REGION OF ATTRACTION ESTIMATES FOR LPV-GAIN SCHEDULED CONTROL SYSTEMS

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Abstract

Recent methods for gain scheduling controller design based on linear parameter-varying (LPV) systems offer a systematic way to obtain a nonlinear controller that covers different operating conditions. However, despite that the LPV synthesis part of the process of obtaining a gain scheduled controller is theoretically straight forward, the nonlinear closed loop system may be unstable for some operating conditions. This property is illustrated by a simple second order autonomous nonlinear system, the well known Van der Pol equation. Furthermore, a region of attraction estimate based on the LPV analysis for the nonlinear system is given.

1 Introduction

One of the most popular controller design methods in practical problems is gain scheduling. This method uses a quasi-stationary heuristic approach to the design of nonlinear controllers. The nonlinear control law is formed by a divide and conquer strategy, leading to a synthesis problem for different operating settings together with a mapping of these to cover a wide range of settings. Due to the heuristics, the method has until the last decade or so received little attention in the academic world, see [9, 7].

One decade ago, linear parameter-varying (LPV) systems, [10] were introduced in the context of gain scheduling. Such systems enable a systematic way of obtaining the controller. The synthesis can incorporate the operating conditions in the scheduling parameter of the system resulting in a controller that is directly parameter dependent, eliminating the explicit mapping of linear controllers.

In parallel to the above mentioned development of LPV system theory, the use of linear matrix inequalities (LMI) in control theory has been developed, see e.g. [3] and references therein. In particular robust $\mathcal{H}_2$, $\mathcal{H}_\infty$ and μ methods fit into this framework of LMI constraints, see e.g. [4]. The Riccati equation for $\mathcal{H}_\infty$ has a corresponding LMI formulation. In the case of full order or state feedback controller synthesis, the problem is convex, and can be solved readily with available numerical LMI software. The combination of the LMI based synthesis methods and the use of LPV systems led to a systematic way of obtaining a gain scheduled controller in a numerically appealing way.

Using LPV synthesis methods means that a nonlinear system has to be formulated as an LPV system. The LPV system description is conservative in the sense that the nonlinearities of the system are captured by the (scheduling) parameter vector, which usually is allowed to take values within a bounded box, and sometimes there are also constraints on the rate of change of the parameter vector. This means that the LPV system not only describes the original nonlinear system, but also all nonlinear systems obtained when changing the parameter vector arbitrarily, as long as its value stays in the bounding box. The goal of the synthesis is to maintain stability and performance for all parameter values in the bounding box, and hence the obtained LPV controller is valid also for the nonlinear system.

The controller synthesis of LPV systems has drawn much attention in the literature. Given an LPV system, the method of obtaining a controller is fairly straight forward. However, the problem of how to end up in an LPV description of the nonlinear system is far from straight forward. A standard anzats to this problem is an approximation of the nonlinear system by mapping Taylor linearizations for different operating conditions. It is clear that such LPV models can deviate much from the nonlinear model, and the LPV design may perform badly or even result in an unstable closed loop system of the original nonlinear system, at least for some operating conditions. This procedure is however motivated under the assumption of a slowly varying parameters. In this paper, only nonlinear systems that can be exactly included by LPV systems will be considered.

The properties of the nonlinear system will be studied in this paper. In particular, asymptotic stability properties of the nonlinear system, in the context of LPV stability analysis, is investigated. Many of the LPV controller synthesis methods compute a Lyapunov function for the closed loop LPV system, which sometimes is quadratic, see e.g. [2] or non-quadratic (parameter dependent), see e.g. [12]. The focus in this paper is on stability of autonomous nonlinear systems, but this problem is closely related to the synthesis problem via the Lyapunov function.

The notation in the paper is standard. We make difference between stability of the nonlinear system and LPV stability. In the later, stability is only considered for parameter trajectories that stay inside the bounding box and without the connection to the (possible) underlying nonlinear system. This means that there is no distinction of whether that parameter is time-varying, resulting in a linear time-varying system, or depend on the state vector, implying a nonlinear system. This is the common approach to LPV gain scheduling in the literature.

