ROBUST POLE-CLUSTERING FOR DESCRIPTOR SYSTEMS A STRICT LMI CHARACTERIZATION

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Abstract

This paper tackles the problem of the characterization of robust pole-clustering for descriptor systems using linear matrix inequalities (LMI). It states a necessary and sufficient condition for a descriptor system to be impulse free and to have its finite poles in a specified convex region of the complex plane. A sufficient condition to guarantee this result in the presence of norm-bounded uncertainties is established. The results are expressed in terms of strict LMIs, thus they are numerically tractable with LMI optimization tools. Numerical examples are given.

1. Introduction

The descriptor formulation (1-2) is known to be much more generic than the usual one since it preserves the physical meaning of the variables and the model can contain a static part and an improper part. Thus physical constraints, static relations and impulsive behaviors can be modeled.
In this paper, we consider a linear time-invariant descriptor systems defined by

$$E\dot{x}(t) = Ax(t) \quad \text{(continuous-time case)} \qquad (1)$$

$$Ex(k+1) = Ax(k) \quad \text{(discrete-time case)} \qquad (2)$$

where $x \in \mathbf{R}^n$ is the descriptor variable, E and A are known real constant $n \times n$ matrices. The matrix E may be rank deficient and we denote its rank by $r = rank(E) \leq n$.
Since [3] and [4] many topics of control have been treated in the descriptor system case. Recently, efforts have been provided to extend the H_∞ theory [9],[13] and the LMI-based approach [11] to descriptor systems. Following [1] it is of interest to note that the analysis tools of the H_∞ theory provide little information about the transient behavior and other temporal characteristics that are linked to the location of the poles of the system. Thus H_∞ analysis and pole clustering are complementary aspects that should be taken into consideration for system analysis and controller design.
Pole placement for the usual state-space representation is a classic problem. Combined H_2/H_∞/pole clustering objective is treated in [1] in which H_2, H_∞ and pole clustering objectives are studied and characterized in terms of LMIs. In [2] the robustness of the pole-clustering control is ensured. Recently, the characterization of pole clustering via LMI has been extended to descriptor systems in [7] and the robustness issue is addressed in [10].

In fact, the LMI formalism is appealing from the practical point of view since it is efficiently and reliably solved by convex optimization algorithms and LMI toolboxes are available in popular control software's [5], [6]. But many results concerning LMI-based control for descriptor systems ([7], [9], [10], [13]) use LMI with equality constraints which often cause numerical problems in computation: checking the equality condition is more tricky because of eventual round off errors [12]. This fact motivates the authors to exclusively express the LMI condition for pole clustering of a matrix pencil in terms of strict LMI (i.e. LMI without any equality constraint) that are more reliably tractable.

The purpose of this paper is firstly to give a necessary and sufficient condition for a descriptor system to have all its dynamic modes in a specified region of the complex plane and to ensure that no impulsive behavior will occur. Secondly we give a sufficient condition to check the robustness of the pole clustering in the presence of norm-bounded uncertainties. The main interests of this paper, compared to [7] and [10] is that a simple and unique formulation of the LMI condition is valid for all regions of the complex plane and that the results are expressed in terms of strict LMI.

The paper is organized as follows. Section 2 recalls some results on descriptor systems, LMI regions and pole clustering. In section 3 the characterization of pole clustering in LMI region for descriptor systems is established and a numerical example is given. In section 4, a sufficient condition for robust pole-clustering is stated.

2. Preliminaries

2.1. Basics on descriptor systems

Herein we recall some useful results concerning descriptor systems, taken from [4].
The system (1) (respectively (2)) has a unique solution, for any initial condition, if it is regular i.e. $det(sE-A)) \neq 0$ (respectively $det(zE-A) \neq 0$). The finite modes correspond to the finite eigenvalues of the matrix pencil (E, A). The system is called stable if and only if the finite modes are stable, i.e. if the finite eigenvalues of (E, A) lie in the open left half-plane (respectively lie in the open unitary disk). The infinite eigenvalues of (E, A) with associated eigenvectors ν satisfying $E.\nu = 0$ are the static modes. The infinite eigenvalues of (E, A) with generalized eigenvectors ν_k satisfying the relations

$Ev_1=0$ and $Ev_k = Av_{k-1}$ $(k\geq 2)$ correspond to impulsive modes (respectively non causality). Impulsive modes may cause impulse terms in the response, even for bounded input, and thus are highly undesirable. A system has no impulsive mode and is called impulse free if and only if (3) (respectively (4)) holds

$$deg(det(sE-A)) = rank(E) \quad (3)$$

$$deg(det(zE-A)) = rank(E) \quad (4)$$

A stable and impulse free descriptor system is called admissible.

