ON THE VALIDITY DOMAIN OF H_∞ CONTROLLERS UNDER SATURATION CONSTRAINTS

Gianni Bianchini[†], **Alberto Tesi**[‡]
[†]Dipartimento di Ingegneria dell'Informazione, Università di Siena,
Via Roma 56, 53100 Siena, Italy - giannibi@dii.unisi.it
[‡]Dipartimento di Sistemi e Informatica, Università di Firenze,
Via di S. Marta 3, 50139 Firenze, Italy - atesi@dsi.unifi.it

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Abstract

The H_∞ disturbance rejection problem for a family of linear systems subject to control input constraints is considered. A class of controllers generalizing the standard Riccati equation-based state feedback is proposed and an estimate of their domain of validity is derived. A simple criterion for tuning controller parameters for validity domain maximization and local performance improvement is presented.

1 Introduction

The problem of controller design for global/local stabilization with performance requirements in the presence of actuator saturation has been addressed by several authors. In [1], the issue of global stabilization with bounded controls is considered, and a design procedure based on passivity arguments is proposed; such design ensures global l_p boundedness of the noise-to-state map, although it does not allow for finite-gain requirements. In [2], global stabilization with finite noise-to-state l_p gain is achieved for systems with input-additive noise via gain scheduled feedback design, while in [3] the non input-additive case is addressed: it is shown that if the open-loop system is critically unstable, then stability with finite noise-state l_p gain is achievable for some p only in a semiglobal setting, while non finite-gain global stabilization is shown to be possible via gain scheduling. The mixed sensitivity problem with saturated control was addressed in [4]. In that paper, global stabilization with l_2 bounded map from noise to input-state pair is achieved. In this paper we investigate the properties of a class of H_∞ controllers for linear single-input systems subject to actuator saturation, with respect to the domain of validity. Our approach follows the spirit of [9],[10]. In those papers, exploiting the HJI approach to nonlinear H_∞ [6], the H_∞ mixed sensitivity problem is addressed for a family of single-input nonlinear systems, and a class of controllers depending on a scalar nonlinear function $m(\cdot)$ is derived. In [9], an estimate of the domain of validity of such controllers is given in closed form, while in [10] sufficient conditions in the shape of a geometric criterion are given for the existence of $m(\cdot)$ yielding a nonlinear controller ensuring the prescribed level of l_2 performance globally. In this paper we exploit a similar condition to investigate the domain of validity of a class of saturated H_∞ controllers for linear systems. The proposed controller class is a generalization of the classical linear state-feedback based on the algebraic Riccati equation approach. For a given controller, a closed form estimate of its domain of validity is derived. Moreover, the controller parameters can be tuned in order to achieve the maximization of the domain of validity within the controller class and to improve local closed loop performance.
The paper is organized as follows. In Section 2 we formulate the problem and give some preliminary results. Section 3 illustrates the main result, whose proof is reported in Section 4. An application example is presented in Section 5. Conclusions are drawn in Section 6.

Notation. $\mathbb{R}^n$: real n-space; $x \in \mathbb{R}^n$: vector of $\mathbb{R}^n$; $\mathbb{R}^{p\times n}$: real $p \times n$-space; $A \in \mathbb{R}^{p\times n}$: matrix of $\mathbb{R}^{p\times n}$; A^T: transpose of A; A^{-1}: inverse of A; $A > 0$ ($A \geq 0$): positive definite (semidefinite) matrix; $\langle \Psi, \Phi \rangle_R = \Psi^T R \Phi$: inner product of Ψ and Φ (with weighting matrix R).

2 Problem statement and preliminary results

Consider the following single-input multi-output system

$$\begin{cases} \dot{x} = Ax + B\,\mathrm{sat}(v) + Ed \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $v \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the exogenous (unknown) disturbance, $y \in \mathbb{R}^p$ is the system output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $E \in \mathbb{R}^n$, $G \in \mathbb{R}^n$, $C \in \mathbb{R}^{p\times n}$, and $\mathrm{sat}(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$\mathrm{sat}(v) = \frac{1}{2}\left(|v + v_{\mathrm{sat}}| - |v - v_{\mathrm{sat}}|\right) \quad ; \quad v_{\mathrm{sat}} > 0 \tag{2}$$

meaning that the control input saturates at a level $\pm v_{\mathrm{sat}}$.
Let us recall the solution of the standard linear state feedback H_∞ disturbance rejection problem for (1) neglecting the input saturation constraint (2) (i.e., setting $v_{\mathrm{sat}} = +\infty$).

