GENERALIZED LFT-BASED REPRESENTATION OF PARAMETRIC UNCERTAIN MODELS

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Abstract

In this paper we introduce a general descriptor-type LFT representation of rational parametric matrices. This generalized representation allows to represent arbitrary rationally dependent multivariate functions in LFT-form. As applications, we develop explicit LFT realizations of the transfer-function matrix of a linear descriptor system whose state space matrices depend rationally on a set of uncertain parameters. The resulting descriptor LFT-based uncertainty models generally have smaller orders than those obtained by using the standard LFT-based modelling approach.

1 Introduction

In modelling parametric uncertainties in linear systems the *linear fractional transformation* (LFT) plays an important role. LFT-based representations are useful to model real parametric uncertainties entering rationally in the system matrices. These models are ready to be used in robust control tools like the structured singular value (also called μ) [1]. LFT-based models are also useful in representing and manipulating multidimensional systems [2].

The main problem of LFT-based uncertainty modelling is the generation of low order LFT-representations. Recall that for a partitioned matrix

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \in \mathbf{R}^{(p_1+p_2)\times(m_1+m_2)}$$

and $\Delta \in \mathbf{R}^{m_1\times p_1}$, the *upper LFT* is defined as

$$\mathcal{F}_u(M,\Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}. \qquad (1)$$

Given a $p_2\times m_2$ real matrix $G(\boldsymbol{\delta})$ depending rationally on k parameters grouped into the real vector $\boldsymbol{\delta} = (\,\delta_1, \delta_2, \ldots, \delta_k\,)$, one wants to represent $G(\boldsymbol{\delta})$ as

$$G(\boldsymbol{\delta}) = \mathcal{F}_u(M,\Delta) \qquad (2)$$

where $M \in \mathbf{R}^{(p_1+p_2)\times(p_1+m_2)}$ and

$$\Delta = \text{diag}\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_k I_{r_k}) \qquad (3)$$

with $p_1 = \sum_{i=1}^{k} r_i$ representing the order of the LFT-representation (2). The well-posedness [1] of the LFT-representation (1) requires that $(I - M_{11}\Delta)$ is invertible for all $\boldsymbol{\delta} \in \Pi$, with Π as the uncertain parameter set defined as

$$\Pi = \{\boldsymbol{\delta} : \delta_i \in [\delta_{i,min}, \delta_{i,max}], i = 1,\ldots,k\}. \qquad (4)$$

Note that representing parameter dependent matrices in an LFT-form is basically equivalent to a multi-dimensional realization problem [2].

There is a basic limitation of realizing arbitrary rational matrices via upper LFTs. Consider the simple case of $G(\boldsymbol{\delta}) = \delta$, which can be immediately realized as

$$G(\boldsymbol{\delta}) = \mathcal{F}_u\left(\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right], \Delta\right) \qquad (5)$$

with $\Delta = \delta$. However, the expression $G(\boldsymbol{\delta}) = 1/\delta$ can not be directly represented as an upper LFT with Δ of the form (3). One way to represent $G(\boldsymbol{\delta}) = 1/\delta$ as an upper LFT is to use in (5) $\Delta = 1/\delta$. However, this approach can not be employed in the case when $G(\boldsymbol{\delta}) = \delta + 1/\delta$.

In practice, to overcome the above difficulty, a *normalization* of uncertainties is performed. Assuming, for example, that $\delta \in [\,\delta_{min}, \delta_{max}\,]$ and $\delta_{nom} := (\delta_{max} + \delta_{min})/2 \neq 0$, then with $\delta_{sl} := (\delta_{max} - \delta_{min})/2$ one obtains

$$\delta = \delta_{nom} + \delta_{sl}\overline{\delta}$$

where $\overline{\delta} \in [\,-1, 1\,]$. With this normalization, we can represent $G(\overline{\boldsymbol{\delta}}) := 1/(\delta_{nom} + \delta_{sl}\overline{\delta})$ as

$$G(\overline{\boldsymbol{\delta}}) = \mathcal{F}_u\left(\left[\begin{array}{c|c} -\delta_{sl}\delta_{nom}^{-1} & -\delta_{sl}\delta_{nom}^{-1} \\ \hline \delta_{nom}^{-1} & \delta_{nom}^{-1} \end{array}\right], \overline{\delta}\right)$$

Note that this approach is not recommended to be used if $0 \in [\,\delta_{min}, \delta_{max}\,]$, because the well-posedness condition is violated. One negative aspect of this approach is that the normalization must be performed as a preliminary operation of the

LFT-based model generation. Since the resulting LFT-models are generated by using symbolic manipulation tools (e.g., [3]), they often tend to have larger orders than those which typically would result when normalization is not performed as the first step. This is why, ideally, the normalization has to be performed as the last step in any LFT-model generation.

