OPTIMAL QUANTIZATION OF SIGNALS FOR SYSTEM IDENTIFICATION

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Abstract

In this paper, we analyse the effect of the quantization of signals used for system identification and show an optimal quantization scheme for minimizing estimation errors under a constraint on the number of subsections of the quantized signals. The optimal quantization scheme has the property that it is coarse near the origin and dense at a distance from it in the definition area of the signals. We also evaluate the estimated parameters and show a trade-off between the quantization error and the noise error under the constraint on the amount of information in the output data.

1 Introduction

The problem of quantization of signals in control systems has a long history, and in recent years this problem has again been discussed actively by several research groups and interesting results have been achieved. Such research activity is certainly related to the recent rapid improvement in the transmission capacity of computer networks. Long-distance automatic control is now realistic with high-speed networks, and the necessity of understanding the effects of transmission limitations on information in control systems has become more widely accepted.

The history of research on the quantization problem in control theory may be summarized as follows. We can see this problem in the books of the 70s (e.g., [2]); however, the approach is elementary and the quantization error is regarded as noise. The turning point is the result by [3], [4], in whose papers the behaviour of control systems, and their stability or state estimation, are analysed in detail. In the last few years, stabilization problems of quantized systems have been actively considered, e.g., [11], [12], [1], [9], [5]. In particular, we pick up the results by Elia & Mitter [5] because their results are related to those in this paper. They considered a form of stabilization of MIMO systems and showed: 1) the coarsest quantization scheme satisfying the stability, 2) an exponential growth rate of the size of the quantization sections: it is dense near the origin and becomes coarse at a distance from it in the space of signals that are quantized, 3) the growth rate can be represented by the poles of the plants.

Compared to this activity in the stabilization or estimation problem, the quantization problem for system identification [6] has not been adequately considered. When a controlled plant with networks is unknown or its system parameters may change during the operation, we need a form of adaptation for the control system. It is also necessary to know the effect of quantization of the I/O data used for the system identification. From this point of view, we consider this problem and give an optimal quantization scheme for minimizing estimation errors under a constraint on the number of levels of the quantized signals. The optimal quantization has a type of dual property to the case of stabilization by [5], that is, the quantization is coarse near the origin of the signals and it is dense at a distance from it. In this paper, we deal with this problem based on the most simple setting and an idealized situation for system identification. The reason is to reveal the essential property of the optimal quantization problem in system identification and assist intuitive understanding of it.

In this paper, we omit the proofs of the lemmas and theorems. Refer the full paper version [7] of this paper for the details.

2 Formulation

Dealing with the general case for the strict treatment of the effect of signal quantization on system identification may result in a discussion with excessive computation and a complexity that hinders the intuitive understanding of the results. In order to reduce such difficulty, there are two possible approaches. The first is to simplify the treatment of the quantization, such as noise, in the classical manner. This approach is effective for the case of sufficiently precise quantization of I/O data, and the result is similar to the usual case of analogue signals. However, this approach cannot reveal the property that is unique to the quantization problem in system identification. The second approach is to consider simple models for system identification, and, in contrast, the effect of quantization is examined strictly. This approach has the disadvantage that it is applicable only for idealized situations. However, it has the potential to reveal the essential property of the quantization in system identification and this is the main purpose of this research.

From the above discussion, in this paper we consider the following scalar systems based on the most simple case of system identification of an MA (moving average) model by the least squares method:

$$y_o(i) = q(y(i)) + w(i) \tag{1}$$
$$y(i) = \phi(i)\theta, \tag{2}$$

where

$$\phi(i) := [\, u(i) \quad u(i-1) \quad \cdots \quad u(i-n+1)\,]$$
$$\theta := [\, \theta_1 \quad \theta_2 \quad \cdots \quad \theta_n\,]^{\mathrm{T}},$$

w is a noise and q is a quantizer of the original analogue output y. Assume $|y| \le \kappa$, and q is defined by

$$q(y) := \mathrm{sgn}(y)\bar{y}_j,\ y \in S_j,\ \bar{y}_j \ge 0, \tag{3}$$

where

$$\begin{aligned} &S_0 := \{y = 0\} \\ &S_j := \{y : d_{j-1} < y \le d_j\},\ j > 0 \\ &S_j := \{y : d_{j-1} \le y < d_j\},\ j < 0 \end{aligned} \tag{4}$$
$$\begin{aligned} &d_0 = 0 < d_1 < d_2 \cdots < d_M = \kappa, \\ &d_{-1} = -d_1,\ d_{-2} = -d_2, \\ &\qquad \ldots,\ d_{-M} = -d_M = -\kappa, \end{aligned} \tag{5}$$

and $\mathrm{sgn}(y)\bar{y}_j$ is the assigned quantized value to the subsection S_j. The quantizer is symmetrical with respect to the origin, and hereafter we may omit references on the negative section if they are obvious from the context.

