A SIMPLE DERIVATION OF ARE SOLUTIONS TO THE STANDARD $\mathcal{H}_\infty$CONTROL PROBLEM BASED ON LMI SOLUTION*

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Abstract

In this paper, a simple derivation of the Riccati equation based solutions to the standard $\mathcal{H}_\infty$control problem, namely the well-known Glover-Doyle solution and DGKF solution is given based on LMI solution. It is hoped that this will be helpful for ordinary control engineers in deepening the understanding of Riccati equation solutions.

1 Introduction

The "two Riccati equation" solution (ARE solution hereafter) to the standard $\mathcal{H}_\infty$control problem was first presented in 1988 in a paper by Glover and Doyle[3] (Glover-Doyle solution hereafter) without proof. The complete proof is given in [5] based on the so-called four-block problem. The derivation of this solution requires quite high level mathematical techniques and is so complicated that, to the knowledge of the authors, no textbooks are able to include this proof. A more system-oriented approach was published in the famous paper by Doyle, Glover, Khargonekar and Francis[1] in 1989. This solution is widely known as the DGKF solution and inspired numerous related researches afterwards. This DGKF solution and Glover-Doyle solution paved the way towards the applications of $\mathcal{H}_\infty$control. Although the approach of [1] is theorectically sophisticated and self-contained, it is still not easy for ordinary control engineers to understand.

Other approaches leading to the ARE solution are the J-spectral factorization approach of [4] and the J-lossless conjugation approach of [7]. However, these approaches are also rather demanding on the readers.

In the mid 1990's, a direct method for a more general $\mathcal{H}_\infty$control problem was proposed in [6, 2] which is based on the use of LMI approach. The derivation of this LMI solution is easier to understand. But more effort of numerical computation is required compared with the ARE solution. So, for standard $\mathcal{H}_\infty$control problems the ARE solution is preferred.

A question arises: is it possible to derive the ARE solution from the LMI solution? This question is asked because of two reasons. First, it is theorectically interesting to know the relationship between the ARE solvability condition and the LMI solvability condition. Second and more importantly, if a simpler derivation of the ARE solution can be found, it will undoubtedly play a role in deepening the understanding of ARE solution for average level engineers and students.

Some effort has been made in this direction in the textbook of Zhou[10] (Sec. 14.2) which derived the ARE solution starting from a Riccati inequality condition under some *simplifying technical assumptions*, namely assumption A2 in the next section is strengthened to $(A,\ B_1)$ is controllable and $(C_1,\ A)$ is observable. A similar proof is given in the lecture notes of Scherer[9] (Theorem 47, p. 139) under the same assumptions. Inspired by thses works, this paper aims at deriving the solvability condition of the ARE solution in the most general form, based on that of the LMI solution. It will be shown that this can be done relatively easily. It is hoped that this result will be useful, at least in an educational sense.

2 Problem Statement

First of all, following the convention a symmetric matrix X will be called a stabilizing solution if it satisfies the following Riccati equation

$$A^TX + XA + XRX + Q = 0,\ R^T = R,\ Q^T = Q$$

and $A + RX$ is stable.

The state space realization of the generalized plant is given

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by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{1}$$

The input of $G(s)$ is $[w^T \;\; u^T]^T$ and the output is $[z^T \;\; y^T]^T$. Let n_u, n_y, n_w, n_z denote respectively the dimensions of u, y, w, z, and n the dimension of the state. Also, $\gamma > 0$ will be used as the norm bound in $\mathcal{H}_\infty$design.

Further, define a linear fractional tranformation (LFT) on G and K as

$$\mathcal{F}_\ell(G,\ K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

in which G_{ij} is a block of a 2×2 partion of G that is compatible with K. This is the closed loop transfer matrix of the LFT connected system $(G,\ K)$.

In the sequel, the abbreviations ARE and ARI will be used for algebraic Riccati equation and algebraic Riccati inequality respetively.

The assumptions made in [1] are the following:

A1 $(A,\ B_2)$ is stabilizable, $(C_2,\ A)$ is detectable.

A2 $(A,\ B_1)$ is stabilizable, $(C_1,\ A)$ is detectable.

