REALIZATION OF NONLINEAR SYSTEMS DESCRIBED BY INPUT/OUTPUT DIFFERENTIAL EQUATIONS: EQUIVALENCE OF DIFFERENT METHODS

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Keywords: nonlinear control system, input-output equation, state space realization, algebraic methods, geometric methods

Abstract

Five different state space realizability conditions for nonlinear single-input single-output high order input-output differential equation are compared and proved to be equivalent. Moreover, the explicit formulas are provided for calculation of the differentials of the state coordinates which can be integrated to obtain the state coordinates iff the necessary and sufficient realizability conditions are satisfied.

1 Introduction

The nonlinear realization problem for i/o differential equations has been the subject of several papers [2, 3, 4, 5, 6] where different approaches were provided to solve the problem. In [5], the intrinsic and coordinate-free necessary and sufficient realizability conditions were formulated in terms of integrability of certain subspaces of one-forms, associated with the i/o model and classified according to their relative degrees. The solution is constructive up to finding the integrating factors and integration of the integrable one-forms. The geometric necessary and sufficient realizability conditions in [6] were formulated in terms of the conditionally invariant distributions for the extended state space system, associated with the i/o equation. Neither [5] nor [6] discuss the computational aspects of the solution. On the other hand, the sufficient realizability conditions obtained independently in [4] and [2] are both algorithm-dependent. The realizability conditions are derived from the algorithm, applied to the i/o differential equation, which constructs such a realization, if it exists. The approach is constructive, up to the solution of the partial differential equations. Though there does not exist, in general, the classical state space realization for nonlinear i/o differential equation, it is always possible to write down the generalized state space realization depending – besides the input – also upon the derivatives of the input. Necessary and sufficient conditions are given in [3] under which there exists a generalized state transformation so that the derivatives of the the input are eliminated, and this result may be viewed as conditions for classical state space realization. Note that the conditions in [3] are expressed as integrability conditions in terms of commutativity of certain vector fields defined by the extended state space system.

The main purpose of this paper is to compare the different realizability conditions and realization algorithms. To keep the presentation simple, and since in [5, 4, 2] only the single-input single-output (SISO) systems are studied, we consider the SISO case. The main purpose of this paper is to demonstrate that the realizability conditions, given in [5] are equivalent to those in [6, 3] and yield the same results, at least for the SISO case. Moreover, we will demonstrate that the algorithms in [4, 2] can be understood as the methods to compute the basis for a subspace of one-forms. As for the differences between the geometric conditions [6] and the involutivity conditions [3] for the multi-input multi-output case, see [3].

2 The realization problem

The realization problem is defined as follows. Given the nonlinear system described by the i/o differential equation where the highest derivative of y appears linearly

$$y^{(n)} = \varphi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}), \tag{1}$$

with $\partial\varphi/\partial u^{(s)} \not\equiv 0$, and $s \leq n$, find, if possible, the state coordinates $x \in I\!R^n$, $x = \psi(y, \ldots, y^{(n-1)}, u, \ldots, u^{(s)})$ such that in these coordinates the system takes the classical state space form

$$\dot{x} = f(x, u), \ \ y = h(x, u), \tag{2}$$

called the realization of (1). The solution of the realization problem in [5, 6, 4, 3] is formulated in terms of the extended state-space system, associated with (1), with the input $v = u^{(s+1)}$, the state $z = [y, \ldots, y^{(n-1)}, u, ..., u^{(s)}]^T \in I\!R^{n+s+1}$ and the vector field $f(z, v)$ defined as

$$\dot{z}_i = z_{i+1}, \ \ i = 1, \ldots, n-1, \ \ \dot{z}_n = \varphi(z) \tag{3}$$

$$\dot{z}_{n+k} = z_{n+k+1} \ \ k = 1, \ldots, s, \ \ \dot{z}_{n+s+1} = v. \tag{4}$$

In many papers on nonlinear control, system (3), (4) is treated as the realization of (1). The disadvantage of the *extended state space realization* is that it uses the $(s+1)$th derivative of control $u^{(s+1)} = v$ explicitly and is not observable. For linear systems it is possible to find extended state coordinate transformation such that the system description in the new coordinates does not involve the explicit differentiation of the input and is observable. Since this is not always possible for nonlinear systems, it is important to characterize the input-output models (1) which do have a classical (observable) state space representation and as well the algorithm to find the state coordinates.