The estimated parameter $\hat{\theta}$ using the least squares method with I/O data $u(i)$ and $y_o(i)$ is given by

$$\hat{\theta} = (U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}\left(\tilde{Y} + W\right), \tag{6}$$

where

$$\begin{aligned} &U := [\, \phi(1)^{\mathrm{T}} \quad \phi(2)^{\mathrm{T}} \quad \cdots \quad \phi(N)^{\mathrm{T}}\,]^{\mathrm{T}} \\ &W := [\, w(1) \quad w(2) \quad \cdots \quad w(N)\,]^{\mathrm{T}} \\ &\tilde{Y} := [\, \tilde{y}(1) \quad \tilde{y}(2) \quad \cdots \quad \tilde{y}(N)\,]^{\mathrm{T}} \\ &\tilde{y}(i) := q(y(i)). \end{aligned} \tag{7}$$

Define the quantization error between $\tilde{y}$ and y by

$$e(i) := \tilde{y}(i) - y(i), \tag{8}$$

and the estimated parameter $\hat{\theta}$ can be written as

$$\begin{aligned} \hat{\theta} &= (U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}(U\theta + E + W) \\ &= \theta + \Delta E + \Delta W, \end{aligned} \tag{9}$$

where

$$\begin{aligned} E &:= [\, e(1) \quad e(2) \quad \cdots \quad e(N)\,]^{\mathrm{T}} \\ \Delta E &:= (U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}E \\ \Delta W &:= (U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}W. \end{aligned} \tag{10}$$

This shows that the estimation error $\hat{\theta} - \theta$ can be evaluated from the magnitudes of ΔE and ΔW. The conventional, and reasonable, method to evaluate ΔW is to show the convergence rates of

$$\begin{aligned} &N(U^{\mathrm{T}}U)^{-1} \overset{N\to\infty}{\longrightarrow} \frac{1}{\sigma^2}I \\ &\frac{1}{N}U^{\mathrm{T}}W \overset{N\to\infty}{\longrightarrow} O \end{aligned} \tag{11}$$

by using the mutual independence of the input signal u and the noise w. This methodology is also basically applicable to the case of ΔE, however, we should note that u and e are not independent in general, and the situation is much more complicated. In order to see this and demonstrate the basic property of the optimal quantization we use the following example as an illustration. For example, a row of $U^{\mathrm{T}}E$ is given by the form:

$$\sum_{i=1}^{N} u(i)e(i). \tag{12}$$

Approximately, the magnitude of e is given as

$$|e| \le \max\{y - d_{j-1}, d_j - y\},\ d_{j-1} < y \le d_j,\ j > 0, \tag{13}$$

and the right hand side of the inequality (13) depends on the setting of the subsection S_j. The sum of the widths of all the subsections S_{-M}, ... , S_M is constant, and therefore, if we set the width of some subsection to be small in order to reduce the quantization error, it causes the widths of the other sections to be large, and the total quantization error may increase as a result. In order to reduce the total quantization errors, it is known from (12) that q should have the following property: the magnitude of $|e(i)|$, which is multiplied by a large $u(i)$, should be small, and conversely, the magnitude of $|e(i)|$, which is multiplied by a small $u(i)$, should be large. This is a basic property of the optimal quantization scheme for reducing the magnitude of ΔE, and in the following we analyse the expectation and the variance of (12).