This paper is organized as follows. In the following section, a motivating example is given to show that stability in the LPV sense does not imply stability in terms of region of attraction for the underlying nonlinear system. This is the case even though the LPV description is an exactly mapping of the nonlinear system. The succeeding section gives a region of attraction estimate, based on the LPV analysis, for both purely quadratic Lyapunov functions and for non-quadratic (parameter dependent) Lyapunov functions. This is illustrated by use of the example in the preceding section. This example also shows the importance of the choice of scheduling parameters in the context of solvability for the LPV system, and obtaining large estimates of the region of attraction for the nonlinear system. Finally, some concluding remarks are given.

2 Motivating Example

It is easy to believe that a closed loop LPV system that satisfies stability (or some other goal) for all parameters varying in the bounding box implies that stability is satisfied for the underlying nonlinear system. In this section, a simple example is given illustrating that this is, in general, not true.

Consider the well known *Van der Pol* equation (with reversed vector field),

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - 0.3(1 - x_1^2)x_2. \end{aligned} \tag{1}$$

This equation (1) is a special case of Liénard's equation, see [5], and it is well known that a limit cycle exists for such systems. This reversed vector field version has the property that all trajectories starting outside this limit cycle diverges and all trajectories starting inside converges to zero, see figure 1.

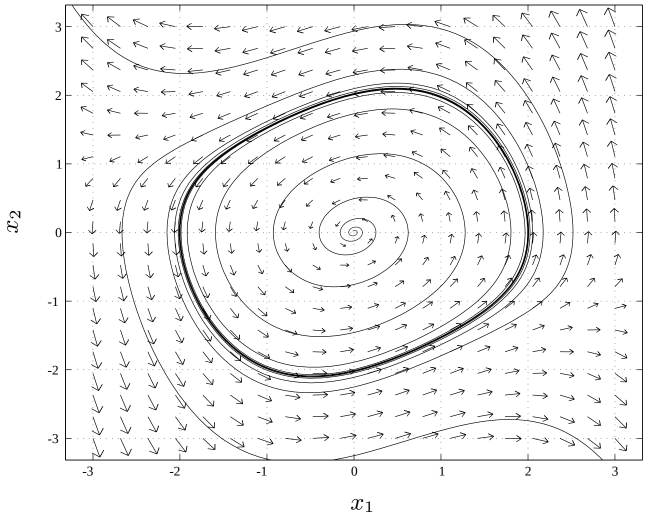

Figure 1: Phase-portrait of the Van der Pol equation with reversed vector field.

One obvious LPV parameterization of (1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -0.3\rho \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2}$$

where

$$\rho = 1 - x_1^2. \tag{3}$$

The only nonlinear term of the right hand side of (1) is hidden in the parameter ρ. Observe that (2) is an exact inclusion of (1) in this sense.

Let $\rho \in \Omega$, where $\Omega = \{\rho \in \mathbb{R} | 0.1 \le \rho \le 1\}$ is the parameter box where ρ takes its values. Now, we want to check stability of the LPV system defined by (2) for all $\rho \in \Omega$. The exact relationship between the scheduling parameter ρ and the states of the system (3) is then neglected. The stability test of (2) can be formulated as a parameterized LMI. If there exist a positive definite symmetric matrix P such that

$$A^T(\rho)P + PA(\rho) < 0,\ \forall \rho \in \Omega \subset \mathbb{R}^p, \tag{4}$$

then the system $\dot{x} = A(\rho)x$ is stable in the LPV sense. This is often referred to as *quadratic stability* to emphasize the use of a quadratic Lyapunov function.

Satisfying condition (4) means that the system $\dot{x} = A(\rho)x$ is stable regardless of whether ρ is a function of time or of state. The parameter ρ may even change arbitrary fast leading to discontinuities in ρ. In this light, the condition (4) is conservative since it guarantees stability for all $\rho \in \Omega$, not only for the possibly underlying nonlinear system. This conservatism might result in that there are no solution P satisfying (4).

For the system (2) where $\rho \in \Omega$ and Ω is according to the set above, it is not hard to find a P such that (4) is satisfied. For example,

$$P = \begin{bmatrix} 33.3483 & -0.5 \\ -0.5 & 33.33 \end{bmatrix}, \tag{5}$$

which corresponds to setting the left hand side of (4) to $-I$ when $\rho = 0.1$. The parameterized LMI (4) is satisfied for all $\rho \in \Omega$ if it is satisfied in the vertexes of Ω, due to convexity of the affine parameter dependence in (4). In this example, the parameterized LMI corresponds to a monotonous function in ρ and the vertex $\rho = 0.1$ is the worst case.