2.2. Basics on pole clustering in LMI regions

Many temporal characteristics of the response of a linear system are linked to the location of its poles. The real part of the pole gives the decay ratio of the associated mode. In the particular case of the second order systems, the maximum overshoot, the frequency of oscillatory modes, the delay time are fully defined by the modulus and the phase of the poles. Thus it is of interest to characterize pole location or pole clustering in prescribed regions of the complex plane. As shown in [1], most of the simple regions of the complex plane that are useful in control applications can be characterized in terms of LMI regions defined below.

Definition 1 [1] *A subset D of the complex plane is called an LMI region if there exist a real symmetric m×m matrix α and a real m×m matrix β such that*

$$D = \{z \text{ complex}: f_D(z) < 0\} \quad (5)$$

where the characteristic function $f_D(z)$ is defined by

$$\begin{aligned} f_D(z) &= \alpha + z\beta + \bar{z}\beta^T \\ &= [\alpha_{kl} + \beta_{kl} z + \beta_{lk}\bar{z}]_{1\leq k,l\leq m} \end{aligned} \quad (6)$$

Notation: $\alpha=[\alpha_{kl}]_{1\leq k,l\leq m}$ means that α is an $m\times m$ matrix (respectively block matrix) with generic entry (respectively block) α_{kl}.

This definition is sufficient to characterize many basic regions like open left half-plane, vertical strips, disks, horizontal strips and conic sectors symmetric with respect to the real axis. Moreover, more complex ones, indeed every convex polygonal regions symmetric with respect to the real axis, can be obtained as the intersection of basic ones [1].

The intersection of two LMI regions D_1 and D_2 is an LMI region which characteristic function is given by

$$f_{D1\cap D2} = diag(f_{D1}, f_{D2}) \quad (7)$$

As an example, consider the region defined by the intersection of a vertical strip and a conic sector centered on the origin. The fact that the poles of a given second-order system lie in a vertical strip guaranties a minimal and a maximal decay ratio and the location in a conic sector ensures a minimal damping ratio.

An usual state-space system is called *D*-stable if and only if its poles (i.e. the eigenvalues of the matrix *A*) lie in *D*, and the following lemma gives an LMI condition to check the *D*-stability of *A*.

Lemma 1 [1] *The matrix A is D-stable if and only if there exists a symmetric n×n matrix X such that*

$$\begin{aligned} &M_D(A,X) < 0,\ X > 0 \\ &\text{with } M_D(A,X) = \alpha\otimes X + \beta\otimes AX + \beta^T\otimes(AX)^T \end{aligned} \quad (8)$$

where ⊗ is the Kronecker product defined by

$$A\otimes B = [A_{ij}B]_{ij}$$

Some properties of the Kronecker product used in this paper are given bellow

$$\begin{aligned} &\alpha\otimes A = \alpha A, \text{ for } \alpha \text{ scalar} \\ &(A+B)\otimes C = A\otimes C + B\otimes C \\ &(A\otimes B)(C\otimes D) = AC\otimes BD \\ &(A\otimes B)^T = A^T\otimes B^T \\ &(A\otimes B)^{-1} = A^{-1}\otimes B^{-1} \end{aligned}$$

3. Strict LMI characterization of pole-clustering

In this section, the concept of *D*-stability is extended to descriptor systems. In the descriptor case, the basic requirement of stability -which only concerns the proper part of the system- is not practically of sufficient interest since it does not imply that impulse behavior is prevented. Thus we need to introduce the *D*-admissibility.

Definition 2 *A descriptor system (E, A) is D-stable (respectively, D-admissible) if and only if its finite poles lie in D (respectively, its finite poles lie in D and it is impulse free).*

The following theorem states a necessary and sufficient condition, in terms of strict LMI, for a system *(E, A)* to be *D*-admissible.

Theorem 1 *A descriptor system (E, A) is D-admissible if and only if there exist a symmetric positive definite n×n matrix P and a (n-r)×(n-r) matrix S such that*

$$\begin{aligned} &\alpha\otimes EPE^T + \beta\otimes APE^T + \beta^T\otimes EPA^T \ldots \\ &+ 1_{mm}\otimes\left(AVSU^T + US^TV^TA^T\right) < 0 \end{aligned} \quad (9)$$

where the real n×(n-r) matrices V and U are of full column rank and composed of bases of Ker(E) and $Ker(E^T)$ respectively

Notation: 1_{ij} denotes the $i\times j$ matrix with all entries set to *1*.