Lemma 1 *Let γ be a positive scalar and suppose $P = P^T > 0$ is a solution of the Riccati equation*

$$A^T P + PA + P\left[\gamma^{-2}EE^T - BB^T\right]P + C^T C = -Q \tag{3}$$

for some $Q = Q^T > 0$. Then, the closed loop system with state feedback

$$v = -B^T P x \tag{4}$$

has l_2 gain from d to $[y \;\; v]'$ less or equal to γ.

We are interested in investigating the H_∞ disturbance rejection problem in the saturated input case ($v_{\rm sat} < +\infty$). Our controller synthesis procedure relies on computing the scalar control input v as a function of an auxiliary output variable. To this purpose, let P be a solution of (3) and introduce the scalar quantity

$$\xi = G^T x \tag{5}$$

where $G = PB$. Moreover, let the control input v be given by

$$v = u(\xi) \tag{6}$$

for some continuous function $u(\cdot)$ such that $u(0) = 0$, to be chosen by the designer. Note that the linear state feedback (4) with non-saturated input represents a special case of (6) when $u(\xi) = -\xi$.

Let $n(\xi)$ be the continuous scalar function given by

$$n(\xi) = u(\xi) - \text{ sat}[u(\xi)]. \tag{7}$$

Then, system (1) can be rewritten in the following nonlinear state space representation

$$\begin{cases} \dot{x} = Ax - Bn(\xi) + Bu + Ed \\ \xi = G^T x \\ y = Cx \end{cases} \tag{8}$$

This allows us to reformulate the H_∞ control problem on (1)-(2) as a standard nonlinear H_∞ problem on (8) where the scalar nonlinearity $n(\cdot)$ is defined by (7). Said another way, once a control law $u = u(\xi)$ has been designed for (8), the resulting dynamics, and hence the related H_∞ performance, is identical to that of system (1)-(2) where $v = u(\xi)$. In particular, since $u(0) = 0$, there exists some neighbourhood Ξ of $\xi = 0$ such that, for all $\xi \in \Xi$, $n(\xi) = 0$ and hence (8) is a linear system and its dynamics is exactly that of system (1) in the non-saturated input case. As a consequence, we have that

$$n(0) = 0 \quad ; \quad \frac{dn}{d\xi}(0) = 0, \tag{9}$$

and hence the origin is an equilibrium point of (8) and the function

$$k_n(\xi) = \frac{n(\xi)}{\xi}. \tag{10}$$

is well defined and such that $k_n(0) = 0$.

The above formulation allows us to exploit the standard HJI approach to nonlinear H_∞ [6] in order to tackle our problem. Such approach is based on the computation of a storage function associated with the origin of (8). The definition of storage function and its connection with state feedback H_∞ control are briefly recalled below.

Definition 1 Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth function such that $V(0) = 0$ and let γ be a positive scalar. Then, $V(x)$ is said to be a *storage function* if the Hamilton-Jacobi-Isaacs (HJI) inequality

$$\frac{\partial V}{\partial x}(x)\left[Ax - Bn(G^T x)\right] + \frac{1}{2}\frac{\partial V}{\partial x}(x)\left[\gamma^{-2}EE^T - BB^T\right]\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}x^T C^T C x \le 0 \tag{11}$$

holds in some neighbourhood of the origin. Moreover, the set W of all x satisfying (11) is said to be the *domain of validity* of V.

Once a storage function has been found, the following well-known result directly provides a state feedback H_∞ controller [6].

Theorem 1 *Consider system (8) and let $\gamma > 0$. Suppose there exists a storage function $V(x)$ and let W be its domain of validity. Then, the closed loop system with the feedback control law*

$$u = -B^T \frac{\partial^T V}{\partial x}(x)$$

has l_2-gain from d to $[y \;\; u]^T$ less than or equal to γ as long as its trajectories lie inside W.

Our aim is now to propose a feedback law of the form $u = u(\xi)$ in (8), and to show that such control law can be associated to a storage function with suitable domain of validity, according to Theorem 1.

Consider a scalar continuous function $m(\sigma) : \mathbb{R} \to \mathbb{R}$, which can be written in the gain form

$$m(\sigma) = k_m(\sigma)\sigma \tag{12}$$

where $k_m(\sigma) : \mathbb{R} \to \mathbb{R}$, and introduce the control law of the form

$$\begin{array}{rcl} u(\xi) \;=\; u_m(\xi) & = & -\xi - B^T P B m(\xi) \\ & = & -\left(1 - k_0^{-1} k_m(\xi)\right)\xi. \end{array} \tag{13}$$

where

$$k_0 = -\frac{1}{B^T P B} = -\frac{1}{G^T P^{-1} G}. \tag{14}$$

Note that expression (13) reduces to the standard linear H_∞ solution (4) for $k_m(\xi) = 0$. In accordance with the equivalence of systems (1) and (8) via (6), we can then view (13) as the H_∞ state feedback for non-saturated input, plus a perturbation depending on a scalar function $k_m(\cdot)$.