One may believe that the nonlinear system (1) is stable, meaning that its trajectories converge to zero for all initial values of the state vector that correspond to the set Ω and the parameter to state relationship (3), that is,

$$-\sqrt{1 - 0.1} \le x_1 \le \sqrt{1 - 0.1}. \tag{6}$$

This is *not* the case, as can be observed in figure 1. All trajectories starting outside the limit cycle diverges, even for x satisfying (6). However, the LPV analysis of the nonlinear system does guarantee local asymptotical stability. As we will see in the next section, the LPV stability analysis can be used to estimate the region of attraction for the underlying nonlinear system.

3 Region of attraction

As the example in the foregoing section illustrates, the stability region (or region of attraction) does not coincide with the part of the state space implicitly defined by the bounding set Ω of the parameter. However, as long as $\rho(x) \in \Omega$ for the stationary point of the nonlinear system it is possible to find an estimate of the region of attraction based on the LPV analysis (synthesis).

In this section the estimate of the region of attraction based on the LPV analysis is given. Recall the definition of the region of attraction.

Definition 3.1 *For a nonlinear system,*

$$\dot{x} = f(x),\ x \in D$$

with the origin as an equilibrium point, let $\phi(t;x)$ denote the solution that starts at initial state x at time $t = 0$. The region of attraction *(or region of asymptotic stability or domain of attraction) is defined as the set,*

$$R_A = \{x \in D | \lim_{t\to\infty} \phi(t;x) = 0\}$$

Finding the exact region of attraction analytically might be difficult or even impossible, see [5]. However it is well known that Lyapunov functions can be used to estimate the region of attraction. It is also well known that the region of attraction, for an asymptotically stable system with respect to the origin, is an open connected set where the origin is an interior point, invariant with respect to the system and that the boundary of R_A is defined by trajectories of the system.

The following theorem gives an estimate of the region of attraction based on the LPV stability analysis. First, the simpler case of quadratic Lyapunov function (parameter independent) is given.

Theorem 3.2 *Consider the nonlinear system,*

$$\dot{x} = f(x) \tag{7}$$

with the exact LPV description,

$$\dot{x} = A(\rho)x,\ \rho = \rho(x). \tag{8}$$

Assume that (8) is LPV stable for $\rho \in \Omega$, with a Lyapunov function $V(x) = x^T P x$. Define the following sets,

$$\mathcal{X} = \{x \in D | \rho(x) \in \Omega\} \tag{9}$$

$$\Gamma_\beta = \{x \in D | V(x) \le \beta\}. \tag{10}$$

If $\Gamma_\beta \subseteq \mathcal{X}$ then $\Gamma_\beta \subseteq R_A$.

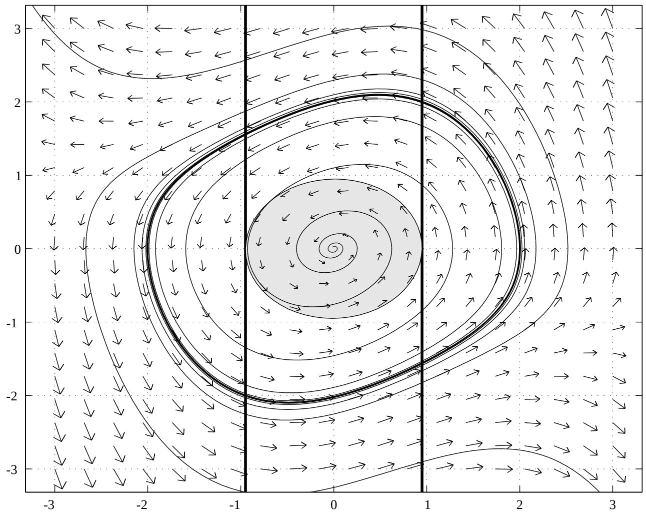

Figure 2: Region of attraction estimate (shaded area) in the region $|x_1| \le \sqrt{1-0.1}$ (solid lines) based on the LPV analysis, using a quadratic Lyapunov function.