Proof of theorem 1 :

Sufficiency : let *v* be any left eigenvector associated with a finite eigenvalue λ, pre- and post-multiplying Equation (9) by $(I_m\otimes v)$ and $(I_m\otimes v^*)$ we have

$$\begin{aligned} \big[&\alpha_{kl} vEPE^T v^* + vA(\beta_{kl}PE^T + VSU^T)v^* \\ &+ (\beta_{lk} vEP + vUS^TV^T)A^T v^* \big]_{1\leq k,l\leq m} < 0. \end{aligned}$$

Since $\lambda vE = vA,\ \bar{\lambda}E^Tv^* = A^Tv^*$, we have

$$\left[(vEP^{1/2})(\alpha_{kl} + \beta_{kl}\lambda + \bar{\lambda}\beta_{lk})(P^{1/2}E^T v^*) \right]_{1\le kl\le m} < 0 .$$

$P>0$ and v is associated with a finite eigenvalue thus it ensures $vE\neq 0$, implying that λ lie in D and consequently (E,A) is D-stable.

Let us prove that Equation (9) implies that the system is impulse free. Computing a singular value decomposition there exist unitary matrices M and N such that

$$MEN = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \text{ and } MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$V = N\begin{bmatrix} 0 \\ I \end{bmatrix} W_1 \text{ and } U = M^T \begin{bmatrix} 0 \\ I \end{bmatrix} W_2$$

where $D=D^T$, W_1 and W_2 non singular matrices. Assuming that Equation (9) holds, this implies

$$\Bigg[\alpha_{kl} M^{-1}\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} N^{-1}PN\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} M + M^{-1}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N^{-1}\left(\beta_{kl} PN \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + N\begin{bmatrix} 0 \\ I \end{bmatrix} W_1 S W_2^T \begin{bmatrix} 0 & I \end{bmatrix} \right) M + M^{-1}\left(\beta_{lk} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} N^{-1}P + \begin{bmatrix} 0 \\ I \end{bmatrix} W_2 S^T W_1^T \begin{bmatrix} 0 & I \end{bmatrix} N^{-1} \right) N \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} M \Bigg]_{1\le k,l\le m} < 0 .$$

With

$$N^{-1}PN = \left(N^{-1}PN\right)^T = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix},$$

it becomes

$$\Bigg[M^T \left(\left(\begin{bmatrix} \alpha_{kl} D P_1 D^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \left(\begin{bmatrix} \beta_{kl} P_1 D & 0 \\ \beta_{kl} P_2^T D & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & W_1 S W_2^T \end{bmatrix} \right) + \left(\begin{bmatrix} \beta_{lk} D P_1 & \beta_{lk} D P_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & W_2 S^T W_1^T \end{bmatrix} \right) \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \right) M \Bigg]_{1\le k,l\le m} < 0 .$$

Since M is non singular we have

$$\begin{bmatrix} \Phi_{kl} & \Theta_{kl} \\ \Psi_{kl} & \Sigma_{kl} \end{bmatrix}_{1\le k,l\le m} < 0$$

with

$$\Phi_{kl} = \alpha_{kl} D P_1 D^T + \beta_{kl}(A_{11}P_1 D + A_{12}P_2^T D) + \beta_{lk}(DP_1 A_{11}^T + DP_2 A_{12}^T)$$

$$\Theta_{kl} = A_{12} W_1 S W_2^T + \beta_{lk}(DP_1 A_{21}^T + DP_2 A_{22}^T)$$

$$\Psi_{kl} = W_2 S^T W_1^T A_{12}^T + \beta_{kl}(A_{21}P_1 D + A_{22}P_2^T D)$$

$$\Sigma_{kl} = A_{22}(W_1 S W_2^T) + (W_1 S W_2^T)^T A_{22}^T .$$

Thus the diagonal blocks and in particular the Σ_{kk} are negative definite implying that A_{22} is of full rank or equivalently that (E, A) is impulse free. This achieves the proof of the sufficiency part.

Necessity : if (E, A) is impulse free, then there exist M and N, two non singular matrices such that $MEN=diag(I, 0)$ and $MAN=diag(J,I)$. Moreover if (E,A) is D-stable then the eigenvalues of J lie in D and from lemma1 there exists a symmetric positive definite P_1 such that

$$\left[\alpha_{kl} P_1 + \beta_{kl} J P_1 + \beta_{lk} P_1 J^T \right]_{1\le kl\le m} < 0 .$$

Then for a sufficiently small ν > 0 there exists a symmetric positive definite matrix P_1 such that

$$\left[\alpha_{kl} P_1 + \beta_{kl} J P_1 + \beta_{lk} P_1 J^T + \frac{v}{2} M_1 M_2^T (M_2 M_2^T)^{-1} M_2 M_1^T \right]_{1\le k,l\le m} < 0 .$$