For $u(\xi)$ defined as in (13), the function $k_n(\xi)$ in (10) becomes

$$k_n(\xi) = -\left(1 - k_0^{-1} k_m(\xi)\right) - \frac{\text{sat}\left[-(1 - k_0^{-1} k_m(\xi))\xi\right]}{\xi}. \tag{15}$$

The key step now is to show that the control law (13) has an associated storage function of the form

$$V_m(x) = \frac{1}{2} x^T P x + \int_0^\xi m(\sigma) d\sigma \tag{16}$$

We have the following result.

Lemma 2 *Let $\gamma > 0$ and suppose $P = P^T > 0$ is a solution of the Riccati equation*

$$A^T P + PA + P\left[\gamma^{-2} EE^T - BB^T\right] P + C^T C = -Q \tag{17}$$

for some $Q = Q^T > 0$. Then, the following statements hold.

1. *$V_m(x)$ is a storage function and its domain of validity is given by*

$$W_m = \left\{x \in \mathbb{R}^n : x^T \bar{Q}(k_n(\xi), k_m(\xi))x \geq 0, V_m(x) \geq 0\right\}, \tag{18}$$

where $\bar{Q}(\kappa_n, \kappa_m)$ is a matrix depending on the two scalar parameters κ_n and κ_m as follows

$$\bar{Q}(\kappa_n, \kappa_m) = Q + \Upsilon(\kappa_n, \kappa_m)G^T + G\Upsilon^T(\kappa_n, \kappa_m), \tag{19}$$

being

$$\begin{aligned}&\Upsilon(\kappa_n, \kappa_m) = \\ &= \kappa_n PB + \kappa_m \left[-A^T G - \gamma^{-2} G^T EPE + G^T BPB\right] \\ &+ G^T B \kappa_n \kappa_m G + \frac{1}{2}\left((G^T B)^2 - \gamma^{-2}(G^T E)^2\right)\kappa_m^2 G.\end{aligned} \tag{20}$$

2. *The feedback controller*

$$u_m(\xi) = -\left(1 - k_0^{-1} k_m(\xi)\right)\xi$$

guarantees that the l_2-gain from d to $[y \quad u]^T$ is less or equal to γ within W_m.

Proof. Statement 1. follows by observing that the HJI inequality (11) reduces to the first inequality in (18) once $V(x) = V_m(x)$ and P is selected according to (3). Statement 2. is a direct consequence of Theorem 1 and (16). ■

Lemma 2 provides a class of controllers parametrized by the scalar function $k_m(\cdot)$, which guarantee the level γ of l_2-performance within the respective domains of validity W_m. Given the required level of l_2-performance, the main result in Section 3 provides a simple criterion for computing an estimate of the domain of validity of the H_∞ control law (13) under the saturation constraint (2) and shows how such estimate can be maximized by a suitable selection of the function $k_m(\cdot)$. Such criterion also provides an upper bound on the minimum level of l_2-performance that the closed loop system is guaranteed to have globally, i.e., in the whole state space.

3 Main result

We will now state the central result of the paper, which provides a simple geometrical criterion for evaluating the domain of validity of the control law (13) under the input saturation constraint (2).
Define the following constants

$$\begin{array}{rcl} h & = & \langle PB, PB\rangle_{Q^{-1}} \\ l & = & \langle K, K\rangle_{Q^{-1}} - (B^T PB)^2 + \gamma^{-2}(B^T PE)^2 \\ \bar{l} & = & \langle PB, K\rangle_{Q^{-1}} \end{array} \tag{21}$$

where the vector K is given by

$$K = -A^T PB - \gamma^{-2} B^T PE \cdot PE + B^T PB \cdot PB. \tag{22}$$

The above quantities depend only on system (1) and the solution P of the linear H_∞ problem (see Lemma 1). Define the matrix

$$T = \begin{bmatrix} 0 & hk_0^{-1} & -h \\ hk_0^{-1} & -(\bar{l}^2 - hl) & -\bar{l} \\ -h & -\bar{l} & -1 \end{bmatrix} \tag{23}$$

and consider the region Ω in the (κ_n, κ_m) plane defined as

$$\Omega = \left\{(\kappa_n, \kappa_m) : \quad \begin{bmatrix} \kappa_n & \kappa_m & 1 \end{bmatrix} T \begin{bmatrix} \kappa_n \\ \kappa_m \\ 1 \end{bmatrix} \leq 0\right\}. \tag{24}$$