Proof. From the assumption of the LPV stability of (8) we have that,

$$A^T(\rho)P + PA(\rho) < 0,\ \forall \rho \in \Omega,\ P > 0$$

It is clear that V is also an Lyapunov function for (7) since,

$$\dot{V}(x) = x^T(A^T(\rho(x))P + PA(\rho(x)))x < 0,\ \forall x \in \mathcal{X} \setminus \{0\}$$

Since Γ_β is bounded and $\dot{V}(x) < 0$ for all $x \in \mathcal{X} \setminus \{0\}$, any trajectories starting in Γ_β at $t = 0$ stays in Γ_β for all $t \ge 0$, and converges to zero as $t \to \infty$ c.f. [5]. Hence, Γ_β is an estimate of the region of attraction. ■

Since $V(x)$ is continuous and positive definite, the set $\Gamma_\beta \ne \emptyset$ and the origin is an interior point of Γ_β, see [5]. Furthermore, the best estimate of R_A is obtained when Γ_β is as large as possible, and from (10) it can be seen that Γ_β becomes larger with increasing β.

The consequence of the theorem is that a region of attraction estimate for the Van der Pol example in previous section is that the ellipsoid associated with the positive definite matrix in (5) is enclosed in the region defined by the inequality (6), see figure 2. This is quite a conservative estimate of R_A. It is not possible to obtain a much larger invariant set that is enclosed in the set (6). A way to reduce this conservatism is to change scheduling variables and to use a more complex Lyapunov function, as described next.

To extend the estimate of R_A, consider a different exact LPV characterization of the Van der Pol equation (1),

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 + 0.3\rho & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

with

$$\rho = x_1 x_2. \tag{12}$$

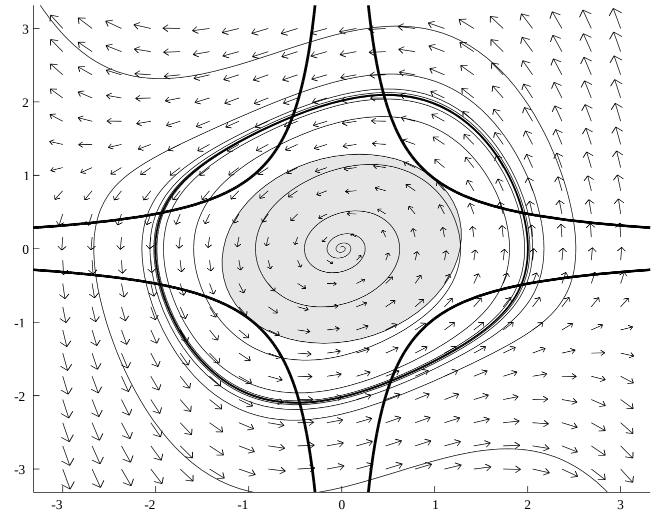

Figure 3: Region of attraction estimate (shaded area) in the region $|x_1 x_2| \leq 0.95$ (solid lines) based on the LPV analysis using a quadratic Lyapunov function.

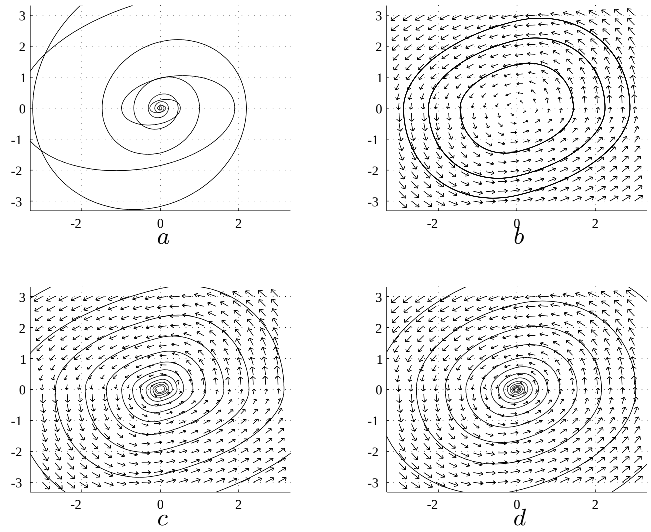

Figure 4: Solution of (11) for $\rho = 1.53$ and $\rho = -1.53$ respectively in a, $\rho = 1.53\mathrm{sign}(x_1 x_2)$ in b, $\rho = 2\mathrm{sign}(x_1 x_2)$ in c and $\rho = \mathrm{sign}(x_1 x_2)$ in d.

