EQUIVALENCE OF DISCRETE-TIME NONLINEAR SYSTEMS TO FEEDFORWARD FORM

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Abstract

The goal of this paper is two-fold. First, given an arbitrary n-dimensional discrete-time nonlinear dynamical system, necessary and sufficient conditions for the existence of a one-dimensional invariant codistribution are obtained. Second, it is shown that the previous conditions can be used iteratively to obtain a nested sequence of n invariant codistributions with the properties that each codistribution contains the previous one and the last one coincides with the cotangent bundle of the state manifold. As a byproduct, necessary and sufficient conditions are obtained for a discrete-time nonlinear dynamical system to be equivalent to the so-called feedforward form.

1 Introduction

Invariant distributions and their dual, invariant codistributions, occupy a prominent place in nonlinear control theory. They have been used to study controllability and observability properties of nonlinear control systems, and to solve various nonlinear synthesis problems. For discrete-time nonlinear control systems, invariant distributions were introduced in [9, 14].

Recently, a generalized notion of invariance has been introduced for both continuous- and discrete-time nonlinear systems. This new notion has been used to solve the dynamic disturbance decoupling problem (DDDP), see [2, 10].

Despite the widespread use of invariant codistributions in control theory, the following question does not seem to have received an answer: given a nonlinear control system, what are all possible invariant codistributions with respect to the system dynamics? Of course answers to special cases of this questions are well known. For instance, checking whether a given codistribution is invariant or not is a simple exercise. Also, explicit methods are available which allow to construct the smallest invariant codistribution containing a given codistribution.

The first goal of this paper is to give a partial answer to the above general question. More specifically, given a discrete-time nonlinear dynamical system, a characterization of all one-dimensional codistributions which are invariant with respect to the system dynamics will be given. As we shall see, the solution of this apparently simple problem is by no means obvious and, moreover, suggests the solution to various equivalence problems.

The second goal of this paper is the characterization of discrete time systems which are equivalent to the so-called feedforward form. This is accomplished by means of an algorithm designed to construct a nested sequence of invariant codistributions. In the case of continuous-time systems, the feedforward form got a geometric interpretation in [5] in terms of a sequence of controlled nonlinear distributions. Unfortunately, no algorithmic procedure is available to compute such a sequence of distributions. In that respect, it is worth noticing that our results for discrete-time systems are stronger that their continuous-time counterpart.

Preliminary results were reported in [4]. The main advance in this paper is the following: it is shown that the accomplishment of the so-called Invariant Codistribution Algorithm (ICA) is independent of the particular basis chosen at each step. Also, some worked examples have been included in order to illustrate the main contributions.

The paper is organized as follows. In Section 2, we adapt the linear algebraic formalism introduced in [3, 8] to the case of uncontrolled systems. In Section 3, the notion of eigenform is presented, as well as its application to the characterization of one dimensional invariant codistributions. In Section 4, the results of the previous Section are used iteratively in order to construct nested sequences of invariant codistributions. In Section 5, it is shown that integrability of these codistributions is a necessary and sufficient condition for equivalence to the so-called feedforward form. Finally, concluding remarks are offered in Section 6.

2 Preliminaries

Throughout the rest of this paper we will make extensive use of the linear algebraic framework introduced in [3, 8].

It will be necessary, however, to adapt this framework to the situation of uncontrolled systems. At some places of the paper, notions from exterior differential systems will be used. For these matters, the reader is referred to [1, 6, 7].

In this paper we will be dealing with discrete-time nonlinear dynamical systems described by the following difference equation:

$$x(t+1) = f[x(t)], \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a real analytic mapping.

Define the operator $\varpi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\nu \mapsto f(\nu),$$

where $f(\cdot)$ is the same mapping as in (1).