In this paper we introduce a generalized LFT-representation which allows to overcome the above difficulties. The *generalized* upper-LFT is defined with

$$M = \left[\begin{array}{c|c|c} M_{10} & M_{11} & M_{12} \\ \hline & M_{21} & M_{22} \end{array}\right]$$

as

$$\mathcal{F}_u(M,\Delta) = M_{22} + M_{21}\Delta(M_{10} - M_{11}\Delta)^{-1}M_{12} \qquad (6)$$

where the submatrix M_{10} is allowed to be generally singular. We call (6) a *descriptor LFT*, in analogy to the generalized state space realizations via descriptor systems [4]. For Δ we assume the more general structure

$$\Delta = \text{diag}\,(\delta_0 I_{r_0}, \delta_1 I_{r_1}, \ldots, \delta_k I_{r_k}) \qquad (7)$$

where δ_0 is a nonzero constant (usually set to 1). Note that the *standard* upper LFT (1) corresponds to $M_{10} = I$ and $r_0 = 0$.

With the generalized upper LFT we can represent $G(\boldsymbol{\delta}) = 1/\delta$ in a descriptor LFT form as

$$G(\boldsymbol{\delta}) = \mathcal{F}_u\left(\left[\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ \hline & & -1 & 0 & 0 \end{array}\right], \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}\right).$$

In this paper we discuss first some algebraic properties of the generalized LFT representations and give explicit formulas for basic operations with LFT-models. We present results showing that after normalization, the descriptor LFT representations can be converted into standard LFT representations. As an application of our approach, we develop explicit LFT realizations for the transfer-function matrix of a linear descriptor system whose matrices depend rationally on a set of uncertain parameters. Our result extends those reported in [5, 6], where only polynomial dependency of the system matrices on a set of uncertain parameters is allowed.

2 Algebraic properties

Since LFT-based representations are similar to transfer-function matrix representation of linear state-space systems, the basic matrix operations like addition/subtraction, multiplication, transposition, inversion as well as column/row concatenation correspond to similar operations performed on the transfer-function matrices of linear systems. These operations underly the methods used to generate LFT-representations of parametric matrices [7]. The following results for descriptor LFT-representations (given without proofs) generalize similar results for standard LFT-representations.

Lemma 2.1. *Let M_1, M_2, and M be the partitioned matrices*

$$M_1 = \left[\begin{array}{c|c|c} E_1 & A_1 & B_1 \\ \hline & C_1 & D_1 \end{array}\right],\; M_2 = \left[\begin{array}{c|c|c} E_2 & A_2 & B_2 \\ \hline & C_2 & D_2 \end{array}\right],$$
$$M = \left[\begin{array}{c|c|c} E & A & B \\ \hline & C & D \end{array}\right],$$

and let Δ_1, Δ_2 and Δ be the corresponding uncertainty matrices. Then, the following results hold:

(i) $\mathcal{F}_u(M_1,\Delta_1) \pm \mathcal{F}_u(M_2,\Delta_2) = \mathcal{F}_u(M_{par},\Delta_{par})$ *(parallel connection), with* $\Delta_{par} = \text{diag}\,(\Delta_1,\Delta_2)$ *and*

$$M_{par} = \left[\begin{array}{cc|cc|c} E_1 & 0 & A_1 & 0 & B_1 \\ 0 & E_2 & 0 & A_2 & \pm B_2 \\ \hline & & C_1 & C_2 & D_1 \pm D_2 \end{array}\right].$$

(ii) $\mathcal{F}_u(M_1,\Delta_1)\mathcal{F}_u(M_2,\Delta_2) = \mathcal{F}_u(M_{ser},\Delta_{ser})$ *(series connection), with* $\Delta_{ser} = \text{diag}(\Delta_1,\Delta_2)$ *and*

$$M_{ser} = \left[\begin{array}{cc|cc|c} E_1 & 0 & A_1 & B_1C_2 & B_1D_2 \\ 0 & E_2 & 0 & A_2 & B_2 \\ \hline & & C_1 & D_1C_2 & D_1D_2 \end{array}\right].$$