By Schur complement this implies

$$\begin{bmatrix} \alpha_{kl} P_1 + \beta_{kl} J P_1 + \beta_{lk} P_1 J^T & -vM_1 M_2^T \\ -vM_2 M_1^T & -2vM_2 M_2^T \end{bmatrix}_{1\le k,l\le m} < 0 ,$$

that can be developed into

$$\Bigg[\alpha_{kl} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \beta_{kl} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \beta_{lk} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} J^T & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} M_2 (-vI) \begin{bmatrix} M_1^T & M_2^T \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} (-vI) M_2^T \begin{bmatrix} 0 & I \end{bmatrix} \Bigg]_{1\le k,l\le m} < 0 .$$

Decomposing N and M by $N=[N_1\ N_2]$ and $M=[M_1^T\ M_2^T]^T$, we have $M_2E=0$, $EN_2=0$ and $U=M_2^T W_M^T$ and $V=N_2W_N$.

$$\Big[\alpha_{kl} MEPE^T M^T + \beta_{kl} MAPE^T M^T + \beta_{lk} MEPA^T M^T + MAN_2 W_N(-W_N^{-1} v W_M^{-1}) W_M M_2 M^T + MM_2^T W_M^T (-W_M^{-T} v W_N^{-T}) W_N^T N_2^T A^T M^T \Big]_{1\le k,l\le m} < 0 .$$

With $P = N\begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} N^T$, $S = -vW_N^{-1}W_M^{-1}$ and since M is non singular it becomes

$$\Big[\alpha_{kl} EPE^T + \beta_{kl} APE^T + \beta_{lk} EPA^T + AVSU^T + US^T V^T A^T \Big]_{1\le k,l\le m} < 0 .$$

Since P is positive definite, that completes the proof. ■

Remark : contrarily to [7] only strict LMI are requested to be solved and thus improves numerical tractability and a unique formulation embraces all LMI regions.

Using the fact that the class of LMI region is invariant under intersection, this theorem generalizes the result of [1] to descriptor systems (let $E=I_n$, then V and U are null and we obtain lemma 1). The D-admissibility can be checked for all convex region of the complex plane provided it is symmetric with respect to the real axis.

Theorem 1 is valid both in the discrete-time case and continuous-time case provided the specified LMI region D is a stability region in the considered case. Thus we can derive the following corollaries characterizing the admissibility.

Corollary 1 *The continuous-time descriptor system (1) is admissible if and only if there exist a symmetric positive definite (n×n) matrix P and a (n-r)×(n-r) matrix S such that*

$$A(PE^T + VSU^T) + (EP + US^T V^T)A^T < 0 \qquad (10)$$

Proof of corollary 1 :
Using definition 1, the system (1) is admissible if and only if it is *D*-admissible where *D* is the open left half-plane which is the LMI region characterized by α=0 and β=1. Applying theorem 1, it is equivalent to Equation (10). ■

Corollary 2 *The discrete-time descriptor system (2) is admissible if and only if there exist a symmetric positive definite (n×n) matrix P and a (n-r)×(n-r) matrix S such that*

$$\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^T & \Phi_1 \end{bmatrix} < 0 \qquad (11)$$

$$\text{with } \Phi_1 = -EPE^T + AVSU^T + US^T V^T A^T$$
$$\Phi_2 = APE^T + AVSU^T + US^T V^T A^T$$

Proof of corollary 2:
Using definition 1, the system (2) is admissible if and only if it is *D*-admissible where *D* is the open unitary disk centered at the origin

$$f_D(z) = \begin{pmatrix} -1 & z \\ \bar{z} & -1 \end{pmatrix} \text{or equivalently } \alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{and } \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then, applying theorem 1, Equation (11) follows. ■

Remark : The admissibility of a continuous-time and discrete-time descriptor system are characterized in terms of strict LMIs in [12] and [14] respectively, but the results were not generalized to all LMI regions .

Numerical example

let us check that the finite poles $\lambda_k=a_k+ib_k$ of a given system *(E, A)* lie in an LMI region defined by the intersection of two basic LMI regions, respectively a vertical strip and a conic sector. The first one ensures a minimal and a maximal decay ratio h_1 and h_2 and the second one ensures a minimal damping ratio $\xi=cos(\theta)$. The vertical strip is defined by $D_1=\{z=x+iy : h_1<x<h_2\}$ and

$$f_{D_1}(z) = \begin{pmatrix} 2h_1 - (z+\bar{z}) & 0 \\ 0 & -2h_2 + (z+\bar{z}) \end{pmatrix}$$

$$\text{or } \alpha_1 = \begin{pmatrix} 2h_1 & 0 \\ 0 & -2h_2 \end{pmatrix} \text{ and } \beta_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

the conic sector is defined by $D_2=\{z=x+iy : x.tan\theta<|y|\}$ and

$$f_{D_2}(z) = \begin{pmatrix} sin\theta(z+\bar{z}) & cos\theta(z-\bar{z}) \\ -cos\theta(z-\bar{z}) & sin\theta(z+\bar{z}) \end{pmatrix}$$

$$\text{or } \alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \beta_2 = \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix}.$$