3 The solutions of the realization problem

3.1 Algebraic solution of the realization problem

In [5] the realization problem is studied using the language of differential forms. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the variables $\{z, v^{(t)},\ t \geq 0\}$. Over the field $\mathcal{K}$ one can define a vector space $\mathcal{E}^* := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$, spanned by the formal differentials of the elements of $\mathcal{K}$. The relative degree r of a one-form $\omega \in \mathcal{E}^*$ is defined to be the least integer such that $\omega^{(r)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}$. If such an integer does not exist, we set $r = \infty$. A decreasing sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}^*$ is defined by

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}, \mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\},\ k \geq 1. \tag{5}$$

Obviously, $\mathcal{H}_k$ is the space of one-forms whose relative degree is greater than or equal to k, and the subspaces $\mathcal{H}_k$ are invariant under state diffeomorphism.

Theorem 1 *[5] The i/o differential equation (1) is locally realizable in the classical state space form (2) iff for $1 \leq k \leq s+2$ the subspaces $\mathcal{H}_k$ defined by (5) for the extended system (3), (4) are completely integrable. The state coordinates can be found by integrating the basis vectors of $\mathcal{H}_{s+2}$.*

In principle, $\mathcal{H}_{s+2}$ can be found using either Definition (5), or the algorithm, given in [1]. However, the algorithm in [1] does not take into account the specific simple structure of the extended system (3), (4). If we take this structure into account, the following recursive algorithm can be obtained to compute the basis of $\mathcal{H}_{s+2}$ from (5).

The algorithm for calculating the basis of $\mathcal{H}_{s+2}$. Define $\omega_i^{[0]} = \omega_i^{[1]} = \mathrm{d}y^{(i-1)}$, $i = 1, \ldots, n$, and calculate recursively for $k = 1, \ldots, s$

$$\omega_i^{[k+1]} = \omega_i^{[k]} - (-1)^k \omega_i^{[k]} \left(L_f^k \frac{\partial}{\partial u^{(s)}} \right) \mathrm{d}u^{(s-k)}, \tag{6}$$

where

$$f = \sum_{i=1}^{n-1} y^{(i)} \frac{\partial}{\partial y^{(i-1)}} + \varphi(\cdot) \frac{\partial}{\partial y^{(n-1)}} + \sum_{i=1}^{s+1} u^{(i)} \frac{\partial}{\partial u^{(i-1)}}$$

is the vector field defined by the extended system (3), (4). The kth step of the algorithm actually means that the one-form $\omega_i^{[k]}$, obtained at the previous step, will be orthogonalized with respect to the vectorfield $L_f^k(\partial/\partial u^{(s)})$. From direct computation we get that $\omega_i^{[k+1]}$ annihilates all the vectorfields $L_f^l(\partial/\partial u^{(s)})$, $l = 0, \ldots, k$. Alternatively, instead of (6), another formula can be derived to compute $\omega_i^{[k]}$ which includes Lie derivatives of one-forms, and not Lie derivatives of vectorfields as in (6)

$$\omega_i^{[k+1]} = \omega_i^{[k]} - \left(L_f^k \omega_i^{[k]} \right) \left(\frac{\partial}{\partial u^{(s)}} \right) \mathrm{d}u^{(s-k)},\ \ k = 1, \ldots, s. \tag{7}$$

Now $\mathcal{H}_k = \mathrm{span}_{\mathcal{K}}\{\omega_1^{[k-1]}, \ldots, \omega_n^{[k-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-k+1)}\}$, $k = 1, \ldots, s+1$, and $\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\omega_1^{[s+1]}, \ldots, \omega_n^{[s+1]}\}$.

Remark. Though it is not necessary to represent the basis of each $\mathcal{H}_k$ through the exact one-forms, in order to keep the calculations more simple, it is advisable to define the new coordinates at each intermediate step after checking the complete integrability of the subspace $\mathcal{H}_k$. This approach also agrees with the algorithm given in subsection 4.3.