(iii) $[\mathcal{F}_u(M_1,\Delta_1)\,\mathcal{F}_u(M_2,\Delta_2)] = \mathcal{F}_u(M_{cc},\Delta_{cc})$ *(column concatenation), with* $\Delta_{cc} = \text{diag}(\Delta_1,\Delta_2)$ *and*

$$M_{cc} = \left[\begin{array}{cc|cc|cc} E_1 & 0 & A_1 & 0 & B_1 & 0 \\ 0 & E_2 & 0 & A_2 & 0 & B_2 \\ \hline & & C_1 & C_2 & D_1 & D_2 \end{array}\right].$$

(iv) $[\mathcal{F}_u(M_1,\Delta_1)^T\,\mathcal{F}_u(M_2,\Delta_2)^T]^T = \mathcal{F}_u(M_{rc},\Delta_{rc})$ *(row concatenation), with* $\Delta_{rc} = \text{diag}(\Delta_1,\Delta_2)$ *and*

$$M_{rc} = \left[\begin{array}{cc|cc|c} E_1 & 0 & A_1 & 0 & B_1 \\ 0 & E_2 & 0 & A_2 & B_2 \\ \hline & & C_1 & 0 & D_1 \\ & & 0 & C_2 & D_2 \end{array}\right].$$

(v) Suppose $\mathcal{F}_u(M,\Delta)$ is a $p\times p$ invertible matrix. Then

$$(\mathcal{F}_u(M,\Delta))^{-1} = \mathcal{F}_u(M_{inv},\Delta_{inv})$$

with $\Delta_{inv} = \text{diag}(I_p,\Delta)$

$$M_{inv} = \left[\begin{array}{cc|cc|c} 0 & 0 & D & C & I_p \\ 0 & E & B & A & 0 \\ \hline & & -I_p & 0 & 0 \end{array}\right].$$

If D is invertible, then we can also choose

$$M_{inv} = \left[\begin{array}{c|c|c} E & A - BD^{-1}C & -BD^{-1} \\ \hline & D^{-1}C & D^{-1} \end{array}\right],\; \Delta_{inv} = \Delta.$$

(vi) Let Q and Z be invertible matrices such that $Z\Delta = \Delta Z$. Then

$$\mathcal{F}_u(M,\Delta) = \mathcal{F}_u(\widetilde{M},\Delta)$$

where

$$\widetilde{M} = \left[\begin{array}{c|c|c} QEZ & QAZ & QB \\ \hline & CZ & D \end{array}\right]$$

(vii) Consider

$$\begin{bmatrix} A(\widetilde{\Delta}) & B(\widetilde{\Delta}) \\ C(\widetilde{\Delta}) & D(\widetilde{\Delta}) \end{bmatrix} = \mathcal{F}_u \left(\left[\begin{array}{c|c|cc} \widetilde{E} & \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ \hline & \widetilde{C}_1 & \widetilde{D}_{11} & \widetilde{D}_{12} \\ & \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{array} \right], \widetilde{\Delta} \right).$$

Then

$$\mathcal{F}_u \left(\left[\begin{array}{c|c|c} E & A(\widetilde{\Delta}) & B(\widetilde{\Delta}) \\ \hline & C(\widetilde{\Delta}) & D(\widetilde{\Delta}) \end{array} \right], \widetilde{\Delta} \right) = \mathcal{F}_u(M, \Delta),$$

with

$$M = \left[\begin{array}{cc|cc|c} \widetilde{E} & 0 & \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ 0 & E & \widetilde{C}_1 & \widetilde{D}_{11} & \widetilde{D}_{12} \\ \hline & & \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{array} \right], \Delta = \begin{bmatrix} \widetilde{\Delta} & 0 \\ 0 & \Delta \end{bmatrix}.$$

Note that by using a descriptor LFT representation, the inverse (see (v) of Lemma 2.1) can be determined in terms of original matrices, without any explicit matrix inversion.

It is possible to express the result of a left fractional factorization in terms of the underlying LFT-representations. The following result is particularly useful when realizing rational parametric matrices in terms of polynomial factorizations.