According to the definition (7) the intersection of the two LMI regions is defined by

$$f_D = \begin{pmatrix} 2h_1 - (z+\bar{z}) & 0 & 0 & 0 \\ 0 & -2h_2 + (z+\bar{z}) & 0 & 0 \\ 0 & 0 & sin\theta(z+\bar{z}) & cos\theta(z-\bar{z}) \\ 0 & 0 & -cos\theta(z-\bar{z}) & sin\theta(z+\bar{z}) \end{pmatrix}$$

For $h_1=-5$, $h_2=-1$ and $\theta=\pi/4$ and the following impulse free matrix pencil *(E, A)*

$$E = \begin{pmatrix} -5 & -1 & -5 & -5 \\ 3 & -6 & 3 & -2 \\ -3 & -1 & -3 & 1 \\ 0 & -1 & 0 & -3 \end{pmatrix} \text{ and } A = \begin{pmatrix} -2 & -6 & 3 & 34 \\ -27 & 20 & -25 & -12 \\ 0 & -4 & 4 & 8 \\ -7 & 2 & -6 & 8 \end{pmatrix}$$

which finite poles are $\sigma_f = \{-3-2i, -3+2i, -4\}$, the LMI is solvable and using the LMI toolbox [5] we obtain the following result

$$P = \begin{pmatrix} 2.9424 & 0.32528 & 0.46319 & -2.7244 \\ 0.32528 & 1.2434 & -0.061769 & -0.21080 \\ 0.46319 & -0.061769 & 2.4520 & -2.3322 \\ -2.7244 & -0.21080 & -2.3322 & 12.587 \end{pmatrix} \text{ and } S = 3.5963$$

Let $h_2=-4$, the pole of the matrix pencil *(E, A)* no longer lie in *D* and as a result the corresponding LMI is not feasible and Matlab returns no solution for *P* and *S*.

4. Strict LMI characterization of robust pole-clustering

In this section, we are looking for an LMI characterization of the robustness of the *D*-admissibility. Consider an uncertain descriptor system given by the following LTI descriptor system (the control input is not mentioned since the study focuses on the dynamics)

$$\begin{aligned} E\dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= \Delta z \end{aligned} \qquad (12)$$

and the norm-bounded uncertainties

$$\Delta \text{ an } (d \times d) \text{ matrix such that } \sigma_{max}(\Delta) \le \gamma^{-1} \qquad (13)$$

the closed-loop system is defined by

$$\begin{aligned} & E\dot{x} = A(\Delta)x \\ & \text{with } A(\Delta) = (A + B(I - \Delta D)^{-1}\Delta C), \sigma_{max}(\Delta) \le \gamma^{-1} \end{aligned} \qquad (14)$$

Thus hereafter, we consider the family of descriptor systems described by the uncertain pencil matrix *(E,A(Δ))* for all admissible *Δ* (i.e. satisfying Equation (14)). The case *Δ=0* is the nominal case and the parameter *γ* corresponds to a level of uncertainty. The main issue of robust *D*-admissibility is to know if, for a given *γ*, the descriptor system family *(E,A(Δ))* remains admissible and if its finite poles remain in the LMI region *D*. As discussed in [2] this formalism is still valid for time-varying uncertainties.

Definition 3 *The uncertain descriptor system family (E,A(Δ)) is robustly D-stable (respectively robustly D-admissible) if the finite eigenvalues of (E,A(Δ)) lie in D (respectively if the finite eigenvalues of (E,A(Δ)) lie in D and (E,A(Δ)) is impulse free) for all admissible Δ.*

Robust D-admissibility is generally difficult to prove, thus a more practical notion is frequently used : the quadratic D-admissibility, defined bellow

Definition 4 [10] *The uncertain descriptor system family $(E,A(\Delta))$ is quadratically D-admissible if there exists a real symmetric $(n\times n)$ matrix $P>0$ and a real $(n\text{-}r)\times(n\text{-}r)$ matrix S such that, for all admissible Δ the following LMI holds*

$$\begin{aligned} M_D(E,A(\Delta),P,S) &= \alpha\otimes E^T PE+\beta\otimes E^T PA(\Delta) \\ &+ \beta^T A(\Delta)^T PE + 1_{mm}\left(VSU^T A(\Delta)+A(\Delta)^T US^T V^T\right)<0 \end{aligned} \tag{15}$$

Obviously, quadratic D-admissibility implies robust D-admissibility. The quadratic D-stability is more conservative since a single pair (P, S) should satisfy the LMI for all admissible Δ's, while robust D-admissibility implies that pairs $(P(\Delta), S(\Delta))$ exist for each admissible Δ. As discussed in [2] for usual systems, the assumption P and S real is not restrictive while much more practical from the computational point of view.