In the following, it will be shown that this choice of the parameter ρ will make it possible to extend the estimate of R_A. This illustrates the importance of the choice of scheduling variables, as both (2) and (11) include the same nonlinear system. An intuitive guideline in this choice is to “hide” as little of the nonlinearities as possible in the parameter and letting the parameter components be as independent as possible.

For $|\rho| \leq \rho_0$, there exists a quadratic Lyapunov function $V(x) = x^T P x$ for the LPV system (11) when $\rho_0 = 1$, estimating the region of attraction R_A according to figure 3.

If ρ_0 increases to $\rho_0 > 1.53$ there cannot exist a quadratic Lyapunov function for the LPV system (11), since ρ might vary in a way such that the solution to (11) diverges to infinity. It can be shown that varying ρ according to $\rho = \rho_0 \mathrm{sign}(x_1 x_2)$ means that the solution to (11) is the worst possible, i.e. the solution that converges slowest ($\rho < 1.53$) to the origin or diverges fastest ($\rho_0 > 1.53$) to infinity, c.f. figure 4.

Setting $\rho = \rho_0 \mathrm{sign}(x_1 x_2)$ implies that ρ is discontinuous and varies infinitely fast for $x_1 = 0$ or $x_2 = 0$, which is possible since there is no bound on the rate of changes of ρ. The result is a switching of ρ between its extreme values. Such systems are commonly referred to as switched or hybrid systems, c.f. [8].

By letting the Lyapunov matrix P depend on ρ, the time derivative of ρ enters the stability conditions,

$$\begin{array}{l} A^T(\rho)P(\rho) + PA(\rho) + \frac{d}{dt}P(\rho) < 0, \\ P(\rho) > 0, \ \forall \rho \in \Omega, \ \forall \dot{\rho} \in \tilde{\Omega} \end{array} \tag{13}$$

by noticing that $\frac{d}{dt}P(\rho) = \frac{\partial}{\partial \rho} P \dot{\rho}$. Restricting $\dot{\rho} \in \tilde{\Omega}$ reduces the possibility of infinitely fast changes of ρ. Hence, by a parameterized quadratic Lyapunov function it is possible to enlarge the estimate of the region of attraction.

Since ρ is a function of x, it means that $\dot{\rho}$ is a function of $\dot{x} = f(x)$ according to $\dot{\rho} = \frac{\partial \rho}{\partial x} f(x)$. Hence, $\dot{\rho} \in \tilde{\Omega}$ adds additional bounds on the state variables x. These bounds must as well be considered in theorem 3.2, resulting in the following theorem,

Theorem 3.3 *Consider the nonlinear system (7), with the exact LPV description (8). Assume that the system (8) is LPV stable with the Lyapunov function $V(x, \rho) = x^T P(\rho) x$ for $\rho \in \Omega$ and $\dot{\rho} \in \tilde{\Omega}$. Define the set,*

$$\tilde{\mathcal{X}} = \{x \in D | \dot{\rho}(x) = \frac{\partial \rho}{\partial x} f(x) \in \tilde{\Omega}\}$$

Let $\mathcal{X}$ and Γ_β be defined according to (9) and (10) respectively. If $\Gamma_\beta \subseteq (\mathcal{X} \cap \tilde{\mathcal{X}})$ then $\Gamma_\beta \subseteq R_A$.

It should be pointed out that if $\tilde{\mathcal{X}} = D$, theorem 3.2 coincide with theorem 3.3. However, we think it is more pedagogical, and since it is common in the LPV synthesis literature, to separate the cases when P is and is not parameterized in ρ.

