1}(z-\zeta)}{F^T G^\perp + \chi E^T G^\perp}). \end{aligned}$$

since $\frac{dG^\perp}{dz} = -G^\perp\frac{dG}{dz}$. But

$$\begin{aligned} \frac{Xu}{F^T G^\perp + \chi E^T G^\perp} = \frac{XG^T}{F^T G^\perp + \chi E^T G^\perp}\cdot \\ \cdot[(\frac{F^T G^\perp}{\dot{\gamma}^T G^\perp}\dot{\gamma} - F) - \chi(E - \frac{E^T G^\perp}{\dot{\gamma}^T G^\perp}\dot{\gamma})] + \\ = X(\frac{G^T\dot{\gamma}}{\dot{\gamma}^T G^\perp} - \frac{G^T(F+\chi E)}{(F+\chi E)^T G}], \end{aligned}$$

therefore:

$$\frac{d}{d\lambda}(\dot\gamma^T G^\perp) = \kappa\nu^T G^\perp + (G^T\dot\gamma)^T X (G^T\dot\gamma) +$$
$$-(\dot\gamma^T G^\perp)(G^T\dot\gamma)^T X\left(\frac{G^T(F+\chi E)}{(F+\chi E)^T G} - \theta\frac{H^{-1}(z-\zeta)}{F^T G^\perp + \chi E^T G^\perp}\right)$$

where $\kappa(\lambda) = \|\ddot\gamma\|$ and $\nu(\lambda)$ is an orthogonal vector to $\dot\gamma(\lambda)$. Remark that if $N(\lambda)$ is an $n\times(n-1)$ orthonormal matrix whose columns generate the subspace orthogonal to $\dot\gamma(\lambda)$, we have that

$$\|\nu^T G^\perp\| \le \|N^T G^\perp\| = (1-\|\dot\gamma G^\perp\|^2)^{\frac{1}{2}} = \|G^T\dot\gamma\|$$

therefore

$$\frac{d}{d\lambda}(\dot\gamma^T G^\perp) \ge -\bar\kappa\|\nu G^\perp\| + m\|G^T\dot\gamma\|^2 +$$
$$-\frac{M}{C}(\dot\gamma^T G^\perp)\|G^T\dot\gamma\|(\|F\| + \chi\|E\| + \theta C_{H^{-1}}\|z-\zeta\|) \ge$$
$$\ge \|G^T\dot\gamma\|(m\|G^T\dot\gamma\| - (\dot\gamma^T G^\perp)\frac{M}{C}(C_F +$$
$$+\chi\|E(0)\| + 2(B_x+B_z) + \frac{\theta}{\varphi(\theta,\chi)}C_{H^{-1}}B_z) - \bar\kappa),$$

by (23), (4), (5), (24) and (25). Furthermore, suppose that $\bar\theta$ is such that $\frac{\bar\theta}{\varphi(\bar\theta,\bar\chi)} \le 2$, that is

$$\bar\theta > 2(1+\frac{2}{R})L_H(C_F + \bar\chi\|E(0)\| + 2(B_x+B_z)), \qquad (32)$$

remark that both (28) and (32) hold if

$$\bar\theta > 2(1+\frac{2}{R})\{(L_F+L_G)(C_F+B_z) + +L_H(B_x+B_z) + L_F + \bar\chi\|E(0)\|\}, \qquad (33)$$

then

$$\frac{d}{d\lambda}(\dot\gamma^T G^\perp) \ge \|G^T\dot\gamma\|(m\|G^T\dot\gamma\| - (\dot\gamma^T G^\perp)\bar A - \bar\kappa) \qquad (34)$$

where $\bar A = \frac{M(C_F+\bar\chi\|E(0)\|+2(B_x+B_z)(1+C_{H^{-1}}))}{i_F\cos\beta - \bar\chi\|E(0)\| - \psi(\bar\theta,\bar\chi)}$. Now suppose, unless of increasing $\bar\theta$, that $\bar A < A$ or equivalently that

$$\psi(\bar\theta,\bar\chi) < B_x + \frac{1}{2}(i_F\cos\beta - \bar\chi\|E(0)\| - B_x) \qquad (35)$$

which is the case if, for instance, (28) holds and $\bar\chi\|E(0)\| + 4(B_x+B_z) < i_F\cos\beta$. If we denote by $\alpha(\lambda)$ the angle between vectors $\dot\gamma(\lambda)$ and $G^\perp(\zeta(\mu^{-1}))$, we have that $\dot\gamma(\lambda)G^\perp(\zeta(\mu^{-1}(\lambda)(\lambda))) = \cos\alpha(\lambda)$; then from (34) we get