The following theorem characterizes the quadratic D-admissibility in terms of strict LMIs and thus generalizes the result obtained in [2] for usual systems. Contrarily to [10] the condition is a strict LMI. The system (12) is a continuous-time system but the result is still valid while considering a discrete time system, provided the LMI region D is a stability region.

Theorem 2 *The uncertain descriptor system family $(E, A(\Delta))$ is quadratically D-admissible if there exist real symmetric positive definite matrices P and P_1 (respectively $(n\times n)$ and $(k\times k)$), a positive real number P_2 and a $(n\text{-}r)\times(n\text{-}r)$ matrix S such that*

$$\begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{22}\end{bmatrix}<0 \tag{16}$$

with

$$\Theta_{11}=\begin{bmatrix} M_D(E,A,P,S) & \beta_1^T\otimes E^T PB & 1_{m1}\otimes VSU^T B\\ \beta_1\otimes B^T PE & -\gamma(P_1\otimes I_d) & 0\\ 1_{1m}\otimes B^T US^T V^T & 0 & -\gamma(P_2 I_d)\end{bmatrix}$$

$$\Theta_{12}=\begin{bmatrix}\beta_2^T P_1\otimes C^T & 1_{m1}P_2\otimes C^T\\ P_1\otimes D^T & 0\\ 0 & P_2 D^T\end{bmatrix} \text{ and}$$

$$\Theta_{22}=\begin{bmatrix}-\gamma(P_1\otimes I_d) & 0\\ 0 & -\gamma(P_2 I_d)\end{bmatrix}$$

where β_1 and β_2 are full column rank $(k\times m)$ matrices such that

Proof of theorem 2:

the proof follows the one of [2], with no loss of generality we assume $\gamma=1$. Since P_2 is positive, Equation (16) implies

$$\begin{bmatrix}-I & D\\ D^T & -I\end{bmatrix}<0$$

thus by Schur complement it is easily seen that $\sigma_{max}(D)<1$ and hence $\sigma_{max}(\Delta D)<1$ for all admissible Δ.

By definition, quadratic D-admissibility holds if and only if $P>0$ and S exist such that, for all $\sigma_{max}(\Delta)\le 1$

$$\begin{aligned} M_D(E,A(\Delta),P,S) &= M_D(E,A,P,S)+\beta\otimes E^T PB(I-\Delta D)^{-1}\Delta C \\ &+\beta^T\otimes\left(B(I-\Delta D)^{-1}\Delta C\right)^T PE+1_{mm} \\ &\otimes\left[VSU^T B(I-\Delta D)^{-1}\Delta C+\left(B(I-\Delta D)^{-1}\Delta C\right)^T US^T V^T\right]<0 \end{aligned}$$

Equivalently, for any nonzero vector h and any admissible Δ the following inequality should hold

$$\begin{aligned} & h^H M_D(E,A,P,S)h+2h^H\left(\left(\beta_1^T\beta_2\right)\otimes\left(E^T PB(I-\Delta D)^{-1}\Delta C\right)\right)h \\ &+2h^H\left(\left(1_{m1}1_{1m}\right)\otimes\left(VSU^T B(I-\Delta D)^{-1}\Delta C\right)\right)h<0 \end{aligned}$$

or equivalently

$$\begin{aligned} & h^H M_D(E,A,P,S)h+2h^H\left(\beta_1^T\otimes E^T PB\right)\left(\beta_2\otimes(I-\Delta D)^{-1}\Delta C\right)h \\ &+2h^H\left(\left(1_{m1}\right)\otimes\left(VSU^T B\right)\right)\left(1_{1m}\otimes(I-\Delta D)^{-1}\Delta C\right)h<0 \end{aligned}$$

For fixed nonzero h, this result amounts to requiring

$$\begin{aligned} & h^H M_D(E,A,P,S)h+2h^H\left(\beta_1^T\otimes E^T PB\right)p_1 \\ &+2h^H\left(\left(1_{m1}\right)\otimes\left(VSU^T B\right)\right)p_2<0 \;\; \text{whenever} \\ & p_1\in S_{1h}:=\left\{\left(\beta_2\otimes(I-\Delta D)^{-1}\Delta C\right)h:\sigma_{max}(\Delta)\le 1\right\} \\ & p_2\in S_{2h}:=\left\{\left(1_{1m}\otimes(I-\Delta D)^{-1}\Delta C\right)h:\sigma_{max}(\Delta)\le 1\right\} \end{aligned}$$