Proposition. *The formulas (6) and (7) are equivalent.*

Proof. By the chain rule, for $r = 1, \ldots, s$, we have

$$\begin{aligned} L_f \left[L_f^{r-1} \omega_i^{[r]} (\partial/\partial u^{(s)}) \right] &= L_f^r \omega_i^{[r]} (\partial/\partial u^{(s)}) \\ &\quad + L_f^{r-1} \omega_i^{[r]} \left(L_f (\partial/\partial u^{(s)}) \right). \end{aligned} \tag{8}$$

As $\omega_i^{[r]} \in \mathcal{H}_{r+1}$ implies $L_f^{r-1} \omega_i^{[r]} \in \mathcal{H}_2$, it follows that $L_f^{r-1} \omega_i^{[r]} (\partial/\partial u^{(s)}) = 0$. From the above and (8) we see that

$$L_f^r \omega_i^{[r]} (\partial/\partial u^{(s)}) = -L_f^{r-1} \omega_i^{[r]} \left(L_f (\partial/\partial u^{(s)}) \right). \tag{9}$$

Repeated application of (9) enables us to write $L_f^k \omega_i^{[k]} (\partial/\partial u^{(s)}) = -L_f^{k-1} \omega_i^{[k]} (L_f (\partial/\partial u^{(s)}) = \ldots = (-1)^k \omega_i^{[k]} (L_f^k (\partial/\partial u^{(s)})$. This establishes the equality of formulas (6) and (7). ∎

3.2 Geometric solution of the realization problem

The realization problem in [6] is studied using the language of vector fields. The increasing sequence of distributions $\{S_k\}$ of $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{(\partial/\partial y), \ldots, (\partial/\partial y^{(n-1)}), (\partial/\partial u), \ldots, (\partial/\partial u^{(s+1)})\}$ is defined by $S_1 = \mathrm{span}_{\mathcal{K}} \left\{ \partial/\partial u^{(s+1)} \right\}$, and

$$S_{k+1} = \bar{S}_k + [f, \bar{S}_k \cap \ker \mathrm{d}y \cap \ker \mathrm{d}u],\ \ k \geq 1 \tag{10}$$

where $\bar{S}$ denotes the involutive closure of the distribution S, and $[f, S]$ denotes the distribution spanned by all Lie brackets $[f, X]$, with X a vector field contained in S. Using the specific structure of the extended state space system (3), (4), it has been proved by van der Schaft [7] that $S_k \subset \ker \mathrm{d}u \cap \ker \mathrm{d}y,\ \ k = 1, \ldots, s+1$, $S_{s+2} \cap \ker \mathrm{d}u \cap \ker \mathrm{d}y = S_{s+1}$. The distribution $S^* = S_{s+2}$ is the minimal involutive distribution S satisfying $[g, S \cap \ker \mathrm{d}y \cap \ker \mathrm{d}u] \subset S,\ \ g \in S$.

Theorem 2 *[6] The i/o differential equation (1) is locally realizable in the classical state space form (2) iff all the distributions $S_1, \ldots, S_{s+2}$ are involutive.*

3.3 Solution in terms of Lie brackets

The realizability conditions in [3] are formulated in terms of the involutivity of the vectorfields.

Theorem 3 *[3] The i/o differential equation (1) is locally realizable in the classical state space form (2) iff for* $0 \le q, r \le s$

$$\left[L_f^q \frac{\partial}{\partial u^{(s)}}, L_f^r \frac{\partial}{\partial u^{(s)}}\right] \equiv 0. \qquad (11)$$

3.4 Algorithmic solution

In [4] the constructive algorithm up to the solution of the partial differential equations has been presented for finding, if possible, the classical state space representation from the input-output differential equation. The starting point for the algorithm is not the input-output equation (1), but the equation where the highest derivative of u appears already linearly

$$\begin{array}{rcl} y^{(n)} & = & \alpha(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s-1)})u^{(s)} \\ & & +\beta((y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s-1)})). \end{array} \qquad (12)$$