A3 $D_{12}^T[C_1 \;\; D_{12}] = [0 \;\; I_{n_u}]$

A4 $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}$

A5 $D_{11} = 0$

Subject to these conditions, the solvability condition for $\mathcal{H}_\infty$control problem is given by the following theorem[1]

Theorem 1 (DGKF Solution) *Assume A1 to A5. The $\mathcal{H}_\infty$control problem $\|\mathcal{F}_\ell(G,\ K)\|_\infty < \gamma$ has a solution iff*
(1) ARE

$$A^T X_\infty + X_\infty A + X_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty + C_1^TC_1 = 0 \tag{2}$$

has a positive semi-definite stabilizing solution X_∞.
(2) ARE

$$AY_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y_\infty + B_1B_1^T = 0 \tag{3}$$

has a positive semi-definite stabilizing solution Y_∞.
(3) $\rho(X_\infty Y_\infty) < \gamma^2$

Meanwhile the most general assumptions on the standard $\mathcal{H}_\infty$control problem made in [3] are the following:

B1 $(A,\ B_2)$ is stabilizable, $(C_2,\ A)$ is detectable.

B2 $D_{12} = [0 \;\; I_{n_u}]^T$, $D_{21} = [0 \;\; I_{n_y}]$.

B3 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

B4 $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

To state Glover-Doyle solution, let us define the following matrices:

$$R := D_{1\bullet}^T D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{n_w} & 0 \\ 0 & 0 \end{bmatrix},\ D_{1\bullet} := [D_{11} \;\; D_{12}]$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}^T - \begin{bmatrix} \gamma^2 I_{n_z} & 0 \\ 0 & 0 \end{bmatrix},\ D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

and decompose D_{11} as

$$D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$$

such that $D_{1122} \in \mathbb{R}^{n_u \times n_y}$.

Glover-Doyle solvability condition for $\mathcal{H}_\infty$control problem is given by the following theorem[3].

Theorem 2 (Glover-Doyle Solution) *Assume B1 to B4. The $\mathcal{H}_\infty$control problem $\|\mathcal{F}_\ell(G,\ K)\|_\infty < \gamma$ has a solution iff*
(1) $\gamma > max\{\sigma_{\max}[D_{1111}, D_{1112},], \sigma_{\max}[D_{1111}^T, D_{1121}^T]\}$
(2) ARE

$$\begin{aligned}&(A - BR^{-1}D_{1\bullet}^TC_1)^TX + X(A - BR^{-1}D_{1\bullet}^TC_1) - \\ &XBR^{-1}B^TX + C_1^T(I - D_{1\bullet}R^{-1}D_{1\bullet}^T)C_1 = 0\end{aligned} \tag{4}$$

has a positive semi-definite stabilizing solution X_∞.
(3) ARE

$$\begin{aligned}&(A - B_1D_{\bullet 1}^T\tilde{R}^{-1}C)Y + Y(A - B_1D_{\bullet 1}^T\tilde{R}^{-1}C)^T - \\ &YC^T\tilde{R}^{-1}CY - B_1(I - D_{\bullet 1}^T\tilde{R}^{-1}D_{\bullet 1})B_1^T = 0\end{aligned} \tag{5}$$

has a positive semi-definite stabilizing solution Y_∞.
(4) $\rho(X_\infty Y_\infty) < \gamma^2$

Lastly, for a more general $\mathcal{H}_\infty$control problem where only B1 is assumed, the solvability condition is given by the next theorem. First of all, define two matrices as follows.

$$N_X = [B_2^T \;\; D_{12}^T]_\perp, \quad N_Y = [C_2 \;\; D_{21}]_\perp$$

Here $A_\perp$ denotes the orthogonal matrix of A, i.e. $AA_\perp = 0$. Or more precisely, $\mathrm{Im}(A_\perp) = \mathrm{Ker}(A)$. Also, when B is a tall matrix $(B^T)_\perp = B_\perp^T$ and $((B^T)_\perp)^T = B_\perp$ will be used for simplicity of presentation.