Observing that

$$\begin{aligned} p_1 &= \left(\beta_2\otimes(I-\Delta D)^{-1}\Delta C\right)h \\ p_2 &= \left(1_{1m}\otimes(I-\Delta D)^{-1}\Delta C\right)h \end{aligned}$$

are respectively the unique solution of the following equations

$$\begin{aligned} & p_1=(I_k\otimes\Delta)q_{1ph}, \text{ with } q_{1ph}=(I_k\otimes D)p_1+(\beta_2\otimes C)h \\ & p_2=(1\otimes\Delta)q_{2ph}, \text{ with } q_{2ph}=Dp_2+(1_{1m}\otimes C)h \end{aligned}$$

an equivalent and simpler characterization of S_{1h} and S_{2h} is

$$\begin{aligned} S_{1h} &= \left\{p_1 : p_1=(I_k\otimes\Delta)q_{1ph}, \sigma_{max}(\Delta)\le 1\right\} \\ S_{2h} &= \left\{p_2 : p_2=(1\otimes\Delta)q_{2ph}, \sigma_{max}(\Delta)\le 1\right\} \end{aligned}$$

Now, $p_1=(I_k\otimes\Delta)q_{1ph}$, $p_2=(1\otimes\Delta)q_{2ph}$ and $\sigma_{max}(\Delta)\le 1$ ensure, for any $(k\times k)$ matrix $P_1>0$ and any $P_2>0$,

$$\begin{aligned} & q_1^H(P_1\otimes I_d)q_1-p_1^H(P_1\otimes I_d)p_1=q_1^H\left(P_1\otimes\left(I-\Delta^H\Delta\right)\right)q_1\ge 0 \\ & q_2^H(P_2\otimes I_d)q_2-p_2^H(P_2\otimes I_d)p_2=q_2^H\left(P_2\otimes\left(I-\Delta^H\Delta\right)\right)q_2\ge 0 \end{aligned}$$

Consequently a sufficient condition for the quadratic D-admissibility is that

$$\begin{aligned} & h^H M_D(E,A,P,S)h+2h^H\left(\beta_1^T\otimes E^T PB\right)p_1 \\ &+2h^H\left(\left(1_{m1}\right)\otimes\left(VSU^T B\right)\right)p_2<0 \end{aligned}$$

whenever

$$\begin{aligned} & q_1^H(P_1\otimes I_d)q_1-p_1^H(P_1\otimes I_d)p_1\ge 0 \\ & q_2^H(P_2\otimes I_d)q_2-p_2^H(P_2\otimes I_d)p_2\ge 0 \end{aligned} \qquad \beta=\beta_1^T\beta_2$$

Or equivalently if

$$v^H\begin{bmatrix} M_D(E,A,P,S) & \beta_1^T\otimes E^T PB & \left(1_{m1}\right)\otimes\left(VSU^T B\right)\\ \beta_1\otimes B^T PE & 0 & 0\\ \left(1_{1m}\right)\otimes\left(B^T US^T V^T\right) & 0 & 0\end{bmatrix}v<0,$$

whenever

$$v^H\left(\begin{bmatrix}\beta_2^T\otimes C^T\\ I_k\otimes D^T\\ 0\end{bmatrix}(P_1\otimes I_d)[\beta_2\otimes C\ I_k\otimes D\ 0]-\begin{bmatrix}0&0&0\\0&P_1\otimes I_d&0\\0&0&0\end{bmatrix}\right)v\geq 0$$

$$v^H\left(\begin{bmatrix}1_{m1}\otimes C^T\\ 0\\ D^T\end{bmatrix}(P_2 I_d)[1_{1m}\otimes C\ 0\ D]-\begin{bmatrix}0&0&0\\0&0&0\\0&0&P_2 I_d\end{bmatrix}\right)v\geq 0$$

$$\text{with } v^T=\begin{bmatrix}h^T & p_1^T & p_2^T\end{bmatrix}.$$

Using an S-procedure argument [8], $M+N_1+N_2<0$ implies that $s^TMs<0$ for all s such that $s^TN_1s\geq 0$ and $s^TN_2s\geq 0$, a sufficient condition for quadratic D-admissibility is

$$\begin{bmatrix}M_D(E,A,P,S) & \beta_1^T\otimes E^TPB & (1_{m1})\otimes(VSU^TB)\\ \beta_1\otimes B^TPE & P_1\otimes I_d & 0\\ (1_{1m})\otimes(B^TUS^TV^T) & 0 & P_2 I_d\end{bmatrix}$$
$$+\begin{bmatrix}\beta_2^T\otimes C^T\\ I_k\otimes D^T\\ 0\end{bmatrix}(P_1\otimes I_d)[\beta_2\otimes C\ I_k\otimes D\ 0]$$
$$+\begin{bmatrix}1_{m1}\otimes C^T\\ 0\\ D^T\end{bmatrix}(P_2 I_d)[1_{1m}\otimes C\ 0\ D]<0.$$