Note that linearity of (1) with respect to the highest derivative of u is the necessary condition for

$$S_3 = \mathrm{span}_{\mathcal{K}} \left\{ \frac{\partial}{\partial u^{(s-1)}} + \frac{\partial y^{(n)}}{\partial u^{(s)}} \cdot \frac{\partial}{\partial y^{(n-1)}}, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}} \right\}$$

to be involutive. The function $\varphi(z)$ in (3) now takes the form $\varphi(z) = \beta(z) + \alpha(z)u^{(s)}$. The algorithm in [4] derives systematically the new extended (generalized) state variables at each step of the algorithm, if this is possible. If the partial differential equation

$$\alpha(\cdot)\frac{\partial r}{\partial y^{(n-1)}} + \frac{\partial r}{\partial u^{(s-1)}} = 0 \qquad (13)$$

has a solution, define the new coordinates $\tilde{z}$ through the solution of this equation as $\tilde{z}_i = z_i$, if $i \neq n$ and $\tilde{z}_n = r(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s-1)})$. In the new coordinates the extended state equations become

$$\begin{array}{rcl} \dot{\tilde{z}}_k & = & \tilde{z}_{k+1}, \;\; k = 1, \ldots, n-2 \\ \dot{\tilde{z}}_{n-1} & = & \tilde{\varphi}_1(\tilde{z}_1, \ldots, \tilde{z}_{n+s}), \;\; \dot{\tilde{z}}_n = \tilde{\varphi}_2(\tilde{z}_1, \ldots, \tilde{z}_{n+s}) \end{array} \qquad (14)$$

and

$$\dot{\tilde{z}}_{n+i} = z_{n+i+1}, \;\; i = 1, \ldots, s, \;\; \dot{\tilde{z}}_{n+s+1} = v \qquad (15)$$

Equation (14) can be regarded as a generalized state space description depending on u and its time derivatives up to order $s-1$. There is thus one differentiation less of the input than in (3). If (13) has no solution, then there does not exist a state space representation for (12).

At the next step, if $\tilde{\varphi}_1(\tilde{z}_1, \ldots, \tilde{z}_{n+s})$ and $\tilde{\varphi}_2(\tilde{z}_1, \ldots, \tilde{z}_{n+s})$ can be put into the form (12), i. e. they are linear in $u^{(s-1)}$, then the same procedure can be repeated to them to produce a new generalized state space representation with $u^{(s-2)}$ as the highest time derivative of the input. From the two functions $\tilde{\varphi}_1(\cdot)$ and $\tilde{\varphi}_2(\cdot)$ we construct partial differential equation $\alpha_1(\cdot)(\partial r/\partial z_{n-1}) + \alpha_2(\cdot)(\partial r/\partial z_n) + (\partial r/\partial z_{n+s-1}) = 0$ where $\tilde{\varphi}_i(\cdot) = \beta_i(\cdot) + \alpha_i(\cdot)u^{(s-1)}$, $i = 1, 2$ and β_i, α_i do not depend on the highest time derivative of the control $u^{(s-1)}$. If this partial differential equation has two independent solutions $r_1(\cdot)$ and $r_2(\cdot)$, then defining the new extended state coordinates by these solutions as follows $\hat{z}_{n-1} = r_1(\cdot), \hat{z}_n = r_2(\cdot)$ and leaving the other coordinates as before, we get the generalized state equations with $u^{(s-2)}$ as the highest time derivative of control. Repeating the same procedure, we finally get the classical state equations, if (13) can be solved at each intermediate step. Note that though obtained independently, the algorithm in [2] practically coincides with the one in [4].

4 Main results: the equivalence of four methods

The main purpose of this section is to prove the equivalence of the three different realizability conditions given in terms of the subspaces $\mathcal{H}_k$ of one-forms, conditionally invariant distributions S_k, and in involutivity of vectorfields as in (11). Moreover, we will demonstrate, that the algorithm-based solutions from [4] and [2] can be understood as the method to compute the basis for the subspaces of one-forms $\mathcal{H}_k$, $k = 3, \ldots, s+2$.

4.1 Relationship of the sequences $\{\mathcal{H}_k\}$ and $\{\mathcal{S}_k\}$

Theorem 4 *Assume that the distributions* S_k, $k = 1, \ldots, s+2$ *are all involutive. Then the subspaces of one-forms* $\mathcal{H}_k$ *annihilate the distributions* S_k*, that is* $\mathcal{H}_k(S_k) = 0$ *for* $k = 1, \ldots, s+2$.