Defining a set for which the parameter ρ in the system (11) is allowed to evolve in,

$$\Omega = \{\rho \in \mathbb{R} | -1.5625 \leq \rho \leq 1.5625\}. \tag{14}$$

and restrict the parameters rate of change $\dot{\rho}$ according to the set,

$$\tilde{\Omega} = \{\dot{\rho} \in \mathbb{R} | -2.63 \leq \dot{\rho} \leq 2.63\}, \tag{15}$$

enable an LPV stability analysis test for (11), using a parameter dependent Lyapunov function (13). To perform this analysis as a parameterized LMI, one must first choose a structure of the parameter ρ dependence in $P(\rho)$. One common way is to mimic the parameter dependence of the system, in this case affine parameter dependence, in the $A(\rho)$ matrix. When such a structure is selected, the LPV stability analysis condition becomes a parameterized LMI. To obtain a finite dimensional LMI problem one can use relaxing strategies such as multi-convexity, see [1]. These types of relaxation methods introduce conservatism. A brute force gridding of the parameter space, and an evaluation of the parameterized version offer the least conservatism, but is computationally demanding. Using such a gridding technique on the set (14), an equidistant grid of 26 points, and optimizing over a $P(\rho)$ matrix with quadratic parameter dependence under the Self-Dual-Minimization package SeDuMi, [11], together with the SeDuMi interface, [6] resulted in the following,

$$\begin{aligned} P &= \begin{bmatrix} 0.4882 & -0.0836 \\ -0.0836 & 0.5118 \end{bmatrix} + \\ &+ \begin{bmatrix} 0.0629 & -0.0167 \\ -0.0167 & -0.0381 \end{bmatrix} \rho + \\ &+ \begin{bmatrix} 0.0163 & -0.0020 \\ -0.0020 & 0.0108 \end{bmatrix} \rho^2. \end{aligned} \tag{16}$$

Since the brute force griding technique does not guarantee that the parameterized LMI is satisfied for all parameters in the set, a post analysis of the result on a denser grid has been performed.

To analyze what estimate of region of attraction the Lyapunov function $V = x^T P(\rho(x,y))x$, with $P(\rho)$ in (16), results in for the nonlinear system (1), the validity region of V must be checked. The validity region of the Lyapunov function is given by $(\mathcal{X} \cap \tilde{\mathcal{X}})$. The set (14) together with (12) defines the unbounded set $\mathcal{X}$ in the x_1x_2-plane

$$|x_1x_2| \leq 1.5625.$$

The set (15) defines, according to (12), the set $\tilde{\mathcal{X}}$,

$$|\dot{\rho}| = |-x_2^2 + x_1(x_1 - 0.3(1 - x_1^2)x_2)| \leq 2.63.$$

The largest level set of V in the intersection of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ is the best estimate of R_A for (1) according to theorem 3.3. This estimate, see figure 5, is larger than in the case of a quadratic Lyapunov function, c.f. figure 3, despite that no optimality is sought between the range of ρ and the range of $\dot{\rho}$ in the LMI problem.

It is easy to increase this estimate even further, using for example a cubic parameter dependence in $P(\rho)$,

$$\begin{aligned} P &= \begin{bmatrix} 0.4860 & -0.0799 \\ -0.0799 & 0.5140 \end{bmatrix} + \\ &+ \begin{bmatrix} 0.0467 & 0.0074 \\ 0.0074 & -0.0481 \end{bmatrix} \rho + \\ &+ \begin{bmatrix} 0.0001 & 0.0016 \\ 0.0016 & 0.0025 \end{bmatrix} \rho^2 + \\ &+ \begin{bmatrix} -0.0049 & -0.0010 \\ -0.0010 & 0.0033 \end{bmatrix} \rho^3 \end{aligned} \tag{17}$$

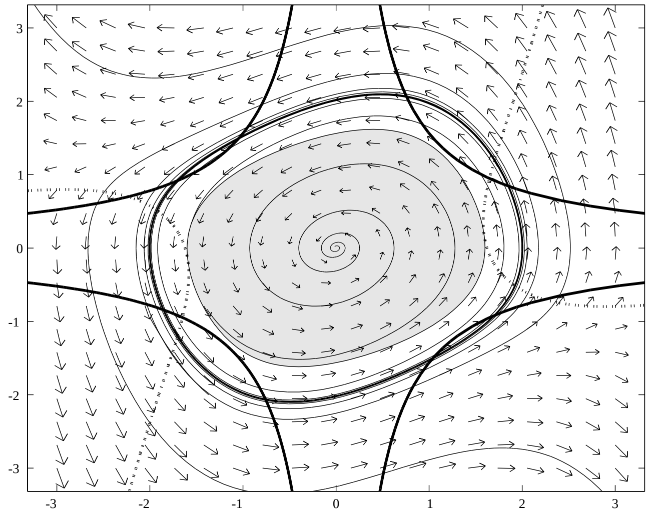

Figure 5: Region of attraction estimate (shaded area) in the region of $|\rho| \leq 1.5625$ (solid lines) and $|\dot{\rho}| \leq 2.63$ (dashed line) based on the parameter dependent Lyapunov LPV analysis.