Let $\mathcal{K}$ denote the field of meromorphic functions of the scalar components of

$$x(0) = [x_1(0), \dots, x_n(0)]^T \in \mathbb{R}^n.$$

The elements of $\mathcal{K}$ can be viewed as functions $\varphi : \mathbb{R}^n \to \mathbb{R}$. Using this interpretation, the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\varphi = \varphi \circ \varpi$. Sometimes, the abridged notation $\varphi^+(\cdot) = \delta\varphi(\cdot)$ is used.

Define the vector space $\mathcal{E} = \mathrm{span}_{\mathcal{K}} \{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The elements of $\mathcal{E}$ are called one-forms. The operator $\delta : \mathcal{K} \to \mathcal{K}$ induces the operator $\Delta : \mathcal{E} \to \mathcal{E}$ in the following way. Let $\omega = \sum_i a_i \mathrm{d}\varphi_i \in \mathcal{E}$. Then

$$\omega^+ = \Delta\omega = \Delta(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^+ \mathrm{d}\varphi_i^+.$$

Throughout the paper it will be assumed that the dynamics of system (1) is reversible. More precisely, we make the following technical assumption

Assumption 1

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial f}{\partial x} = n.$$

Assumption 1 implies that the subset $\mathcal{S} \subset \mathbb{R}^n$ where the Jacobian matrix $\frac{\partial f}{\partial x}$ losses rank is of measure zero in $\mathbb{R}^n$. Assumption 1 also guarantees that the mapping $\delta : \mathcal{K} \to \mathcal{K}$ is well defined. It is satisfied for discrete-time systems which arise from sampling a continuous time system [11]. The following example displays the type of pathologies that can appear for non reversible systems.

Example 1 Consider the discrete-time nonlinear system

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 \\ x_3^+ &= x_1 x_2. \end{aligned} \tag{2}$$

Easy computations show that system (2) does not satisfy Assumption 1. Define the function $\mu = \frac{1}{x_3 + x_1 x_2} \in \mathcal{K}$. It can be checked that the forward-shift μ^+ is not defined.

Under Assumption 1, the mapping $\varpi : \mathbb{R}^n \to \mathbb{R}^n$ is well defined and invertible. Therefore, the backward-shift operator $\delta^{-1} : \mathcal{K} \to \mathcal{K}$ exists and is defined by $\delta^{-1}\varphi = \varphi \circ \varpi^{-1}$. Sometimes, the abridged notation $\varphi^-(\cdot) = \delta^{-1}\varphi(\cdot)$ will be used.

The operator $\Delta^{-1} : \mathcal{E} \to \mathcal{E}$ is defined in the following way. Let $\omega = \sum_i a_i \mathrm{d}\varphi_i \in \mathcal{E}$. Then

$$\omega^- = \Delta^{-1}\omega = \Delta^{-1}(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^- \mathrm{d}\varphi_i^-.$$

Given a codistribution or subspace

$$\Omega = \mathrm{span}_{\mathcal{K}} \{\omega_1, \dots, \omega_r\} \subset \mathcal{E},$$

define $\Omega^+ = \Delta\Omega = \mathrm{span}_{\mathcal{K}} \{\omega^+ \mid \omega \in \Omega\}$.

Definition 1 A codistribution $\Omega \subset \mathcal{E}$ is said to be invariant with respect to the dynamics (1) if $\Omega^+ \subset \Omega$.

3 One-dimensional invariant codistributions

The goal of this Section is to give necessary and sufficient conditions for the existence of a one-dimensional codistribution which is invariant with respect to the dynamics (1). The solution of this apparently simple problem constitutes the fundamental brick upon which solutions to different equivalence problems can be obtained.