Proof. Consider the subspace $\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\}$ which is obviously an annihilator of $S_1 = \mathrm{span}_{\mathcal{K}}\left\{(\partial/\partial u^{(s+1)})\right\}, \mathcal{H}_1(S_1) = 0$. Next, we will show that $\mathcal{H}_2(S_2) = 0$. According to formula (10)

$$S_2 = \mathrm{span}_{\mathcal{K}} \left\{ \frac{\partial}{\partial u^{(s+1)}}, \left[f, \frac{\partial}{\partial u^{(s+1)}}\right] \right\}.$$

We denote for an arbitrary subspace of one-forms $\mathcal{H} = \mathrm{span}_{\mathcal{K}}\{\omega_i, i = 1, \ldots, p\}$, $L_f\mathcal{H} = \mathrm{span}_{\mathcal{K}}\{L_f\omega_i,\ i = 1, \ldots, p\}$. By (10)

$$\begin{array}{rcl} \mathcal{H}_2(S_2) & = & \mathcal{H}_2(S_1 + [f, S_1]) = \mathcal{H}_2(S_1) + \mathcal{H}_2([f, S_1]) \\ & = & \mathcal{H}_2(S_1) + L_f\mathcal{H}_2(S_1) \end{array}$$

. Since both $\mathcal{H}_2 \subset \mathcal{H}_1$ and $L_f\mathcal{H}_2 \subset \mathcal{H}_1$, we have $\mathcal{H}_2(S_2) = 0$. We can proceed analogously taking into account (10) and the fact that $\mathcal{H}_{k+1} \subset \mathcal{H}_k$, $L_f\mathcal{H}_{k+1} \subset \mathcal{H}_k$, $k = 1, \ldots, s+1$ to prove the theorem. ∎

Remark. If S_k is not involutive, then the annihilator of S_{k+1} is not $\mathcal{H}_{k+1}$, but the largest completely integrable subset of $\mathcal{H}_{k+1}$.

Corollary 1 The subspaces $\mathcal{H}_k$, $k = 1, \ldots, s+2$, defined by (5) for the extended system (3), (4) are completely integrable iff the distributions S_k, $k = 1, \ldots, s+2$ defined by (10) for the extended system (3), (4) are involutive.

4.2 The relationship between the condition in terms of one-forms and Lie brackets

Theorem 5 *Integrability of the subspaces $\mathcal{H}_k$, $k = 1, \ldots, s+2$ is equivalent to condition (11).*

Proof. Assume that all the subspaces $\mathcal{H}_k$ are integrable which will yield that all the subspaces S_k, $k = 1, \ldots, s+2$ are involutive. Then one can represent S_k, for $k = 1, \ldots, s+2$ as

$$S_k = \mathrm{span}_{\mathcal{K}}\left\{L_f^{k-1}\frac{\partial}{\partial u^{(s+1)}}, \ldots, L_f\frac{\partial}{\partial u^{(s+1)}}, \frac{\partial}{\partial u^{(s+1)}}\right\}. \qquad (16)$$

From (16) and the involutivity of S_k, (11) follows.

Assume now that (11) holds, and define the distributions

$$\sigma_k = \mathrm{span}_{\mathcal{K}}\left\{L_f^{k-1}\frac{\partial}{\partial u^{(s+1)}}, \ldots, \frac{\partial}{\partial u^{(s+1)}}\right\} \qquad (17)$$

which are by (11) involutive for $k = 1, \ldots, s+2$. We will show that $\mathcal{H}_k$ is a maximal annihilator of the distribution σ_k which implies the complete integrability of $\mathcal{H}_k$. From the definition of $\mathcal{H}_1$, we have $\mathcal{H}_1((\partial/\partial u^{(s+1)})) = 0$, which means that $\mathcal{H}_1(\sigma_1) = 0$. The remaining part of the proof is by induction on k. We show that $\mathcal{H}_k(\sigma_k) = 0$ implies $\mathcal{H}_{k+1}(\sigma_{k+1}) = 0$. Note that from $\mathcal{H}_k(\sigma_k) = 0$ we get $\dot{\mathcal{H}}_{k+1}(\sigma_k) = 0$ since $\mathcal{H}_{k+1} \subset \mathcal{H}_k$. Then from $\mathcal{H}_{k+1}(\sigma_k) = 0$ we get $\dot{\mathcal{H}}_{k+1}(\sigma_k) = -\mathcal{H}_{k+1}([f, \sigma_k])$. By (17) $\sigma_{k+1} = \sigma_k + [f, \sigma_k]$. As $\dot{\mathcal{H}}_{k+1} \subset \mathcal{H}_k$, $\dot{\mathcal{H}}_{k+1}(\sigma_k) = 0$ which implies that $\mathcal{H}_{k+1}([f, \sigma_k]) = 0$ and this is the desired conclusion.