From (23) and (24) it follows that Ω contains the origin and its boundary $\partial\Omega$ is described by a simple geometrical curve. Indeed, it is not difficult to show that such curve is always a hyperbole with an asymptote at $\kappa_m = k_0$. Introduce the half plane

$$\Pi = \{(\kappa_n, \kappa_m) : \quad \kappa_m > \kappa_0\} \tag{25}$$

and the intersection

$$\Omega' = \Omega \cap \Pi, \tag{26}$$

which is delimited in the (κ_n, κ_m) plane by the upper branch of $\partial\Omega$ and contains the origin.
Given a positive scalar $\bar{\xi}$, denote with $S_{\bar{\xi}}$ the following subset of the state space

$$S_{\bar{\xi}} = \left\{x \in \mathbb{R}^n \quad : \quad -\bar{\xi} \leq B^T Px \leq \bar{\xi}\right\}. \tag{27}$$

We have the following main result. The complete proof requires some preliminary technical lemmas and is deferred to Section 4 to improve readability.

Theorem 2 *Given P evaluated as in Lemma 1 for given γ, compute the region Ω' according to (23), (24), (25), and (26). Let $k_m(\xi)$ be a scalar continuous function and $\bar{\xi} > 0$. Consider the curve $\mathcal{C}_{\bar{\xi}}$ in the (κ_n, κ_m) plane described parametrically as*

$$\mathcal{C}_{\bar{\xi}} = \quad \begin{cases} \kappa_n = k_n(\xi) \\ \kappa_m = k_m(\xi) \end{cases} \quad -\bar{\xi} \leq \xi \leq \bar{\xi}, \tag{28}$$

with $k_n(\xi)$ as in (15) and suppose that

$$\mathcal{C}_{\bar{\xi}} \subset \Omega'. \tag{29}$$

Then, the domain of validity W_m of the controller

$$u_m(x) = -\left(1 - k_0^{-1} k_m(\xi)\right)\xi \tag{30}$$

is such that $S_{\bar{\xi}} \subseteq W_m$, i.e., $u_m(\xi)$ guarantees the level γ of l_2-performance within $S_{\bar{\xi}}$.

Remark 1 It can be shown (see [9]) that if the point $(k_n(\pm\bar{\xi}), k_m(\pm\bar{\xi}))$ of the (κ_n, κ_m) plane belongs to $\partial\Omega$, then the boundary ∂W_m of the domain of validity is tangent to the boundary of $S_{\bar{\xi}}$.

The above theorem provides estimates of the domain of validity of the control law (13) associated to a given choice of $k_m(\xi)$. The least conservative of such estimates is clearly associated to the maximal $\bar{\xi}$ for which (28)-(29) hold, i.e., for $(k_n(\bar{\xi}), k_m(\bar{\xi})) \in \partial\Omega$. It now makes sense to look for a constructive criterion for the existence of a scalar function $k_m(\xi)$ satisfying (28)-(29) for given $\bar{\xi}$. This would allow us to achieve the maximization (in the estimate sense) of the domain of validity of the proposed control law, once the values of the required l_2 gain γ and saturation level v_{sat} are given. Indeed, we have the following result.

Theorem 3 *There exists $k_m(\xi)$ such that the curve $\mathcal{C}_{\bar{\xi}}$ defined in (28) satisfies $\mathcal{C}_{\bar{\xi}} \subset \Omega'$ if, and only if, there exists a scalar $\bar{k}_m > k_0$ such that*

$$(\bar{k}_n, \bar{k}_m) \in \Omega'$$

where $\bar{k}_n = -(1 - k_0^{-1}\bar{k}_m) + v_{\text{sat}}\bar{\xi}^{-1}$.
Moreover, under the above condition, $k_m(\xi)$ can be given as

$$k_m(\xi) = \begin{cases} k_m^0 & 0 \le \xi \le \xi_{\text{sat}}^0 \\ \dfrac{(\bar{k}_m - k_m^0)(-1 + v_{\text{sat}}\xi^{-1}) + \bar{k}_n k_m^0}{\bar{k}_n - k_0^{-1}(\bar{k}_m - k_m^0)} & \xi > \xi_{\text{sat}}^0 \end{cases} \tag{31}$$

where

$$\xi_{\text{sat}}^0 = v_{\text{sat}}(1 - k_0^{-1}k_m^0)^{-1}. \tag{32}$$

and k_m^0 is a free parameter which must be chosen such that $(0, k_m^0) \in \Omega'$.