which adds degrees of freedom to the parameterized LMI problem (13). This freedom can be used to increase the set for which we allow the parameter to vary in,

$$|\rho| = |x_1x_2| \leq 1.69.$$

and its rate of change,

$$|\dot{\rho}| = |-x_2^2 + x_1(x_1 - 0.3(1 - x_1^2)x_2)| \leq 3.$$

The resulting estimate of the region of attraction, see figure 6, is closer to the trajectories that describes the true region of attraction, and the level curve of the Lyapunov function is close to the shape of the boundary trajectory of the region of attraction.

It should be noted that both stability conditions (parameter independent and parameter dependent LPV stability) do guarantee local stability near the stationary point of the original nonlinear system. In the parameter independent case, closed level curves Γ_β can always be found due to positiveness of the quadratic Lyapunov function, as long as the set $\mathcal{X}$ contains the origin, which is natural since the origin is the stationary point of the nonlinear system. In the parameter dependent case, closed level curves can always be found if the origin is included in the set $\mathcal{X} \cap \tilde{\mathcal{X}}$. Since stationarity of the origin implies that $\dot{\rho} = 0$ according to

$$\dot{\rho} = \frac{\partial \rho}{\partial x} f(0) = 0,$$

it is also natural to say that the origin is included in $\tilde{\mathcal{X}}$. Hence, there will alway exist a non-empty estimate Γ_β of the region of attraction for the nonlinear system, based on the LPV analysis, for reasonable $\mathcal{X}$ and $\tilde{\mathcal{X}}$.

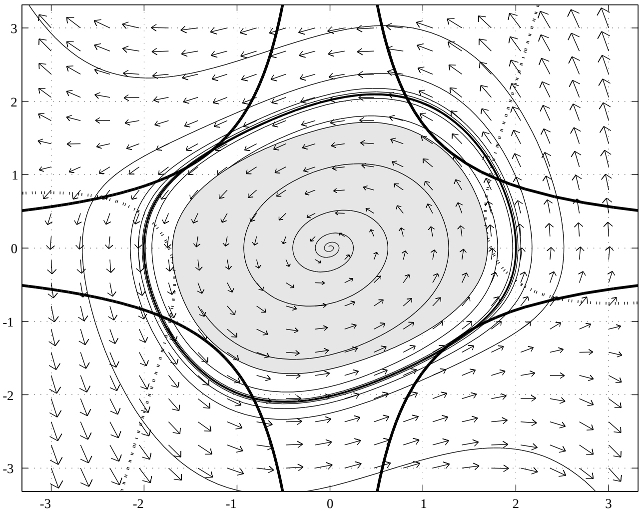

Figure 6: Region of attraction estimate (shaded area) based on the parameter dependent (cubic) Lyapunov LPV analysis. Dashed lines describe the area for which $|\dot{\rho}| \leq 3$ and thick lines for which $|\rho| \leq 1.69$

4 Conclusions

In LPV based gain scheduling controller synthesis, it is easy to believe that stability for the closed loop LPV system implies stability in the sense of region of attraction for the underlying nonlinear system. This is in general not true. However, an estimate of the region of attraction for the nonlinear system can be computed, based on the Lyapunov function obtained for the LPV system. These results are formalized in this paper by two theorems.

As this paper illustrates, nonlinear systems can be exactly included by many different LPV systems, by different choices of the (scheduling) parameter. It is also illustrated that this particular choice of parameter plays an important roll when estimating the region of attraction based on the LPV stability condition. This choice also affects the solvability of the LPV stability problem.

In many applications of nonlinear systems, there are also measurable input signals that can not be manipulated. Here, only autonomous nonlinear systems are considered. However, in such case of a measurable input signal, the parameter vector would incorporate this signal. The result of this paper would still hold in such a case, and the estimate of the region of attraction is not affected by the exogenous signal.

Ongoing work is to extend this result to input-output properties, such as induced $\mathcal{L}_2$-norm, of an LPV system and its relation to the underlying nonlinear system.

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