4.3 Realization algorithm

In this subsection we will demonstrate that the algorithm from [4] or [2], recalled in subsection 3.4, constructs the exact basis vectors for the subspace of one-forms $\mathcal{H}_k$, $k \geq 3$, whenever possible. Note that the basis vectors for $\mathcal{H}_1$ are always exact by definition and the basis vectors for $\mathcal{H}_2$ are exact by the specific structure of (3), (4). Let us consider the first step of the algorithm which requires to find the solution for the partial differential equation (13), that is for the equation

$$-L_f\frac{\partial}{\partial u^{(s)}} = \alpha(\cdot)\frac{\partial}{\partial y^{(n-1)}} + \frac{\partial}{\partial u^{(s-1)}} = 0. \qquad (18)$$

In case this equation is solvable, the solution $r(\cdot)$ defines the new state coordinate $\tilde{z}_n = r(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s-1)})$ and in case there is no solution, the algorithm stops, meaning that the i/o equation cannot be transformed into the state space form. We will demonstrate that equation (18) has a solution iff $\mathcal{H}_3$ is integrable and that $\mathrm{d}r \in \mathcal{H}_3$.

In order to define $\mathcal{H}_3$, we calculate, $L_f\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{(1)}, \ldots, \mathrm{d}y^{(n)}, \mathrm{d}u^{(1)}, \ldots, \mathrm{d}u^{(s)}\}$ where $L(\mathcal{H}_2)$ in $\mathrm{d}y^{(n)} = L(\mathcal{H}_2) + (\partial y^{(n)}/\partial u^{(s)})\mathrm{d}u^{(s)}$ has to be understood as the linear combination of one-forms contained in $\mathcal{H}_2$. According to (5), to get a basis element of $\mathcal{H}_3$, $\mathrm{d}y^{(n-1)}$ has to be replaced by another one-form $\omega_n^{[2]}$ in such a way that $L_f\omega_n^{[2]} \in \mathcal{H}_2$. A natural choice is (see also (6))

$$\begin{aligned}\omega_n^{[2]} &= \mathrm{d}y^{(n-1)} + \left(\mathrm{d}y^{(n-1)}L_f\frac{\partial}{\partial u^{(s)}}\right)\mathrm{d}u^{(s-1)} \\ &= \mathrm{d}y^{(n-1)} - \alpha(\cdot)\mathrm{d}u^{(s-1)}\end{aligned} \qquad (19)$$

Note that the one-form $\omega_n^{[2]}$ annihilates $L_f(\partial/\partial u^{(s)})$ in (18). So, if the one-form $\omega_n^{[2]}$ is exact, the solution of (18) can be obtained by integrating $\omega_n^{[2]}$. Of course, the one-form (19) is not necessarily an exact one-form. Iff $\mathcal{H}_3$ is completely integrable, then it is possible to find the integrating factors A_i, $i = 0, \ldots, n-1$, B_j, $j = 0, \ldots, s-2$ such that $A_{n-1}\omega_n^{[2]} + \sum_{i=0}^{n-2} A_i\mathrm{d}y^{(i)} + \sum_{j=0}^{s-2} B_j\mathrm{d}u^{(j)}$ is exact and equal to $\mathrm{d}r$, r being the solution of (18). In the similar manner it can be demonstrated that the second step of the algorithm constructs the exact basis vectors for $\mathcal{H}_4$, if possible and so on. ■

Acknowledgement

This work was partially supported by the Estonian Science Foundation Grant nr. 5405.

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