To begin with, we need to introduce some notation. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping which defines the system dynamics (1). Define the mapping $\mathcal{C}(\delta) : \mathcal{K}^n \to \mathcal{K}^n$ by

$$\mathcal{C}(\delta) = \left[\frac{\partial f_j}{\partial x_i}\delta\right] = \left[\frac{\partial f}{\partial x}\delta\right]^T. \tag{3}$$

In the rest of the paper, $[\mathrm{d}x]$ stands for the column vector $[\mathrm{d}x_1, \dots, \mathrm{d}x_n]^T$. With this notation, it is easy to see that $[\mathrm{d}x^+] = [\frac{\partial f}{\partial x}][\mathrm{d}x]$. Since $\{\mathrm{d}x_1, ..., \mathrm{d}x_n\}$ is a basis for $\mathcal{E}$, any one-form $\omega \in \mathcal{E}$ can be written as

$$\omega = \sum_{i=1}^{i=n} a_i \mathrm{d}x_i = [a_1, \dots, a_n][\mathrm{d}x] = [a][\mathrm{d}x].$$

Note that $\omega^+ = [a^+][\mathrm{d}x^+] = [\mathrm{d}x]^T \mathcal{C}(\delta)[a]^T$. Finally, define the family of operators $\Gamma_\lambda(x, \delta) = [\mathcal{C}(\delta) - \lambda I]$, parameterized by a function $\lambda \in \mathcal{K}$.

Definition 2 (Eigenform) A one-form $\omega \in \mathcal{E}$ is said to be an eigenform if there exists a function $\lambda \in \mathcal{K}$ such that $\omega^+ = \lambda\omega$.

Clearly, if ω is an eigenform, then $\Omega = \mathrm{span}_{\mathcal{K}}\{\omega\}$ is a one-dimensional invariant codistribution. Therefore, the characterization of one-dimensional invariant codistributions is equivalent to the characterization of eigenforms.

Theorem 1 A one-form $\omega = [a][\mathrm{d}x] \in \mathcal{E}$ is an eigenform if and only if there exists a function $\lambda \in \mathcal{K}$ such that $[a] \in \ker \Gamma_\lambda(x, \delta)$.

Theorem 1 provides a complete characterization of all one-dimensional codistributions which are invariant with respect to the dynamics of the system (1). From a practical point of view, the problem has been reduced to that of finding a function $\lambda \in \mathcal{K}$ such that the operator $\Gamma_\lambda(x, \delta) : \mathcal{K}^n \to \mathcal{K}^n$ becomes singular. This problem can be tackled by usual Gaussian elimination thanks to the following technical Lemma, proven in [12].

Lemma 1 Let $\mathcal{K}[\delta]$ denote the ring of polynomials in the operator δ whose coefficients belong to the field $\mathcal{K}$. Then, for all $a(\delta), b(\delta) \in \mathcal{K}[\delta]$ there exist polynomials $p(\delta), q(\delta) \in \mathcal{K}[\delta]$ such that $p(\delta)a(\delta) + q(\delta)b(\delta) = 0$.

The following simple example serves to illustrate the typical procedure.

Example 2 Consider the following discrete-time (linear) system

$$\begin{array}{rcl} x_1^+ & = & x_2 \\ x_2^+ & = & -x_1. \end{array} \tag{4}$$

For system (4) the family of operators $\Gamma_\lambda(x, \delta)$ is given by

$$\Gamma_\lambda(x, \delta) = \begin{bmatrix} -\lambda & -\delta \\ \delta & -\lambda \end{bmatrix}.$$

The operator $\Gamma_\lambda(x, \delta)$ can be brought to a triangular form by performing elementary row operations. Straightforward computations show that, whenever $\lambda \neq 0$, it holds that

$$\begin{bmatrix} -1 & 0 \\ -\delta & -\lambda^+ \end{bmatrix} \Gamma_\lambda(x, \delta) = \begin{bmatrix} \lambda & \delta \\ 0 & \delta^2 + \lambda\lambda^+ \end{bmatrix}.$$

At this point, the computation of $\ker \Gamma_\lambda(x, \delta)$ amounts to solve the difference equation $a_2^{++} + \lambda\lambda^+ a_2 = 0$ in the unknown a_2, and then solve the equation $\lambda a_1 + a_2^+$ in the unknown a_1. In general, the solutions to these equations are not unique. Table 1 displays various solutions, corresponding to different choices of the parameter λ.