Theorem 3 (LMI Solution) *Assume B1. The $\mathcal{H}_\infty$control problem $\|\mathcal{F}_\ell(G,\ K)\|_\infty < \gamma$ has a solution iff*

(1) LMI

$$\begin{bmatrix} N_X^T & \\ & I_{n_w} \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1X & -\gamma I_{n_z} & D_{11} \\ B_1^T & D_{11}^T & -\gamma I_{n_w} \end{bmatrix} \times \begin{bmatrix} N_X & \\ & I_{n_w} \end{bmatrix} < 0 \quad (6)$$

has a positive definite solution X.
(2) LMI

$$\begin{bmatrix} N_Y^T & \\ & I_{n_z} \end{bmatrix} \begin{bmatrix} YA + A^TY & YB_1 & C_1^T \\ B_1^TY & -\gamma I_{n_w} & D_{11}^T \\ C_1 & D_{11} & -\gamma I_{n_z} \end{bmatrix} \times \begin{bmatrix} N_Y & \\ & I_{n_z} \end{bmatrix} < 0 \quad (7)$$

has a positive definite solution Y.
(3) X and Y satisfy

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad \text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_K + n \quad (8)$$

in which n_K is the degree of controller $K(s)$.

3 ARE and ARI

In the derivation of the ARE solution from the LMI solution, the relationship between ARE and ARI is essential. The following lemma provides such a relationship between ARE and ARI. This lemma is adopted from [8] (Theorem 9.1.3) and is known as a comparison theorem. See also Theorem 14.4 of [10] for a proof subject to the condition that $(A,\ B)$ is controllable.

Lemma 1 *Suppose $(-A,\ B)$ is stabilizable and $Q^T = Q$. Then the following statements are equivalent.*
(1) There exists a matrix $X^T = X$ satisfying

$$XA + A^TX + XBB^TX + Q < 0. \quad (9)$$

(2) There exists a matrix $X_\infty^T = X_\infty$ satisfying

$$X_\infty A + A^TX_\infty + X_\infty BB^TX_\infty + Q = 0 \quad (10)$$

such that $-(A + BB^TX_\infty)$ is stable.

Further, these two matrices satisfy the relation

$$X_\infty > X.$$

This lemma is further extended to the following general case in which $(A,\ B)$ is only assumed to be controllable on the imaginary axis. This proposition plays a central role in the derivation of the ARE solutions from the LMI solution.

Proposition 1 *Set a matrix function as*

$$G(X, A, B, Q) = XA^T + AX + XQX + BB^T. \quad (11)$$

Suppose $(A,\ B)$ is controllable on the imaginary axis and $Q^T = Q$. Then the following statements are equivalent.

1. *There exists a matrix $\hat{X} > 0$ satisfying*

$$G(\hat{X}, A, B, Q) < 0. \quad (12)$$

2. *There exists a matrix $X \geq 0$ satisfying*

$$G(X, A, B, Q) = 0 \quad (13)$$

such that $A + XQ$ is stable.

Further, these two matrices satisfy the relation

$$\hat{X} > X.$$

(Proof) This proposition is obviously true when $(A,\ B)$ only has antistable uncontrollable modes by Lemma 1. So only the case in which $(A,\ B)$ has both stable and antistable uncontrollable modes needs to be addressed. Without loss of generality, it can be assumed that $(A,\ B)$ and Q are decomposed as

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad (14)$$
$$Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}$$

in which $(-A_1,\ B_1)$ is stabilizable and A_2 is stable.

(1) $\Rightarrow$ (2) Partition $\hat{X}^{-1}$ as

$$\hat{X}^{-1} = \begin{bmatrix} \hat{X}_1^{-1} & * \\ * & * \end{bmatrix} \Rightarrow$$
$$A_1^T\hat{X}_1^{-1} + \hat{X}_1^{-1}A_1 + \hat{X}_1^{-1}B_1B_1^T\hat{X}_1^{-1} + Q_1 < 0.$$

According to Lemma 1 there exists a matrix $X_1^{-1} > \hat{X}_1^{-1} > 0$ satisfying

$$A_1^TX_1^{-1} + X_1^{-1}A_1 + X_1^{-1}B_1B_1^TX_1^{-1} + Q_1 = 0$$
$$\Rightarrow G(X_1, A_1, B_1, Q_1) = 0$$

and $-(A_1 + B_1B_1^TX_1^{-1})$ is stable.