First, we define subsets Φ_j of the regression vector ϕ associated with the subsection S_j by

$$\Phi_j := \{\phi : y = \phi\theta \in S_j\}. \tag{14}$$

Next, we consider a variable transformation:

$$y \;=\; \phi\theta = \phi T \cdot T^{-1}\theta =: \phi'\theta',\ T^{-1}\theta = \begin{bmatrix} 1 \\ O \end{bmatrix}, \tag{15}$$

then, Φ_j is represented as

$$\Phi_j = \{\phi : {\phi'}_1 \in (d_{j-1},\ d_j]\},\ j > 0. \tag{16}$$

Corresponding to this transformation, the estimated parameter $\hat{\theta}$ is also transformed as follows:

$$\begin{aligned} \hat{\theta}' &:= T^{-1}\hat{\theta} \\ &= T^{-1}(U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}(U\theta + E + W) \\ &= (T^{\mathrm{T}}U^{\mathrm{T}}UT)^{-1}T^{\mathrm{T}}U^{\mathrm{T}}(UT \cdot T^{-1}\theta + E + W) \\ &= \theta' + \Delta E' + \Delta W' \\ \Delta E' &:= (U'^{\mathrm{T}}U')^{-1}U'^{\mathrm{T}}E, \\ \Delta W' &:= (U'^{\mathrm{T}}U')^{-1}U'^{\mathrm{T}}W,\ U' := UT. \end{aligned} \tag{17}$$

Note 2.1 In the remainder of this paper, we assume that the transformation matrix T is given. Of course, this assumption is a contradiction in the system identification problem because T is defined from the true θ. However, the main purpose of this paper is not to propose a concrete system identification method, but to show the essential property of the optimal signal quantization, and therefore, we consider this ideal situation. Moreover, if necessary for dealing with realistic problems, we can use an approximated transformation matrix from non-optimal quantized data and revise it by iterative experiments.

Next, define

$$I_j := \{\phi'_1 : \phi'_1 \in (d_{j-1},\ d_j]\}\ ,\ j > 0, \tag{18}$$

then, the subsections S_j, Φ_j, and I_j correspond to each other, and the probability distribution of y depends only on that of ϕ'_1. Therefore, in order to analyse the probability distribution of y, the variable ϕ'_1 and its subsection I_j are convenient to deal with, and hereafter, we mainly discuss the problem using them. We assume a probability distribution of ϕ'_1 as follows.

Assumption 2.1 *ϕ'_1 obeys a uniform distribution in $[-\kappa,\ \kappa]$.*

We denote the probability distribution of Assumption 2.1 by $F(\phi'_1)$.

The expectation of $\Delta E'$ should be zero,

$$\mathrm{E}(u' \cdot e) = \sum_{j=-M}^{M} \left(\int_{I_j} u' \cdot e\, dF \right) =: \sum_{j=-M}^{M} \mathrm{E}_{I_j}(u' \cdot e) = 0.$$

With this in mind, we consider the next optimal quantization problem for the signals for system identification.

Problem 2.1 *For the system (2) with Assumption 2.1, give a quantizer q that minimizes the variance of (12) such that $\mathrm{E}_{I_j}(\Delta E') = 0\ (\forall j)$ under the constraint of quantization number of $[-\kappa,\ \kappa]$.*

In the following we may omit the symbol "prime", such as:

$$\theta' \to \theta,\ \hat{\theta}' \to \hat{\theta},\ u' \to u$$
$$U' \to U,\ \Delta E' \to \Delta E,\ \Delta W' \to \Delta W,$$

in order to simplify representations except for the case of necessary specifications.

3 Optimal Quantization

As described in Section 2, the quantization scheme of $[-\kappa,\ \kappa]$ on y is essentially equal to that on ϕ'_1 and it is composed of the setting of the subsections $I_{-M}, \ldots, I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots, I_M$, and the quantized values

$$\begin{aligned} & q(y),\ y \in S_j \\ &\quad = \ q(\phi'_1),\ \phi'_1 \in I_j \\ &\quad = \ \tilde{y}_j \end{aligned} \tag{19}$$

for each subsection I_j. We consider I_j $(= (d_{j-1},\ d_j])$, I_{j+1} $(= (d_j,\ d_{j+1}])$ where their boundaries d_j, d_{j+1} have a relation (see Fig. 1):

$$d_j = r_j d_{j+1},\ r_j \in [0, 1]. \tag{20}$$

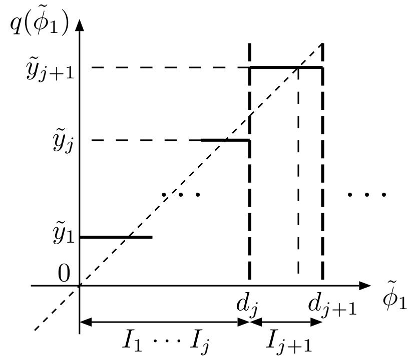

Fig. 1 The quantization scheme of q

We obtain the following result.