Proof. *[If]* We can suppose, without loss of generality, that $k_m(\xi)$ is a scalar function with even symmetry, i.e., $k_m(-\xi) = k_m(\xi)$. This allows us to restrict the support of the curve $\mathcal{C}_{\bar{\xi}}$ in (28) to $\xi \in [0, \bar{\xi}]$ without altering the condition of Theorem 2. We also note that in order for (29) to hold, it is required that $k_m(\xi) > k_0$. Then (15) implies the existence of $\xi_{\text{sat}} > 0$ such that $k_n(\xi) = 0$ for all $\xi \in [0, \xi_{\text{sat}}]$, and ξ_{sat} satisfies

$$\xi_{\text{sat}} = v_{\text{sat}}(1 - k_0^{-1}k_m(\xi_{\text{sat}}))^{-1}. \tag{33}$$

Clearly, ξ_{sat} is the value of ξ for which the control input reaches the saturation limit $u_m(\xi) = -v_{\text{sat}}$. Moreover, we have $k_n(\xi) \le 0$ for all $\xi \ge \xi_{\text{sat}}$ and $k_n(\xi) < 0$ for sufficiently large ξ. Then, (15) implies that for $\bar{\xi} \ge \xi_{\text{sat}}$ we have

$$k_n(\bar{\xi}) = -(1 - k_0^{-1}k_m(\bar{\xi})) + v_{\text{sat}}\bar{\xi}^{-1}. \tag{34}$$

It is easy to see from (24) that the curve $\partial\Omega$ in the (κ_n, κ_m) plane for $\kappa_m > k_0$ is a one-valued function $\kappa_n = f(\kappa_m)$ which does not switch convexity.
Since $(0, k_m^0) \in \Omega'$ and $(\bar{k}_n, \bar{k}_m) \in \Omega'$, the line segment $\mathcal{S}$ in the (κ_n, κ_m) plane

$$\mathcal{S} \ : \ \kappa_m = \bar{k}_m + \frac{(\bar{k}_m - k_m^0)}{\bar{k}_n}(\kappa_n - \bar{k}_n) \ ; \ \bar{k}_n \le \kappa_n \le 0 \tag{35}$$

is entirely contained in Ω'. Now, define $k_m(\xi)$ as in (31)-(32). Let $k_n(\xi)$ be defined accordingly as in (15) and consider the corresponding curve $\mathcal{C}_{\bar{\xi}}$ in (28). It is easily seen that $k_m(0) = k_m^0$, $k_m(\bar{\xi}) = \bar{k}_m$, $k_n(\bar{\xi}) = \bar{k}_n$ and moreover, by (34), $\mathcal{C}_{\bar{\xi}} \equiv \mathcal{S}$.
[Only if] If some $k_m(\xi)$ generates according to (28) a curve $\mathcal{C}_{\bar{\xi}}$ satisfying $\mathcal{C}_{\bar{\xi}} \subset \Omega'$, it must hold that $(\bar{k}_n, \bar{k}_m) \in \Omega'$. ■

Clearly, there always exists k_m^0 such that $(0, k_m^0) \in \Omega'$, since $(0, 0) \in \Omega'$. It thus becomes a question of enforcing the condition (a):$(\bar{k}_n, \bar{k}_m) \in \Omega'$ or, in order to compute the tightest estimate $S_{\bar{\xi}}$ (see Remark 1), the condition (b):$(\bar{k}_n, \bar{k}_m) \in \partial\Omega$. This amounts to looking for a pair of scalars $(\bar{k}_m, \bar{\xi})$ satisfying, respectively

$$\begin{aligned} &\text{(a)} \quad \begin{cases} \bar{k}_n = -(1 - k_0^{-1}\bar{k}_m) + v_{\text{sat}}\bar{\xi}^{-1} \ge f(\bar{k}_m) \\ \bar{k}_m > k_0 \\ \bar{k}_n < 0 \end{cases} \quad \text{or} \\ &\text{(b)} \quad \begin{cases} \bar{k}_n = -(1 - k_0^{-1}\bar{k}_m) + v_{\text{sat}}\bar{\xi}^{-1} = f(\bar{k}_m) \\ \bar{k}_m > k_0 \\ \bar{k}_n < 0 \end{cases} \end{aligned} \tag{36}$$

Once this is accomplished, the control law (13) with $k_m(\xi)$ as in (31) guarantees the prescribed level γ of l_2-performance within a domain of validity W_m an estimate of which is given by $S_{\bar{\xi}}$ in (27).

Remark 2 Note that $\bar{\xi}$ solving (36) for $\bar{k}_m = 0$ yields estimates of the domain of validity of the standard linear H_∞ solution $u = -B^T Px$.

Remark 3 In solving (36) (b), the value of $\bar{k}_m$ can be chosen in general so as to maximize the corresponding $\bar{\xi}$.