Lemma 2.2. *Let $[N(\boldsymbol{\delta})\ D(\boldsymbol{\delta})] = \mathcal{F}_u(M, \Delta)$ be defined with*

$$M = \left[\begin{array}{c|c|cc} E & A & B_N & B_D \\ \hline & C & D_N & D_D \end{array} \right], \tag{8}$$

and assume that $D(\boldsymbol{\delta})$ is $p \times p$ and invertible. Then

$$(D(\boldsymbol{\delta}))^{-1} N(\boldsymbol{\delta}) = \mathcal{F}_u(M_{lf}, \Delta_{lf}) \tag{9}$$

with

$$M_{lf} = \left[\begin{array}{cc|cc|c} 0 & 0 & D_D & C & D_N \\ 0 & E & B_D & A & B_N \\ \hline & & -I_p & 0 & 0 \end{array} \right], \quad \Delta_{lf} = \begin{bmatrix} I_p & 0 \\ 0 & \Delta \end{bmatrix}. \tag{10}$$

If D_D is invertible we can also choose $\Delta_{lf} = \Delta$ and

$$M_{lf} = \left[\begin{array}{c|c|c} E & A - B_D D_D^{-1} C & B_N - B_D D_D^{-1} D_N \\ \hline & D_D^{-1} C & D_D^{-1} D_N \end{array} \right].$$

Proof. Using *(v)* and *(ii)* of Lemma 2.1, we have

$$(D(\boldsymbol{\delta}))^{-1} N(\boldsymbol{\delta}) = \mathcal{F}_u(M_M, \Delta_M)$$

where $\Delta_M = \mathrm{diag}(I_p, \Delta, \Delta)$ and

$$M_M = \left[\begin{array}{c|c|c} E_M & A_M & B_M \\ \hline & C_M & D_M \end{array} \right] := \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & D_D & C & C & D_N \\ 0 & E & 0 & B_D & A & 0 & 0 \\ 0 & 0 & E & 0 & 0 & A & B_N \\ \hline & & & -I_p & 0 & 0 & 0 \end{array} \right].$$

We now apply a similarity transformation to M_M, yielding a transformed matrix $\widetilde{M}_M$. Consider the transformation matrices Q and Z given by

$$Q = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}, \quad Z = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix}$$

with the identity matrix I of the same size as Δ. It is easy to see that $Z\Delta_M = \Delta_M Z$, thus applying *(vi)* of Lemma 2.1, we obtain

$$\widetilde{M}_M = \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & D_D & C & 0 & D_N \\ 0 & E & 0 & B_D & A & 0 & B_N \\ 0 & 0 & E & 0 & 0 & A & B_N \\ \hline & & & -I_p & 0 & 0 & 0 \end{array} \right]$$

By evaluating $\mathcal{F}_u\left(\widetilde{M}_M, \Delta_M\right)$ directly, we see that this expression reduces to $\mathcal{F}_u(M_{lf}, \Delta_{lf})$, with M_{lf}, Δ_{lf} as defined in (10).

The result for invertible D_D can be proven similarly (see also [8]).

■

The following lemma (given without proof) gives the dual result for a right fractional factorization.

Lemma 2.3. *Let $[N(\boldsymbol{\delta})^T\ D(\boldsymbol{\delta})^T]^T = \mathcal{F}_u(M, \Delta)$ be defined with*

$$M = \left[\begin{array}{c|c|c} E & A & B \\ \hline & C_N & D_N \\ & C_D & D_D \end{array} \right],$$

and assume that $D(\boldsymbol{\delta})$ is $p \times p$ and invertible. Then

$$N(\boldsymbol{\delta})(D(\boldsymbol{\delta}))^{-1} = \mathcal{F}_u(M_{rf}, \Delta_{rf})$$

with

$$M_{rf} = \left[\begin{array}{cc|cc|c} 0 & 0 & D_D & C_D & -I_p \\ 0 & E & B & A & 0 \\ \hline & & D_N & C_N & 0 \end{array} \right], \quad \Delta_{rf} = \begin{bmatrix} I_p & 0 \\ 0 & \Delta \end{bmatrix}.$$

If D_D is invertible we can also choose

$$M_{rf} = \left[\begin{array}{c|c|c} E & A - B D_D^{-1} C_D & B D_D^{-1} \\ \hline & C_N - D_N D_D^{-1} C_D & D_N D_D^{-1} \end{array} \right], \quad \Delta_{rf} = \Delta.$$

3 LFT-realization procedure

Using the results of section 2, we can directly build LFT-representations of arbitrary rational parametric matrices along the lines of the procedure suggested in [7]. The advantage of using generalized LFT-representations is that the obligatory normalization of parameters (see next section) can be performed at the end of the realization, thus the order of the LFT-representation is not artificially increased by intrinsically more complicated symbolic manipulations.