It is not difficult to verify that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$
$$G(X, A, B, Q) = \begin{bmatrix} G(X_1, A_1, B_1, Q_1) & 0 \\ 0 & 0 \end{bmatrix} = 0$$

holds and

$$A + XQ = \begin{bmatrix} A_1 + X_1Q_1 & * \\ 0 & A_2 \end{bmatrix}$$
$$= \begin{bmatrix} -X_1(A_1 + B_1B_1^TX_1^{-1})^TX_1^{-1} & * \\ 0 & A_2 \end{bmatrix}$$

is stable. To prove $\hat{X} > X$, let $\epsilon > 0$ and define a matrix

$$X_\epsilon = \begin{bmatrix} X_1 & \\ & \epsilon I \end{bmatrix}.$$

Then obviously $X_\epsilon \geq X$ and

$$X_\epsilon^{-1} = \begin{bmatrix} X_1^{-1} & \\ & \epsilon^{-1} I \end{bmatrix}.$$

When ϵ is sufficiently small, $X_\epsilon^{-1} > \hat{X}^{-1}$ holds. Thus

$$\hat{X} > X_\epsilon \geq X.$$

(1) $\Leftarrow$ (2) Multiplying $[0 \ \ I]^T$ to the right of Eq.(13) yields a Sylvester equation

$$(A + XQ)X \begin{bmatrix} 0 \\ I \end{bmatrix} + X \begin{bmatrix} 0 \\ I \end{bmatrix} A_2^T = 0.$$

Since both $(A+XQ)$ and A_2 are stable, this equation has a unique solution

$$X \begin{bmatrix} 0 \\ I \end{bmatrix} = 0.$$

Therefore, X must have a structure of

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_1 \geq 0.$$

Then it is clear that $G(X_1, A_1, B_1, Q_1) = 0$ and $-(A_1 + B_1 B_1^T X_1^{-1}) = X_1(A_1 + X_1 Q_1)^T X_1^{-1}$ is stable. If $X_1 > 0$, then there must be a matrix $\hat{X}_1 > X_1 > 0$ satisfying

$$G(\hat{X}_1, A_1, B_1, Q_1) < 0 \tag{15}$$

by Lemma 1. Now let

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \\ & \hat{X}_2 \end{bmatrix}$$

in which $\hat{X}_2$ is a compatible matrix and is assumed satisfying

$$\hat{X}_2^{-1} A_2 + A_2^T \hat{X}_2^{-1} + Q_2 + \epsilon^{-1} I = 0. \tag{16}$$

Then

$$G(\hat{X}, A, B, Q) =$$
$$\begin{bmatrix} G(\hat{X}_1, A_1, B_1, Q_1) & (A_{12} + \hat{X}_1 Q_{12})\hat{X}_2 \\ \hat{X}_2 (A_{12} + \hat{X}_1 Q_{12})^T & A_2 \hat{X}_2 + \hat{X}_2 A_2^T + \hat{X}_2 Q_2 \hat{X}_2 \end{bmatrix}$$

holds. Therefore,

$$\begin{bmatrix} I & \\ & \hat{X}_2^{-1} \end{bmatrix} G(\hat{X}, A, B, Q) \begin{bmatrix} I & \\ & \hat{X}_2^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} G(\hat{X}_1, A_1, B_1, Q_1) & A_{12} + \hat{X}_1 Q_{12} \\ (A_{12} + \hat{X}_1 Q_{12})^T & -\epsilon^{-1} I \end{bmatrix} \tag{17}$$

is obtained. When $\epsilon > 0$ is small enough, Eq.(16) has a unique positive definite solution and the right side of Eq.(17) becomes negative definite by Schur complement argument. Hence, $G(\hat{X}, A, B, Q) < 0$. Obviously $\hat{X} > X$.