Remark 4 If for some $\bar{k}_m$ condition (36) (a) is found to hold for all $\bar{\xi} \in [0, +\infty)$, then the control law (13) with $k_m(\xi)$ as in (31) ensures the level γ of l_2-performance globally. Unfortunately, there is no evidence that the open-loop l_2 gain γ_{ol} can be improved globally under the saturation constraint by means of the proposed design. On the other hand, condition (36) (a) can be exploited in order to compute an upper bound $\bar{\gamma} \ge \gamma_{\text{ol}}$ on the l_2 gain that the closed loop system is guaranteed to exhibit globally once a level of l_2-performance $\gamma < \gamma_{\text{ol}}$ is enforced locally. To this purpose, it is enough to verify (36) (a) for all $\bar{\xi} \in [0, +\infty)$ (i.e., setting $v_{\text{sat}} = 0$) where $\bar{k}_m$ is assigned the previously designed value and Ω (i.e., $f(\cdot)$) is computed for γ equal to the testing value $\bar{\gamma}$.

Remark 5 The free parameter k_m^0, which has to be chosen such that $(0, k_m^0) \in \Omega'$ in order to ensure the prescribed l_2-performance, influences the feedback gain when the system operates in the linear region. Therefore, such parameter can be tuned in order to improve the transient response properties in that region.

4 Proof of Theorem 2

Our first goal is to characterize the geometrical shape of the region in the (κ_n, κ_m) plane where the matrix $\bar{Q}(\kappa_n, \kappa_m)$ in (19), which plays a key role in the definition of W_m (see (18)), is positive semidefinite. To this purpose, we need the following auxiliary result, see [10] for the proof.

Lemma 3 *Consider the one-parameter family of $n \times n$ matrices*

$$\bar{R}(\kappa) = R + \kappa(\Psi\Phi^T + \Phi\Psi^T) \tag{37}$$

where $R \in \mathbb{R}^{n\times n}$, $R = R^T > 0$, $\Phi \in \mathbb{R}^n$, $\Psi \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. Then, $\bar{R}(\kappa) \ge 0$ if and only if

$$1 + 2\langle\Psi, \Phi\rangle_{R^{-1}}\,\kappa + \left(\langle\Psi, \Phi\rangle_{R^{-1}}^2 - \langle\Psi, \Psi\rangle_{R^{-1}}\langle\Phi, \Phi\rangle_{R^{-1}}\right)\kappa^2 \ge 0. \tag{38}$$

Exploiting the above lemma, we have a first result concerning $\bar{Q}(\kappa_n, \kappa_m)$.

Lemma 4 *The following conditions are equivalent*

1. $\bar{Q}(\kappa_n, \kappa_m) \geq 0$
2. $\langle G, G\rangle_{Q^{-1}} \langle \Upsilon(\kappa_n, \kappa_m), \Upsilon(\kappa_n, \kappa_m)\rangle_{Q^{-1}} - \left(1 + \langle \Upsilon(\kappa_n, \kappa_m), G\rangle_{Q^{-1}}\right)^2 \leq 0$

Proof. It is easily verified that $\bar{Q}(\kappa_n, \kappa_m)$ in (19) has exactly the form in (37) once $R = Q$, $\Psi = \Upsilon(\kappa_n, \kappa_m)$, $\Phi = G$, and $\kappa = 1$. Hence, the proof follows by observing that in this case (38) reduces to (4). ■

Lemma 4 provides a convenient equivalent expression for $\bar{Q}(\kappa_n, \kappa_m) \geq 0$. The next step is to show that (4) defines a very simple geometrical constraint on the parameters κ_n and κ_m.

Indeed, exploiting the expression (20) of $\Upsilon(\kappa_n, \kappa_m)$ and after some straightforward though tedious manipulations, it turns out that condition (4) simplifies to the quadratic form

$$-(\bar{l}^2 - hl)\kappa_m^2 + 2hk_0^{-1}\kappa_n\kappa_m - 2h\kappa_n - 2\bar{l}\kappa_m - 1 \leq 0, \quad (39)$$

which is equivalent to

$$\left[\begin{array}{ccc} \kappa_n & \kappa_m & 1 \end{array}\right] T \left[\begin{array}{c} \kappa_n \\ \kappa_m \\ 1 \end{array}\right] \leq 0, \quad (40)$$

with T as in (23).

The next result follows directly from Lemma 4, the definition of Ω in (24), and the equivalence of (4) and (40).

Lemma 5 *The following conditions are equivalent:*

$$1. \quad \bar{Q}(\kappa_n, \kappa_m) \geq 0; \quad (41)$$

$$2. \quad (\kappa_n, \kappa_m) \in \Omega. \quad (42)$$

Let us now characterize the condition $V_m(x) \geq 0$ in the definition of W_m (18).