Each one of the choices displayed in Table 1 defines a eigenform $\omega = a_1 \mathrm{d}x_1 + a_2 \mathrm{d}x_2$ and, consequently, a codistribution $\Omega = \mathrm{span}_{\mathcal{K}}\{\omega\}$ which is invariant with respect to the dynamics of the system (4).

Table 1: Possible choices of coefficients for system (4)

Parameter λ	Coefficient a_1	Coefficient a_2
1	x_1	x_2
1	$-x_2$	x_1
-1	$-x_1$	x_2
-1	x_2	x_1

4 Nested sequences of invariant codistributions

In this Section an algorithm will be presented which allows to construct a sequence of invariant codistributions with the property that their dimensions increase by one at each step. Applications of this construction will be presented in the following section.

The tangent linear system associated to the discrete-time nonlinear system (1) is given by $[\mathrm{d}x^+] = [\frac{\partial f}{\partial x}][\mathrm{d}x]$. In order to develop the Algorithm, an alternative representation of the tangent linear system will be presented.

Let $\{\omega_1, \ldots, \omega_n\}$ be an arbitrary basis of the space $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. Then, necessarily, there exist coefficients $a_{ij} \in \mathcal{K}$, such that $\omega_i^+ = \sum_{j=1}^n a_{ij}\omega_j$, for $i = 1, \ldots, n$. Define $\omega = [\omega_1, \ldots, \omega_n]^T$. Then the above relations can be written in the following matrix form:

$$\omega^+ = \begin{bmatrix} \omega_1^+ \\ \vdots \\ \omega_n^+ \end{bmatrix} = [a_{ij}] \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} = A\,\omega.$$

4.1 Invariant Codistribution Algorithm (ICA)

step n

If $\ker \Gamma_\lambda(x, \delta) = 0$, then this step can not be accomplished and the algorithm terminates. Otherwise, pick $[a]^T \in \ker \Gamma_\lambda(x, \delta)$, and define

$$\omega_n = [a][\mathrm{d}x] = \sum_{j=1}^n a_j \mathrm{d}x_j.$$

Choose $n-1$ one-forms $\{\omega_1^n, \ldots, \omega_{n-1}^n\}$ such that

$$\mathrm{span}_{\mathcal{K}}\{\omega_1^n, \ldots, \omega_{n-1}^n, \omega_n\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}.$$

Let $A_{n-1} \in \mathcal{K}^{(n-1)\times(n-1)}$ be the unique matrix such that

$$\begin{bmatrix} (\omega_1^n)^+ \\ \vdots \\ (\omega_{n-1}^n)^+ \end{bmatrix} \equiv A_{n-1} \begin{bmatrix} \omega_1^n \\ \vdots \\ \omega_{n-1}^n \end{bmatrix} \mod \{\omega_n\}.$$

step i, for $i = n-1, \ldots, 2$

Define $\Gamma_\lambda^i(x, \delta) = [A_i^T\delta - \lambda I]$. If $\ker \Gamma_\lambda^i(x, \delta) = 0$, then this step can not be accomplished and the algorithm terminates. Otherwise, pick $[a]^T \in \ker \Gamma^i_\lambda(x, \delta)$, and define

$$\omega_i = [a] \begin{bmatrix} \omega_1^{i+1} \\ \vdots \\ \omega_i^{i+1} \end{bmatrix} = \sum_{j=1}^{i} a_j \omega_j^{i+1}.$$

Choose $i-1$ one-forms $\{\omega_1^i, \ldots, \omega_{i-1}^i\}$ such that

$$\operatorname{span}_{\mathcal{K}} \left\{\omega_1^i, \ldots, \omega_{i-1}^i, \omega_i\right\} = \operatorname{span}_{\mathcal{K}} \left\{\omega_1^{i+1}, \ldots, \omega_i^{i+1}\right\}.$$