Finally, $X_1 > 0$ is proved by reductive absurdity. Suppose instead that $\mathrm{Ker}(X_1)$ is not empty. Then there exists a matrix T such that $\mathrm{Ker}(X_1) = \mathrm{Im}(T)$. Hence there holds

$$\begin{aligned} &0 = T^T G(X_1, A_1, B_1, Q_1) T = T^T B_1 B_1^T T \\ &\Rightarrow B_1^T T = 0. \end{aligned} \tag{18}$$

Further

$$0 = G(X_1, A_1, B_1, Q_1)T = X_1 A_1^T T$$

which implies that $\mathrm{Ker}(X_1)$ is A_1^T-invariant. So there is a matrix Λ such that

$$A_1^T T = T\Lambda. \tag{19}$$

Eqs.(19) and (18) imply that the eigenvalues of $-\Lambda$ are uncontrollable modes of $(-A_1, \ B_1)$ and must be stable because $(-A_1, \ B_1)$ is stabilizable. However, due to the stability of $A_1 + X_1 Q_1$ and

$$T^T(A_1 + X_1 Q_1) = T^T A_1 = \Lambda^T T,$$

Λ must be stable. This is a contradiction. Thus $X_1 > 0$ must be true. $\diamond$

4 From LMI to DGKF

In this section, LMI solvability condition is reduced to an ARI first, then Proposition 1 is applied to derive the solvability of Theorem 1.

By assumptions A3 and A4, there exist matrices $D_{12\perp}^T$ and $D_{21\perp}$ such that $[D_{12\perp}^T \ D_{12}]$, $[D_{21\perp} \ D_{21}^T]$ are unitary, i.e.

$$\begin{bmatrix} D_{12\perp} \\ D_{12}^T \end{bmatrix} [D_{12\perp}^T \ D_{12}] = I \tag{20}$$

$$\begin{bmatrix} D_{21\perp}^T \\ D_{21} \end{bmatrix} [D_{21\perp} \ D_{21}^T] = I. \tag{21}$$

Further, it is not difficult to show that

$$N_X = [B_2^T \ D_{12}^T]_\perp = \begin{bmatrix} I & 0 \\ -D_{12} B_2^T & D_{12\perp}^T \end{bmatrix} \tag{22}$$

$$N_Y = [C_2 \ D_{21}]_\perp = \begin{bmatrix} I & 0 \\ -D_{21}^T C_2 & D_{21\perp} \end{bmatrix} \tag{23}$$

holds. Therefore, the left side of LMI (6) becomes

$$\begin{bmatrix} AX + XA^T - \gamma B_2 B_2^T & X C_1^T D_{12\perp}^T & B_1 \\ D_{12\perp} C_1 X & -\gamma I & 0 \\ B_1^T & 0 & -\gamma I \end{bmatrix}.$$

Noting $D_{12}^T C_1 = 0$ and $D_{12\perp}^T D_{12\perp} + D_{12} D_{12}^T = I$,

$$C_1^T D_{12\perp}^T D_{12\perp} C_1 = C_1^T C_1$$

holds. Then by Schur complement argument, LMI (6) is equivalent to

$$AX + XA^T + \gamma\left(\gamma^{-2} B_1 B_1^T - B_2 B_2^T\right) + \gamma^{-1} X C_1^T C_1 X < 0$$
$$\Leftrightarrow \qquad (X/\gamma)^{-1} A + A^T (X/\gamma)^{-1} +$$
$$(X/\gamma)^{-1}\left(\gamma^{-2} B_1 B_1^T - B_2 B_2^T\right)(X/\gamma)^{-1} + C_1^T C_1 < 0.$$

According to Proposition 1 this condition is equivalent to that ARE (2) has a stabilizing solution X_∞ and

$$(X/\gamma)^{-1} > X_\infty \geq 0. \tag{24}$$

Analogously, it can be proved that LMI (7) is equivalent to that ARE (3) has a stabilizing solution Y_∞ and

$$(Y/\gamma)^{-1} > Y_\infty \geq 0. \tag{25}$$

Finally, if a full order controller is considered, i.e. $n_K = n$, then the rank condition

$$\text{rank}\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_K + n = 2n$$

holds automatically. Further

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \Leftrightarrow X \geq Y^{-1} \Leftrightarrow \rho(X^{-1}Y^{-1}) \leq 1.$$

This condition turns out to be equivalent to

$$\rho(X_\infty Y_\infty) < \gamma^2$$

due to Eqs.(24) and (25). Thus Theorem 1 is proven.