Given a scalar $\bar{\xi} > 0$, consider the region $S_{\bar{\xi}}$ of the state space (27). We have the following result.

Lemma 6 *Given $\bar{\xi} > 0$, if $(k_n(\xi), k_m(\xi)) \in \Pi$ for $-\bar{\xi} \leq \xi \leq \bar{\xi}$, then $V_m(x) \geq 0$ for all $x \in S_{\bar{\xi}}$.*

Proof. Let $k_m(\xi) = -\frac{1}{G^TP^{-1}G} + \epsilon_m(\xi)$ where $\epsilon_m(\xi) > 0$ for all $-\bar{\xi} \leq \xi \leq \bar{\xi}$. We have

$$\begin{aligned} V_m(x) &= \frac{1}{2}x^TPx + \int_0^{G^Tx} m(\sigma)d\sigma = \frac{1}{2}x^TPx + \int_0^{G^Tx} \sigma k_m(\sigma)d\sigma \\ &= \frac{1}{2}x^TPx - \int_0^{G^Tx} \frac{\sigma}{G^TP^{-1}G}d\sigma + \int_0^{G^Tx} \sigma\epsilon_m(\sigma)d\sigma \\ &\geq \frac{1}{2}x^TPx - \frac{1}{2}\frac{(G^Tx)^2}{G^TP^{-1}G} = \frac{1}{2}x^T\left(P - \frac{GG^T}{G^TP^{-1}G}\right)x \end{aligned}$$

To show that $V_m(x) \geq 0\ \forall x \in S_{\bar{\xi}}$ it suffices to prove that the matrix $P - \frac{GG^T}{G^TP^{-1}G}$ is positive semidefinite. From a standard determinantal result we have

$$\det\left(P - \kappa\frac{GG^T}{G^TP^{-1}G}\right) = (1-\kappa)\det(P).$$

Hence, since $P > 0$, the above expression implies that $P - \frac{GG^T}{G^TP^{-1}G} \geq 0$. ■

To complete the proof of Theorem 2, we observe that if $\mathcal{C}_{\bar{\xi}} \subset \Pi$, then Lemma 6 guarantees that the second inequality in (18) holds for all $x \in S_{\bar{\xi}}$. Moreover, if $\mathcal{C}_{\bar{\xi}} \subset \Omega$, then Lemma 5 ensures that $\bar{Q}(k_n(G^Tx), k_m(G^Tx)) \geq 0$ for all $x \in S_{\bar{\xi}}$ and therefore also the first inequality in (18) holds in $S_{\bar{\xi}}$. ■

5 Example

Consider system (1) where

$$A = \left[\begin{array}{cc} 0 & 1 \\ -0.2 & -2 \end{array}\right], B = \left[\begin{array}{c} 1 \\ -1 \end{array}\right], E = \left[\begin{array}{c} 1 \\ 0 \end{array}\right], C = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]'$$

The open loop is shown to have l_2 gain from d to y equal to $\gamma_{\rm ol} = 10$. Let us consider the H_∞ disturbance attenuation problem

$$\|[y\ \ u]'\|_2 \leq \gamma\|d\|_2$$

with $\gamma = 5$. Solving the Riccati equation (3) for

$$Q = \left[\begin{array}{cc} 5 & 0 \\ 0 & 5 \end{array}\right]$$

yields

$$P = \left[\begin{array}{cc} 4.98 & 2.54 \\ 2.54 & 2.58 \end{array}\right]$$

and the corresponding feedback controller (4) which solves the problem for the non-saturated input case ($v_{\rm sat} = \infty$). Let us enforce the constraint $v_{\rm sat} = 1$. The values of matrix T in (23) and k_0 in (14) turn out to be

$$T = \left[\begin{array}{ccc} 0 & -2.97 & -1.19 \\ -2.97 & -5.25 & -2.75 \\ -1.19 & -2.75 & -1 \end{array}\right] \ ; \ k_0 = -0.4.$$