Let $A_{i-1} \in \mathcal{K}^{(i-1)\times(i-1)}$ be the unique matrix such that

$$\begin{bmatrix} (\omega_1^i)^+ \\ \vdots \\ (\omega_{i-1}^i)^+ \end{bmatrix} \equiv A_{i-1} \begin{bmatrix} \omega_1^i \\ \vdots \\ \omega_{i-1}^i \end{bmatrix} \bmod \{\omega_i, \ldots, \omega_n\}.$$

step 1

Pick $\omega_1 = \omega_1^2$. It follows that $\{\omega_1, \ldots, \omega_n\}$ is a basis of $\operatorname{span}_{\mathcal{K}} \{\mathrm{d}x\}$.

Theorem 2 There exist a sequence of invariant codistributions

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n,$$

with $\dim \Omega_i = (n+1) - i$, if and only if all the steps of Algorithm 1 can be accomplished.

It is important to strengthen that the accomplishment of the Algorithm is independent of the particular choice of the forms $\{\omega_1, ..., \omega_k\}$ at each step. This assertion is precisely stated in the following Proposition.

Proposition 1 Assume that

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n$$

and

$$\Omega_k \neq \tilde{\Omega}_k \supset \Omega_{k+1} \supset \cdots \supset \Omega_n$$

are two chains of invariant codistributions, with $\dim \Omega_i = n+1-i$. Then there exist invariant codistributions $\tilde{\Omega}_1 \supset \tilde{\Omega}_2 \supset \cdots \supset \tilde{\Omega}_{k-1}$ such that $\tilde{\Omega}_{k-1} \supset \tilde{\Omega}_k$, $\dim \tilde{\Omega}_j = n+1-j$ for any $j = 1, ..., k-1$.

In plain words, Proposition 1 means that if at some step of the ICA there are more than one choice for the codistribution Ω_k, and if the algorithm can be completed by one choice, then it can also be completed by any other choice.

Proposition 1 may be interpreted by the lattice in Figure 1.

The following example illustrates the application of the ICA.

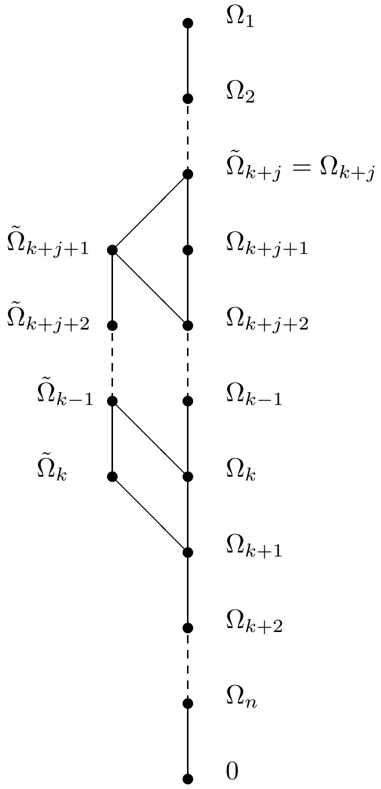

Figure 1: Structure of invariant codistributions

Example 3 Consider the following discrete-time nonlinear system

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 \\ x_3^+ &= x_3 + x_1 x_2. \end{aligned} \tag{5}$$

For system (5) the family of operators $\Gamma_\lambda(x, \delta)$ is given by

$$\Gamma_\lambda(x, \delta) = \begin{bmatrix} -\lambda & -\delta & x_2\delta \\ \delta & -\lambda & x_1\delta \\ 0 & 0 & \delta - \lambda \end{bmatrix}.$$

Now we proceed to apply ICA to system (5).