5 From LMI to Glover-Doyle

The derivation of Glover-Doyle solution is a little bit more involved and needs some preparations. If D_{11} is assumed as a zero matrix, then the presentation of the following derivation process would be greatly simplified. However, for completeness, Glover-Doyle solution will be derived in its general form.

Only the equivalence between ARE (4) and LMI (6) will be shown. The equivalence between ARE (5) and LMI (7) follows by duality. And the equivalence between the spectral radius condition and Eq. (8) has been proved in the previous subsection.

The following matrix inversion formulae will be used extensively in the sequel

$$\begin{bmatrix} X & Y \\ Y^T & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -Y^T & I \end{bmatrix}\begin{bmatrix} (X-YY^T)^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}$$
$$(I-AB)^{-1}A = A(I-BA)^{-1}$$
$$B(I-AB)^{-1} = (I-BA)^{-1}B.$$
$$(A-BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}$$

As in the preceding subsection, due to assumption B2 there exist $D_{12\perp}$, N_X satisfying Eqs. (20), (22) respectively. Hence, LMI (6) reduces to

$$\begin{bmatrix} \underline{A}X + X\underline{A}^T - \gamma B_2B_2^T & XC_1^TD_{12\perp}^T & \underline{B_1} \\ D_{12\perp}C_1X & -\gamma I & D_{12\perp}D_{11} \\ \underline{B_1}^T & D_{11}^TD_{12\perp}^T & -\gamma I \end{bmatrix} < 0 \tag{26}$$

in which (22) and $D_{12\perp}^TD_{12\perp} = I$, $D_{12\perp}D_{12} = 0$ have been used and matrices $\underline{A}$, $\underline{B_1}$ are defined as

$$\underline{A} = A - B_2D_{12}^TC_1,\ \underline{B_1} = B_1 - B_2D_{12}^TD_{11}.$$

Since $D_{12\perp} = [I\ \ 0]$, it is easy to see that

$$\begin{bmatrix} -\gamma I & D_{12\perp}D_{11} \\ D_{11}^TD_{12\perp}^T & -\gamma I \end{bmatrix} < 0 \Leftrightarrow$$
$$\gamma > \sigma_{\max}[D_{1111}, D_{1112}].$$

So the following two matrices are positive definite

$$\underline{R} := \gamma^2I - D_{11}^TD_{12\perp}^TD_{12\perp}D_{11} > 0$$
$$E := I - \gamma^{-2}D_{12\perp}D_{11}D_{11}^TD_{12\perp}^T > 0.$$

Noting $D_{12}^TD_{12} = I$ and $D_{12}D_{12}^T + D_{12\perp}^TD_{12\perp} = I$, it is obtained that

$$R^{-1} = \begin{bmatrix} I & 0 \\ -D_{12}^TD_{11} & I \end{bmatrix}\begin{bmatrix} -\underline{R}^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -D_{11}^TD_{12} \\ 0 & I \end{bmatrix}.$$

Then, some routine calculations yield

$$\begin{aligned} &I - D_{1\bullet}R^{-1}D_{1\bullet}^T = D_{12\perp}^TE^{-1}D_{12\perp} \geq 0 \\ &A - BR^{-1}D_{1\bullet}^TC_1 = \underline{A} - \underline{B_1}\underline{R}^{-1}D_{11}^TD_{12\perp}^TD_{12\perp}C_1 \\ &BR^{-1}B^T = B_2B_2^T - \underline{B_1}\underline{R}^{-1}\underline{B_1}^T \end{aligned} \tag{27}$$

where $E^{-1} = I + D_{12\perp}D_{11}\underline{R}^{-1}D_{11}^TD_{12\perp}^T$ has been used in the calculation of the first equation.