Proposition 3.1 *The optimal ratios r_j that minimize the sum of the variances of $I_1, I_2, \ldots, I_M$, and also $I_{-1}, I_{-2}, \ldots, I_{-M}$, are given by solving the following optimization problem iteratively.*

$$\begin{aligned} r_j &:= \arg\min_{r \in [0,\,1]} f_j(r) && (21) \\ f_j(r) &:= f_{j-1}^{\min} r^5 - 18(1-r)^5 + 45(1+r)^2(1-r)^3 \\ &\quad + 5(1-r)^7(1+r)^{-2} && (22) \\ f_j^{\min} &:= f_j(r_j), \quad f_0^{\min} = 32 && (23) \end{aligned}$$

The optimal value of the variance is given by

$$V_M(u \cdot e) := \sum_{j=-M}^{M} V_{I_j} = \frac{1}{2160} \kappa^4 f_M^{\min}. \tag{24}$$

We call this optimal quantization scheme $\mathrm{Q_{opt}}$.

Every ratio r_j can be explicitly given by (21) $\sim$ (23) iteratively, however, understanding the properties of r_j is not straightforward from (21) $\sim$ (23) directly. In this paper, we show the asymptotic characteristics of the optimal ratios r_j $(j = 1, 2, \ldots)$ and related quantities. We can derive the following series of lemmas.

Lemma 3.1

$$r_j < r_{j+1},\ \forall j > 0 \tag{25}$$
$$r_j \to 1,\ j \to \infty \tag{26}$$

Lemma 3.2

$$|I_j| > |I_{j+1}|,\ \forall j > 0 \tag{27}$$

Lemma 3.2 shows that the optimal quantization scheme $\mathrm{Q_{opt}}$ has the property that it is coarse near the origin of y and becomes dense near the boundaries of $[-\kappa,\ \kappa]$. This property is, in some sense, a dual to the result of the quantization problem for stabilization by [5], that is, the coarsest quantization scheme for stabilization is dense around the origin and becomes coarse at a distance from the origin.

Next, consider the unboundedness of $\prod_{j=1}^{\infty} \frac{1}{r_j}$. If it is bounded and $\prod_{j=1}^{\infty} \frac{1}{r_j} = \gamma < \infty$, then this causes a contradiction of the optimality of $\mathrm{Q_{opt}}$, that is, when a region $[-\gamma,\ \gamma]$ of $\tilde{\phi}_1$ is quantized, the width of I_1, for example, is never smaller than one even if the number of quantization levels increases to infinity. Of course, this is not true and $\prod_{j=1}^{\infty} \frac{1}{r_j}$ is therefore unbounded. The next lemma gives a strict proof of this.

Lemma 3.3

$$\prod_{j=1}^{\infty} \frac{1}{r_j} = \infty \tag{28}$$

From Lemma 3.1 to Lemma 3.3, we know the outline of the quantization of the region $[-\kappa,\ \kappa]$. Next, consider the evaluation of the magnitude of $\Delta E'$ with respect to the number of quantization levels M, and the following lemma shows an asymptotic characteristics of $f_M^{\min}$.

Lemma 3.4

$$f_M^{\min} \to \Psi_a^b(M),\ M \to \infty \tag{29}$$

where $a = -5 \cdot 3^{-\frac{5}{2}}$ *and* $b = \frac{3}{2}$*, and* $\Psi_a^b(m)$ *is a function of* m *defined as the solution of the following recurrence formula with an appropriate initial number* $\psi(0) = K$.

$$\psi(m) - \psi(m-1) = a\psi^b(m-1) \tag{30}$$

By approximating the difference equation (30) with a differential equation

$$\frac{d\tilde{\psi}(m)}{dm} = (a+\nu)\tilde{\psi}^b(m) \geq a\tilde{\psi}^b(m) + o(\tilde{\psi}^b(m)), \tag{31}$$

where $\nu > 0$ is an appropriate constant number, then, we obtain

$$\tilde{\psi}(m) = \{(-b+1)(a+\nu)m\}^{\frac{1}{-b+1}}. \tag{32}$$

From (24) and the convexity of the function (32), the variance $V_M(u \cdot e)$ at sufficiently large M satisfies

$$\begin{aligned} V_M(u \cdot e) &\leq \frac{1}{2160}\kappa^4 \Big((-3/2+1) \\ &\qquad \times (-5 \cdot 3^{-\frac{5}{2}} + \nu)(M-1)\Big)^{\frac{1}{-3/2+1}} \\ &= A\kappa^4 (M-1)^{-2}, \\ A &:= \frac{1}{2160}\left(5 \cdot 2^{-1} \cdot 3^{-\frac{5}{2}} - 2^{-1}\nu\right)^{-2}. \end{aligned} \tag{33}$$

(33) shows a relation between the optimal variance and the number of quantization levels. In the following section this result is used to evaluate the magnitude of ΔE.