To provide estimates of the validity domain of the control law (13) for $k_m(\xi)$ as in (31) we need to find $(\bar{k}_m, \bar{\xi})$ according to (36)(b). In order to select $\bar{k}_m$ such that the control law yields the largest validity domain estimate, we find it convenient (see Remark 3) to compute, from the first equation in (36)(b), the value of $\bar{\xi}$ as a function of $\bar{k}_m$. In Figure 1, the solid line represents $\bar{\xi}(\bar{k}_m)$, while the dashed line represents the value $\bar{\xi}_{\rm sat}(\bar{k}_m)$ of ξ at which the control input enters saturation when choosing $k_m(\xi) \equiv \bar{k}_m$, computed according to (33). Choosing $\bar{k}_m$ such that $\bar{\xi}(\bar{k}_m) < \bar{\xi}_{\rm sat}(\bar{k}_m)$ does not yield any significant estimate, since the control input (13) with $k_m(\xi)$ as in (31) does not even saturate for $0 \leq \bar{\xi}(\bar{k}_m)$ (and (36)(b) does not hold since $\bar{k}_n \geq 0$). It turns out that the largest validity domain estimate $S_{\bar{\xi}}$ is achieved with the pair $(\bar{k}_m = \bar{k}_m^{\max} = -0.2, \bar{\xi} = \bar{\xi}_{\max} = 2.42)$. The validity domain estimate for the feedback controller (4) (the standard linear H_∞) is computed by taking $\bar{k}_m = 0$ yielding $\bar{\xi} = \bar{\xi}(0) = 1.72$.

According to Remark 5, the additional free parameter k_m^0 can be chosen in order to obtain an improvement of the transient response properties. In Figure 2, the transient response from the initial condition $x_0 = [1\ \ 1]'$ (which lies close to the boundary of $S_{\bar{\xi}}$) is shown for $k_m^0 = 0.2, \bar{k}_m = -0.2$ ($y(t)$) and for $k_m^0 = \bar{k}_m = -0.2$ ($y'(t)$)). Note that the expression of $u_m(\xi)$ (13) for $k_m^0 \neq \bar{k}_m$ turns out to be nonlinear while for $k_m^0 = \bar{k}_m$ it is linear (see (31)). In this case, the first control law outperforms the second one as far as the transient behaviour is concerned, while maintaining the same l_2 gain within the region $S_{\bar{\xi}}$ as it is clear from the inspection of Figure 3. Indeed, both curves $\mathcal{C}$ ($k_m^0 \neq \bar{k}_m$, nonlinear $u_m(\xi)$) and $\mathcal{C}'$ ($k_m^0 = \bar{k}_m$, linear $u_m(\xi)$) lie inside the region Ω' for $-\bar{\xi} \leq \xi \leq \bar{\xi}$ as expected.

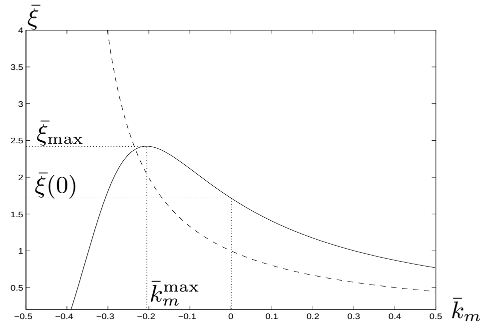

Figure 1: $\bar{\xi}(\bar{k}_m)$ (solid line), $\bar{\xi}_{\rm sat}(\bar{k}_m)$ (dashed line)

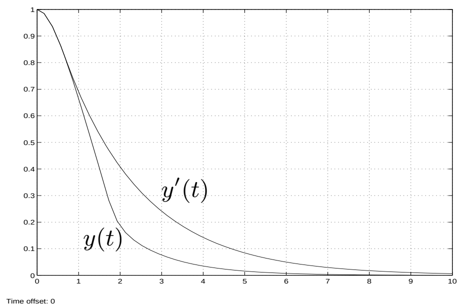

Figure 2: Transient responses: $y(t)$ ($k_m^0 \neq \bar{k}_m$, nonlinear $u_m(\xi)$) and $y'(t)$ ($k_m^0 = \bar{k}_m$, linear $u_m(\xi)$)

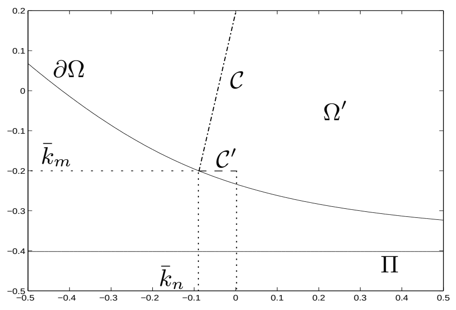

Figure 3: Curves $\mathcal{C}$ ($k_m^0 \neq \bar{k}_m$, nonlinear $u_m(\xi)$) and $\mathcal{C}'$ ($k_m^0 = \bar{k}_m$, linear $u_m(\xi)$)

6 Conclusion

In this paper we presented an approach to H_∞ controller design for a family of linear plants subject to input saturation. An analysis of the properties of the proposed controller class was performed with respect to their domain of validity. A method for computing controller parameters in order to maximize the domain of validity as well as to improve local closed loop performance was derived. Finally, an application example was presented to illustrate the proposed results.

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