Lemma 2 *Subject to assumptions B2, B3 and $\gamma > \sigma_{\max}[D_{1111}, D_{1112}]$, the pair $((I - D_{1\bullet}R^{-1}D_{1\bullet}^T)^{1/2}C_1, A - BR^{-1}D_{1\bullet}^TC_1)$ is observable on the imaginary axis.*

(Proof) First of all, the following equation

$$\begin{bmatrix} I & 0 \\ 0 & D_{12\perp} \\ 0 & D_{12}^T \end{bmatrix}\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}\begin{bmatrix} I & 0 \\ -D_{12}^TC_1 & I \end{bmatrix} = \begin{bmatrix} \underline{A} - j\omega I & B_2 \\ D_{12\perp}C_1 & 0 \\ 0 & I \end{bmatrix}$$

shows that $\begin{bmatrix} \underline{A} - j\omega I \\ D_{12\perp}C_1 \end{bmatrix}$ has full column rank. Then the conclusion follows from equation

$$\begin{bmatrix} A - BR^{-1}D_{1\bullet}^TC_1 - j\omega I \\ (I - D_{1\bullet}R^{-1}D_{1\bullet}^T)^{1/2}C_1 \end{bmatrix} = \begin{bmatrix} I & -\underline{B_1}\underline{R}^{-1}D_{11}^TD_{12\perp}^T \\ 0 & E^{-1/2} \end{bmatrix}\begin{bmatrix} \underline{A} - j\omega I \\ D_{12\perp}C_1 \end{bmatrix}$$

immediately. ◇

In the following, LMI (26) is reduced to an ARI first, then Proposition 1 is invoked to derive ARE (4).

Now define a matrix as

$$\hat{X} := \gamma X^{-1} > 0.$$

An equivalent LMI

$$\begin{bmatrix} \hat{X}\underline{A} + \underline{A}^T\hat{X} - \hat{X}B_2B_2^T\hat{X} & * & * \\ D_{12\perp}C_1 & -I & * \\ \gamma^{-1}\underline{B_1}^T\hat{X} & \gamma^{-1}D_{11}^TD_{12\perp}^T & -I \end{bmatrix} < 0$$

is obtained by multiplying γX^{-1} to the first row block and the first column block of the matrix on the left side of (26), then dividing the whole matrix by γ. Since

$$\begin{bmatrix} -I & \gamma^{-1}D_{12\perp}D_{11} \\ \gamma^{-1}D_{11}^TD_{12\perp}^T & -I \end{bmatrix}^{-1} = -\begin{bmatrix} I & 0 \\ \gamma^{-1}D_{11}^TD_{12\perp}^T & I \end{bmatrix}\begin{bmatrix} E^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & * \\ 0 & I \end{bmatrix},$$

the LMI above is equivalent to the following Riccati inequality

$$\hat{X}\underline{A} + \underline{A}^T\hat{X} - \hat{X}(B_2B_2^T - \gamma^{-2}\underline{B_1}\underline{B_1}^T)\hat{X} + (*)^TD_{12\perp}^TE^{-1}D_{12\perp}(C_1 + \gamma^{-2}D_{11}\underline{B_1}^T\hat{X}) < 0.$$

Substitution of $\gamma^{-2}D_{11}^TD_{12\perp}^TE^{-1}D_{12\perp}C_1 = \underline{R}^{-1}D_{11}^TD_{12\perp}^TD_{12\perp}C_1$ and $\underline{R}^{-1} = \gamma^{-2}I + \gamma^{-4}D_{11}^TD_{12\perp}^TE^{-1}D_{12\perp}D_{11}$ as well as the equations in (27) gives

$$(A - BR^{-1}D_{1\bullet}^TC_1)^T\hat{X} + \hat{X}(A - BR^{-1}D_{1\bullet}^TC_1) - \hat{X}BR^{-1}B^T\hat{X} + C_1^T(I - D_{1\bullet}R^{-1}D_{1\bullet}^T)C_1 < 0.$$

Then invoking Proposition 1 and Lemma 2 shows that this is equivalent to that ARE (4) has a stabilizing solution $X_\infty \geq 0$ and $\hat{X} > X_\infty$. This ends the proof.

6 Conclusion

A simple derivation of the Riccati equation based solutions to the standard $\mathcal{H}_\infty$control problems, i.e. DGKF solution and Glover-Doyle solution, has been obtained in this note, starting from the LMI solution for general $\mathcal{H}_\infty$control problems. It is hoped that this will serve as a guide in the understanding of ARE based $\mathcal{H}_\infty$control theory.

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