$$(-\sin\alpha)\dot\alpha \ge \sin\alpha(m\sin\alpha - \bar A\cos\alpha - \bar\kappa), \quad \forall\lambda\in[0,\lambda_M[$$

which implies that

$$\dot\alpha \le -m\sin\alpha + \bar A\cos\alpha - \bar\kappa \quad \text{a. e. on}[0,\lambda_M[$$

that is

$$\frac{d}{d\lambda}(\alpha-\bar\alpha) \le -(m^2+\bar A^2)^{\frac{1}{2}}\sin(\alpha-\bar\alpha) + \bar\kappa \quad \text{a. e. on } [0,\bar\lambda_M[$$

where $\bar\alpha = \arctan\frac{\bar A}{m}$. Therefore $\alpha(\lambda) \le \hat\alpha < \frac{\pi}{2},\ \forall\lambda\in[0,\bar\lambda[$ where $\hat\alpha$ is any real number such that

$$\max\{(\alpha(0)-\bar\alpha), \arcsin((m^2+\bar A^2)^{-\frac{1}{2}}\bar\kappa)\} \le \hat\alpha - \bar\alpha < \frac{\pi}{2} - \bar\alpha;$$

remark that $\alpha(0) = \arccos(\dot\gamma(0)^T G^\perp(0)) < \frac{\pi}{2}$, by (15) and $(m^2+\bar A^2)^{-\frac{1}{2}}\bar\kappa < \sin(\frac{\pi}{2}-\bar\alpha)$, by (16), being $(1+(\frac{\bar A}{m}^2))^{-\frac{1}{2}} = \sin(\frac{\pi}{2} - \arctan\frac{\bar A}{m}) = \sin(\frac{\pi}{2}-\bar\alpha)$. Since $\arcsin((m^2+\bar A^2)^{-\frac{1}{2}}\bar\kappa) + \arctan\frac{\bar A}{m} < \arcsin((m^2+A^2)^{-\frac{1}{2}}\bar\kappa + \arctan\frac{A}{m} < \arcsin((1+\frac{A^2}{m^2})^{-\frac{1}{2}} + \arctan\frac{A}{m} = \frac{\pi}{2}$ (being $\bar A < A$ and $\frac{\bar\kappa}{m} < 1$) we can take $\hat\alpha = \max\{\arcsin((m^2+A^2)^{-\frac{1}{2}}\bar K) + \arctan\frac{A}{m}, \alpha(0)\}$, in other words $\cos\alpha(\lambda) \ge \cos\hat\alpha = R$ (see definition (20)) therefore (22) holds and the theorem is proved.□.

Remark 1 *Suppose that $4(B_x+B_z) < i_F\cos\beta$, from the proof of theorem 3 we deduce the following procedure to determine the values of the gains $\bar\chi$ and $\bar\theta$:*

1) take $\bar\chi$ such that $\bar\chi\|E(0)\| + 4(B_x+B_z) < i_F\cos\beta$

2) take $\bar\theta > 2(1+\frac{2}{R})\{(L_F+L_G)(C_F+B_z) + L_H(B_x+B_z) + L_F + \bar\chi\|E(0)\|\}$ where R is given by (20).

4 Simulation

Consider the simplified airplane model (1) with $\bar\theta_1 = \bar\theta_2 = 0$ and add the bounded noise terms $e_{x_1}, e_{x_2}, e_{x_3}, e_{z_1}, e_{z_2}$ to the state equation:

$$\begin{cases} \dot Q = v(t)\begin{pmatrix}\cos(z_1(t))\cos(z_2(t))\\ \sin(z_1(t))\cos(z_2(t))\\ \sin(z_2(t))\end{pmatrix} + \\ +\begin{pmatrix}-\sin z_1(t) & -\cos z_1(t)\sin z_2(t)\\ \cos z_1(t) & -\sin z_1(t)\sin z_2(t)\\ 0 & \cos z_2(t)\end{pmatrix}u(t) + \begin{pmatrix}e_{x_1}\\ e_{x_2}\\ e_{x_3}\end{pmatrix}\\ \dot z = A^{-1}(z)u(t) + \begin{pmatrix}e_{z_1}\\ e_{z_2},\end{pmatrix}\\ x_1(0)=10,\ x_2(0)=0,\ x_3(0)=0,\ z_1(0)=0,\ z_2(0)=0. \end{cases} \qquad (36)$$

where $v = 150\frac{\mathrm{m}}{\mathrm{s}}$, $d = 10$m, the noise terms are sine function with frequency 4 Hz and $\|e_{x_1}\|, \|e_{x_2}\|, \|e_{x_3}\| < 0.1\frac{\mathrm{m}}{\mathrm{s}^2}$, $\|e_{z_1}\|, \|e_{z_2}\| < 0.01\frac{\mathrm{rad}}{\mathrm{s}^2}$. System (36) is in the form (10).