4 Evaluation of the Error Terms

Using the results in the previous section, we evaluate the magnitude of the error term ΔE based on the approach in [10]. First, we evaluate the magnitude of $(U^{\mathrm{T}}U)^{-1}$.

Lemma 4.1 [10] *Suppose that* u_i *are i.i.d. random variables with* $\mathbf{E}(u_i) = 0$, $\mathbf{V}(u_i) = \sigma_u^2$, $\mathbf{V}(u_i^2) = \eta$*. Then, for any reliability index* β_1*, where* $1 - \beta_1 > 0$*, and* $\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta} + (n-1)\sigma_u^2\right) > 0$*, the following inequality is satisfied.*

$$\mathbf{Pr}\left(\|(U^{\mathrm{T}}U)^{-1}\|_1 \geq \epsilon_1\right) \leq \beta_1 \tag{34}$$

$$\epsilon_1 := \frac{1}{\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta} + (n-1)\sigma_u^2\right)} \tag{35}$$

When $u(i)$ has a uniform distribution: $u_i \in [-\kappa, \kappa]$, that is, $\sigma_u^2 = \frac{1}{3}\kappa^2$, $\eta = \frac{4}{45}\kappa^4$, then,

$$\epsilon_1 = \frac{1}{\kappa^2\left(\frac{1}{3}N - n\left(\sqrt{\frac{4}{45}} + \frac{1}{3}(n-1)\right)\sqrt{\frac{N}{\beta_1}}\right)}.$$

By employing Lemma 4.1, we can evaluate the magnitude of ΔE in the following proposition.

Proposition 4.1 *Suppose that* u_i *are i.i.d. random variables of a uniform distribution in* $[-\kappa, \kappa]$*. Moreover,* $\tilde{y}$ *is the output of the quantizer* $q(y)$ *defined by (3)* $\sim$ *(5), (21)* $\sim$ *(23). Then, for reliability indices* β_1, β_2*, a length of data* N *and the number of quantization levels* $2M$ *in* $[-\kappa,\ \kappa]$*, where* $1 - \beta_1 - \beta_2 > 0$, $M \gg 1$*, and* $\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta} + (n-1)\sigma_u^2\right) > 0$*, the following inequality holds.*

$$\mathbf{Pr}\left(\|\Delta E\|_\infty \leq \epsilon_1\epsilon_2\right) \geq 1 - \beta_1 - \beta_2 \tag{36}$$

$$\epsilon_1 := \frac{1}{\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta} + (n-1)\sigma_u^2\right)} \tag{37}$$

$$\epsilon_2 := \frac{A^{\frac{1}{2}}\kappa^2}{M-1}\sqrt{\frac{nN}{\beta_2}} \tag{38}$$

From this proposition, we know that the convergence rate of the error term ΔE has an order of M^{-1} at sufficiently large M and of $N^{-1/2}$. Approximately, the total amount of information on the quantized output transmitted from identified systems to the observers is $\log_2 2M \times N$ using a binary coding. Therefore, under a constraint of such a total amount of information, a large M is preferable to large N. Of course, this fact is valid only for the error term ΔE and the situation is different for the noise error term ΔW. We introduce the result for ΔW in the following proposition.

Proposition 4.2 [10] *Suppose that* u_i *and* w_i *are i.i.d. random variables with* $\mathbf{E}(u_i) = 0$, $\mathbf{V}(u_i) = \sigma_u^2$*, and* $\mathbf{V}(w_i) = \sigma_w^2$*, respectively. Then, for reliability indices* β_1, β_2*, and a* *length of data N, where $1-\beta_1-\beta_2>0$, and $\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta}+(n-1)\sigma_u^2\right)>0$, the following inequality holds.*

$$\mathbf{Pr}\left(\|\Delta W\|_\infty \le \epsilon_1\epsilon_2\right) \ge 1-\beta_1-\beta_2 \tag{39}$$

$$\epsilon_1 := \frac{1}{\sigma_u^2 N - n\sqrt{\frac{N}{\beta_1}}\left(\sqrt{\eta}+(n-1)\sigma_u^2\right)} \tag{40}$$

$$\epsilon_2 := \sigma_u\sigma_w\sqrt{\frac{nN}{\beta_2}} \tag{41}$$

By combining Proposition 4.1 and Proposition 4.2, we conclude there exists a trade-off between ΔE and ΔW for reducing the total identification error under the constraint of the amount of information transmitted from the identified systems to the observers.