Finally by Schur complement and pre- and post-multiplying by $diag(I_{mn}, I_{kd}, I_d, P_1, P_2)$ the proof is completed. ■

This result generalizes the result of [2], for usual state-space systems. Let $E=I_n$, $V=0$ and $V=0$ by deleting the third and fifth rows and columns then theorem 3.3 of [2] follows.

A sufficient condition for robust admissibility in continuous-time case is derived from theorem 2.

Corollary 3 *The continuous-time uncertain descriptor system family (E, A(Δ)) is robustly admissible if there exist a symmetric positive definite n×n matrix P, real positive numbers* P_1 *and* P_2 *and a (n-r)×(n-r) matrix S such that*

$$\begin{pmatrix}E^TPA+A^TPE+VSU^TA+A^TUS^TV^T & E^TPB & VSU^TB & P_1C^T & P_2C\\ B^TPE & -\gamma P_1I_d & 0 & P_1D^T & 0\\ B^TUS^TV^T & 0 & -\gamma P_2I_d & 0 & P_2D^T\\ CP_1 & DP_1 & 0 & -\gamma P_1I_d & 0\\ CP_2 & 0 & DP_2 & 0 & -\gamma P_2I_d\end{pmatrix}<0$$

Proof of corollary 3:

Let D be the open left half-plane, applying theorem 2, corollary 3 follows. ■

Remark: applying theorem 2, with D defined by the unitary disk centered at the origin, a similar result can be established for discrete-time descriptor systems.

5. Conclusion

In this paper we have studied the D-admissibility of a pencil matrix which can be envisaged as the extension of the concept of D-stability to the descriptor systems. A necessary and sufficient LMI condition to test the D-admissibility has been established. The robustness of the D-admissibility faced to norm-bounded uncertainties is envisaged. A sufficient LMI condition for robust D-admissibility has been established. For the sake of numerical tractability, the results are exclusively given in terms of strict LMIs, i.e. without any equality constraint.

Further works are to be done to derive controllers achieving (robust) pole-clustering of the controlled system.

References

[1] M. Chilali and P. Gahinet, H∞ Design with pole placement constraints: an LMI approach, IEEE Trans. Automat. Contr., Vol. 41, n° 3, pp. 358-367, 1996

[2] M. Chilali, P. Gahinet and P. Apkarian, Robust pole placement in LMI regions, IEEE Trans. Automat. Contr., Vol. 44, n° 12, pp. 2257-2270, 1999

[3] D. Cobb, Descriptor variable systems and optimal state regulation, IEEE Trans. Automat. Contr., Vol. 28, n° 5, pp.601-611, 1983

[4] L. Dai, Singular control systems, Springer 1989

[5] P Gahinet, A. Nemirovski, A.J. Laub and M. Chilali, The LMI Control Toolbox, The MathWorks Inc., 1995

[6] L. El Ghaoui, R. Nikoukah and F. Delebecque, LMITOOL: a package for LMI optimization, Proc. CDC, pp. 3096-3101,1995

[7] K.L. Hsiung and L. Lee, Pole clustering characterization via LMI for descriptor systems, Proc. CDC, pp. 1313-1314, 1997

[8] U.T. Jönsson, A Lecture on the S-Procedure, Lecture note at the Royal Institute of Technology, Sweden, 2001

[9] A. Kawamoto and T Katayama, Standard H∞ control problem for descriptor systems, Proc. CDC, pp. 4130-4133, 1997

[10] J.L. Cheng and L. Lee, Robust pole-clustering for descriptor systems with norm-bounded uncertainties, Proc. ACC, pp. 2953-2954, 2001

[11] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda, H∞ control for descriptor systems : A matrix inequalities approach, Automatica, vol. 33, n° 4, pp. 669-673, 1997

[12] E. Uezato and M. Ikeda, Strict LMI conditions for stability, robust stabilization and H∞ control of descriptor systems, Proc. CDC, pp. 4092-4097, 1999

[13] H.S. Wang, C.F. Yung and F.R. Chang, Bounded real lemma and H∞ control for descriptor systems, Proc. ACC, pp. 2115-2119, 1997

[14] G. Zhang and Y. Jia, New results on discrete-time bounded real lemma for singular systems: strict matrix inequality conditions, Proc. ACC, pp. 634-638, 2002

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