step 3 First apply elementary row operations to bring the operator $\Gamma_\lambda(x, \delta)$ into triangular form. Define the unimodular matrix

$$B(\delta) = \begin{bmatrix} -1 & 0 & 0 \\ -\delta & -\lambda^+ & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that

$$B(\delta)\Gamma_\lambda(x, \delta) = \begin{bmatrix} \lambda & \delta & -x_2\delta \\ 0 & \delta^2 + \lambda\lambda^+ & x_1\delta^2 - \lambda^+ x_1 \delta \\ 0 & 0 & \delta - \lambda \end{bmatrix}.$$

Choosing $\lambda = -1$, it follows that the vector $[a_1, a_2, a_3]^T$, with $a_1 = x_2$, $a_2 = x_1$, and $a_3 = 0$ annihilates the operator $\Gamma_\lambda(x, \delta)$. Therefore, we choose $\omega_3 = x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2$. We complete a basis of $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$ by $\omega_1^3 = x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2$, and $\omega_2^3 = \mathrm{d}x_3$. Straightforward computations show that $(\omega_1^3)^+ = \omega_1^3$, and $(\omega_2^3)^+ = \omega_2^3 + \omega_3$. Therefore,

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

step 2 The family of operators $\Gamma_\lambda^2(x,\delta)$ is given by

$$\Gamma_\lambda^2(x,\delta) = \begin{bmatrix} \delta-\lambda & 0 \\ 0 & \delta-\lambda \end{bmatrix}.$$

It is easy to see that the vector $[a_1, a_2]^T$ annihilates the operator $\Gamma_\lambda^2(x,\delta)$, whenever $a_1 = a_2 = \alpha$ and $\lambda = \frac{\alpha^+}{\alpha}$, $\alpha \in \mathcal{K}$ being a free parameter. Choose for instance $\alpha = 1$. Therefore, the form ω_2 is defined by

$$\omega_2 = \omega_1^3 + \omega_2^3 = \mathrm{d}x_3 + x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2.$$

A basis of $\mathrm{span}_{\mathcal{K}}\left\{\omega_1^3, \omega_2^3\right\}$ can be completed by taking $\omega_1^2 = \mathrm{d}x_3$.

step 1. Pick $\omega_1 = \omega_1^2 = \mathrm{d}x_3$.

Since all the steps of Algorithm 1 can be accomplished, the sequence of invariant codistributions $\Omega_1 \supset \Omega_2 \supset \Omega_3$ exists, and is defined as follows:

$$\begin{aligned} \Omega_3 &= \mathrm{span}_{\mathcal{K}}\{\omega_3\} \\ &= \mathrm{span}_{\mathcal{K}}\{x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2\} \\ \Omega_2 &= \mathrm{span}_{\mathcal{K}}\{\omega_2, \omega_3\} \\ &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_3 + x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2, x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2\} \\ \Omega_1 &= \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2, \omega_3\} \\ &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}. \end{aligned} \tag{6}$$

The non uniqueness of the sequences of invariant codistributions stated in Proposition 1 is illustrated by Example 4 below.

Example 4 Consider system

$$\begin{aligned} x_1^+ &= x_3 \\ x_2^+ &= \alpha(x_2 - x_3x_1) \\ x_3^+ &= -\alpha x_1, \end{aligned} \tag{7}$$

where $\alpha \in \mathbb{R}$. Application Step 3 of ICA may yield to different choices for the eigenform ω_3, amongst: the exact eigenform $\omega_3 := \mathrm{d}(x_1x_3)$ associated to $\lambda = -\alpha$, and the non integrable eigenform $\tilde{\omega}_3 = \mathrm{d}x_2 - x_3\mathrm{d}x_1$ which is associated to $\lambda = \alpha$. Both eigenforms define one-dimensional invariant codistributions. In each case, there are many choices for the forms ω_2 and $\tilde{\omega}_2$ which define the invariant codistributions Ω_2, $\tilde{\Omega}_2$. Some possible choices are $\omega_2 = \mathrm{d}x_1$, $\omega_2 = \mathrm{d}x_2$, $\omega_2 = \mathrm{d}x_3$, $\tilde{\omega}_2 = \mathrm{d}(x_1x_3)$.