5 Conclusion

In this paper, we showed an optimal quantization scheme for system identification. The quantization has the property that it is coarse near the origin of the signals and dense at a distance from it in the region of interest. This shows a form of duality against the quantization problem in system stabilization given in [5].

From the result of this paper, we know that the difference between a uniform quantization scheme and the optimal one becomes trivial when the number of quantization levels is large. In this sense, the non-uniform optimal quantization in this paper is efficient when used for the condition of low capacity signal transmission in real systems.

Another important topic, which is not discussed in this paper, is the problem of the coding of the quantized data. When we consider the coding of signals and their code length, the optimal quantization scheme may be different from that in this paper.

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A Appendix

Proof of Lemma 4.1 [10]

The diagonal elements of $U^{\mathrm{T}}U$ are in the form of

$$u_{-k+1}^2 + u_{-k+2}^2 + \cdots + u_{-k+N}^2.$$

From the assumption that every signal u_i is independent, then,

$$\begin{aligned}\mathbf{E}((U^{\mathrm{T}}U)_{ii}) &= \mathbf{E}(u_{-k+1}^2 + u_{-k+2}^2 + \cdots + u_{-k+N}^2)\\ &= \sum_{j=1}^{N}\mathbf{E}(u_{-k+j}^2)\\ &= N\sigma_u^2.\end{aligned} \tag{42}$$

The variance can be calculated as

$$\mathbf{V}((U^{\mathrm{T}}U)_{ii}) = \sum_{j=1}^{N}\mathbf{V}(u_{-k+j}^2) = N\eta. \tag{43}$$

On the other hand, the non-diagonal elements $(U^{\mathrm{T}}U)_{ij}$ $(i \neq j)$ are in the form of

$$u_{-k+1}u_{-l+1} + u_{-k+2}u_{-l+2} + \cdots + u_{-k+N}u_{-l+N},\ k \neq l.$$

Then, their expectations are given by

$$\begin{aligned}\mathbf{E}((U^{\mathrm{T}}U)_{ij}) &= \mathbf{E}(u_{-k+1}u_{-l+1}+u_{-k+2}u_{-l+2}+\cdots \\ &\quad + u_{-k+N}u_{-l+N}) \\ &= \sum_{m=1}^{N}\mathbf{E}(u_{-k+m}u_{-l+m}) \\ &= 0. \qquad (44)\end{aligned}$$

The variance is given by noting that $\mathbf{E}((u_{k+m}u_{l+m})\times(u_{k+n}u_{l+n}))=0$, even if $u_{l+m}=u_{k+n}$ or $u_{k+m}=u_{l+n}$.

$$\begin{aligned}\mathbf{V}((U^{\mathrm{T}}U)_{ij}) &= \mathbf{E}((u_{-k+1}u_{-l+1}+u_{-k+2}u_{-l+2}+\cdots \\ &\quad + u_{-k+N}u_{-l+N})^2) \\ &= \sum_{m=1}^{N}\mathbf{E}(u_{-k+m}^2u_{-l+m}^2),\ k\neq l \\ &= N\sigma_u^4 \qquad (45)\end{aligned}$$

Here we decompose $U^{\mathrm{T}}U$ as

$$U^{\mathrm{T}}U=(U^{\mathrm{T}}U-N\sigma_u^2I)+N\sigma_u^2I,$$

and by employing the norm inequality we obtain

$$\|U^{\mathrm{T}}U\|_1\geq\|N\sigma_u^2I\|_1-\|U^{\mathrm{T}}U-N\sigma_u^2I\|_1. \qquad (46)$$