The reference trajectory is an helix with arc-length parametrization:

$$x_1 = r\cos(\frac{\lambda}{r}\sqrt{1-\alpha})$$
$$x_2 = r\sin(\frac{\lambda}{r}\sqrt{1-\alpha})$$
$$x_3 = \alpha\lambda.$$

where r is the radius and $0 < \alpha < 1$ is a form parameter. We chose $r = 20$ and $\alpha = 0.5$.

Figure 4 shows the reference and the output trajectory and the error function $E(t) = x(t) - \gamma(\mu(t))$.

5 Conclusions:

We have proposed a dynamic controller which combines a feedforward inversion based action with a feedback error correction for the control of a suitable class of nonlinear uncertain system. We have investigated its robust behaviour with respect to the initial error and the noise terms bounds and have provided a convergence result which depends on the curvature of the path to be followed.

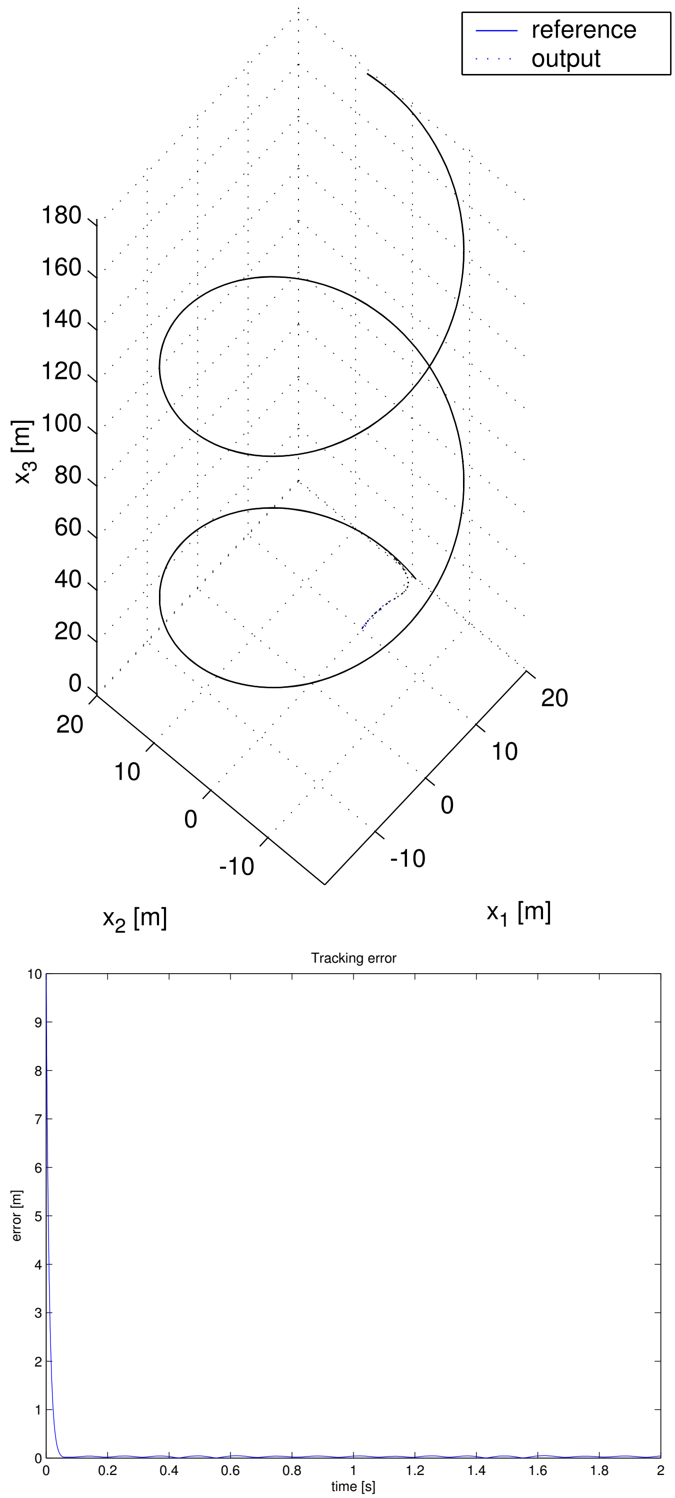

Figure 3: Helix example.

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