5 Equivalence to feedforward form

Definition 3 System (1) is equivalent to feedforward form if there exists a local change of coordinates $z = \varphi(x_1, ..., x_n)$ such that

$$\begin{aligned} z_1(t+1) &= f_1(z_1, ..., z_n) \\ z_2(t+1) &= f_2(z_2, ..., z_n) \\ &\vdots \\ z_n(t+1) &= f_n(z_n) \end{aligned}$$

In the continuous-time case, a nice geometric characterization of those systems that are equivalent to feedforward form can be found in [5], and is recasted below in a dual form.

Theorem 3 System (1) can be transformed into feedforward form if and only if there exists a sequence of completely integrable codistributions

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n$$

such that $\dim \Omega_i = n + 1 - i$.

Corollary 1 System (1) is equivalent to feedforward form if all the steps of Algorithm 1 can be accomplished and the set of forms $\{\omega_1, \ldots, \omega_n\}$ thereby identified satisfy

$$\mathrm{d}\omega_i \wedge \omega_i \equiv 0 \mod \{\omega_{i+1}, \ldots, \omega_n\}.$$

Example 5 Consider system (5), and the sequence of invariant codistributions $\Omega_1 \supset \Omega_2 \supset \Omega_3$ obtained in Example (3). It can be easily checked that the codistributions Ω_1, Ω_2 and Ω_3 are completely integrable. Therefore, system (5) is equivalent to feedforward form. The corresponding change of coordinates is obtained by integration of the one-forms ω_1, ω_2, ω_3. This leads to the change of coordinates $z_1 = x_3$, $z_2 = x_3 + x_1x_2$, $z_3 = x_1x_2$. In z coordinates, system (5) becomes:

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_2 - z_3 \\ z_3^+ &= -z_3, \end{aligned}$$

which is in feedforward form.

It should be strengthened that Corollary 1 provides only sufficient conditions for equivalence to feedforward form. The main obstacle to obtain necessary and sufficient conditions is the fact that the sequences of invariant codistributions constructed by an application of the ICA are not unique. The following simple example illustrates this situation.

Example 6 Consider again system (4), and define the following codistributions:

$$\begin{aligned} \Omega_2 &= \mathrm{span}_{\mathcal{K}}\{x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2\} \\ \tilde{\Omega}_2 &= \mathrm{span}_{\mathcal{K}}\{x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2\} \\ \Omega_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}. \end{aligned}$$

From Example 2, it follows that Ω_1, Ω_2, and $\tilde{\Omega}_2$ are invariant codistributions. Therefore, different applications of the ICA would lead to the sequences $\Omega_1 \supset \Omega_2$ or $\Omega_1 \supset \tilde{\Omega}_2$.

The pathology exhibited by Example 6 is not a consequence of the application of ICA. It comes from the fact that a given nonlinear system can be equivalent to two different feedforward forms, through the appropriate change of coordinates.

6 Perspectives and concluding remarks

In this paper we have introduced the notion of eigenform. This notion allows to give a characterization of one-dimensional codistributions which are invariant with respect a given discrete-time nonlinear system. We have also presented an algorithm that allows to construct sequences of invariant codistributions. As an application of these technical developments, explicit sufficient conditions for equivalence to feedforward form have been obtained. It is interesting to note that this is an improvement with respect to the same problem in the continuous-time case.

A natural continuation of this work would be the characterization of nested sequences of controlled invariant codistributions for discrete-time nonlinear control systems.

Finally, it is worth mentioning that equivalence to feedforward form can be used for the design of stabilizers for discrete-time nonlinear systems [13].

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