The value of the first term of the right hand side in (46) is $N\sigma_u^2$, and in the second term, by employing Chebyshev's inequality with (42) and (44), we obtain

$$\mathbf{Pr}\left(|(U^{\mathrm{T}}U-N\sigma_u^2I)_{ij}|\geq\sqrt{\frac{\mathbf{V}((U^{\mathrm{T}}U)_{ij})}{r}}\right)\leq r,$$

and

$$\begin{aligned}&\mathbf{Pr}\left(\sum_{j=1}^{n}|(U^{\mathrm{T}}U-N\sigma_u^2I)_{ij}|\right.\\ &\qquad\left.\geq\sqrt{\frac{\mathbf{V}((U^{\mathrm{T}}U)_{ii})}{r}}+(n-1)\sqrt{\frac{\mathbf{V}((U^{\mathrm{T}}U)_{ij})}{r}}\right)\\ &= \mathbf{Pr}\left(\sum_{j=1}^{n}|(U^{\mathrm{T}}U-N\sigma_u^2I)_{ij}|\right.\\ &\qquad\left.\geq\sqrt{\frac{N}{r}}\left(\sqrt{\eta}+(n-1)\sigma_u^2\right)\right)\leq nr.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{Pr}\left(\|U^{\mathrm{T}}U-N\sigma_u^2I\|_1=\max_i\sum_{j=1}|(U^{\mathrm{T}}U-N\sigma_u^2I)_{ij}|\right.\\ &\qquad\left.\geq\sqrt{\frac{N}{r}}\left(\sqrt{\eta}+(n-1)\sigma_u^2\right)\right)\leq n^2r.\end{aligned}$$

Noting that

$$\begin{aligned}\|(U^{\mathrm{T}}U)^{-1}\| &= \frac{1}{\inf_x\frac{\|U^{\mathrm{T}}Ux\|}{\|x\|}} \\ &= \frac{1}{\inf_x\frac{\|\sigma_u^2NI+(U^{\mathrm{T}}U-\sigma_u^2NI)x\|}{\|x\|}} \\ &\leq \frac{1}{\sigma_u^2N-\sup_x\frac{\|(U^{\mathrm{T}}U-\sigma_u^2NI)x\|}{\|x\|}},\end{aligned}$$

this means

$$\begin{aligned}&\mathbf{Pr}\left(\|(U^{\mathrm{T}}U)^{-1}\|_1\right.\\ &\qquad\left.\geq\frac{1}{N\sigma_u^2-\sqrt{\frac{N}{r}}\left(\sqrt{\eta}+(n-1)\sigma_u^2\right)}\right)\leq rn^2.\end{aligned}$$

By denoting $\beta_1:=rn^2$ for simplicity, we obtain the statement.
□

Proof of Proposition 4.1

First evaluate the magnitude of $U^{\mathrm{T}}E$. Its i-th element $(U^{\mathrm{T}}E)_i$ is of form

$$u_1e_1+u_2e_2+\cdots+u_Ne_N.$$

From the independence of u_i and (33), the expectation and the variance of $(U^{\mathrm{T}}E)_i$ are given as:

$$\mathbf{E}((U^{\mathrm{T}}E)_i)=0,\ \mathbf{V}((U^{\mathrm{T}}E)_i)\leq NA\kappa^4(M-1)^{-2}$$

Then by Chebyshev's inequality, we obtain

$$\mathbf{Pr}\left(|U^{\mathrm{T}}E|_i\geq\sqrt{\frac{A\kappa^4N}{r(M-1)^2}}\right)\leq r,$$

for a reliability index r, and therefore the following inequality is deduced:

$$\mathbf{Pr}\left(\|U^{\mathrm{T}}E\|_\infty=\max_i|U^{\mathrm{T}}E|_i\geq\epsilon_2\right)\leq\beta_2,$$

where $\beta_2:=nr$. Combine $(U^{\mathrm{T}}U)^{-1}$ and $U^{\mathrm{T}}E$ using a norm inequality:

$$\|(U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}E\|_\infty\leq\|(U^{\mathrm{T}}U)^{-1}\|_1\|U^{\mathrm{T}}E\|_\infty,$$

and this gives

$$\begin{aligned}&\mathbf{Pr}\left(\|(U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}E\|_\infty\leq\epsilon_1\epsilon_2\right)\\ &\quad\geq\mathbf{Pr}\left(\|(U^{\mathrm{T}}U)^{-1}\|_1\leq\epsilon_1\text{ and }\|U^{\mathrm{T}}E\|_\infty\leq\epsilon_2\right).\end{aligned}$$

Therefore we prove the lemma. □