An alternative way to avoid the preliminary normalization has been proposed in [8], where we build an LFT-representation for a rational parametric matrix $G(\boldsymbol{\delta})$ by starting from a fractional representation $G(\boldsymbol{\delta}) = (D(\boldsymbol{\delta}))^{-1}N(\boldsymbol{\delta})$, with $D(\boldsymbol{\delta})$ and $N(\boldsymbol{\delta})$ as multivariate polynomial matrices. After realizing $[N(\boldsymbol{\delta})\ D(\boldsymbol{\delta})]$ as a standard LFT-representation, we can perform the normalization (without increasing the order) and employ Lemma 2.2 to obtain a realization of $G(\boldsymbol{\delta})$. Although this approach is well-suited to realize individual parametric matrices, it has some limitation when solving more complicated problems (as for example that presented in Section 5).

4 Normalization

To obtain at the end a standard LFT-representation ready to be used in μ-analysis, a *normalization* of the parameters must be usually performed. This amounts to replace δ_i with $\delta_{i,nom} + \delta_{i,sl}\overline{\delta}_i$, where $\delta_{i,nom}$ and $\delta_{i,sl}$ are such that $|\overline{\delta}_i| \leq 1$, for $i = 1, \ldots, k$. The normalized parameter vector is given by $\overline{\boldsymbol{\delta}} = (\overline{\delta}_1, \ldots, \overline{\delta}_k)$. To perform the normalization, we have to replace Δ by $\Delta_{nom} + \Delta_{sl}\overline{\Delta}$ in the final LFT-representation, where

$$\Delta_{nom} = \mathrm{diag}(I_{r_0}, \delta_{1,nom}I_{r_1}, \ldots, \delta_{k,nom}I_{r_k}) \qquad (11)$$
$$\Delta_{sl} = \mathrm{diag}(0_{r_0}, \delta_{1,sl}I_{r_1}, \ldots, \delta_{k,sl}I_{r_k}). \qquad (12)$$

The following result provides formulas to express $G(\overline{\boldsymbol{\delta}})$ in terms of the LFT representation of $G(\boldsymbol{\delta})$.

Lemma 4.1. *Let $G(\boldsymbol{\delta}) = \mathcal{F}_u(M, \Delta)$ with*

$$M = \left[\begin{array}{c|c|c} E & A & B \\ \hline & C & D \end{array}\right].$$

If $(E - A\Delta_{nom})$ is invertible, then

$$G(\overline{\boldsymbol{\delta}}) = \mathcal{F}_u\left(M, \Delta_{nom} + \Delta_{sl}\overline{\Delta}\right) = \mathcal{F}_u\left(\overline{M}, \overline{\Delta}\right),$$

where

$$\overline{M} = \left[\begin{array}{c|c|c} I & \overline{A} & \overline{B} \\ \hline & \overline{C} & \overline{D} \end{array}\right]$$

with

$$\begin{aligned}
\overline{A} &= (E - A\Delta_{nom})^{-1}A\Delta_{sl} \\
\overline{B} &= (E - A\Delta_{nom})^{-1}B \\
\overline{C} &= C(\Delta_{nom}(E - A\Delta_{nom})^{-1}A + I)\Delta_{sl} \\
\overline{D} &= C\Delta_{nom}(E - A\Delta_{nom})^{-1}B + D
\end{aligned}$$

The order of the resulting normalized standard LFT representation is the same as the order of the original descriptor LFT representation. When applying the LFT-realization procedure of the previous section, the resulting LFT-representation (M, Δ) has the following particular form

$$M = \left[\begin{array}{c|c|c} E & A & B \\ \hline & C & D \end{array}\right] = \left[\begin{array}{cc|cc|c} 0_{r_0} & 0 & A_{11} & A_{12} & B_1 \\ 0 & I & A_{21} & A_{22} & B_2 \\ \hline & & C_1 & C_2 & D \end{array}\right], \qquad (13)$$
$$\Delta = \mathrm{diag}(I_{r_0}, \delta_1 I_{r_1}, \ldots, \delta_k I_{r_k}). \qquad (14)$$

For this particular realization, we have the following specialization of Lemma 4.1, which shows that the normalization can lead to a lower order LFT realization.

Corollary 4.1. *Let $G(\boldsymbol{\delta}) = \mathcal{F}_u(M, \Delta)$ with M and Δ given in (13) and (14), respectively, and let $\Delta = \Delta_{nom} + \Delta_{sl}\overline{\Delta}$, where Δ_{nom} and Δ_{sl} have the forms in (11) and (12), respectively. Then $G(\overline{\boldsymbol{\delta}}) = \mathcal{F}_u(\overline{M}, \overline{\Delta})$ with*

$$\overline{M} = \left[\begin{array}{c|c|c} I & \overline{A}_{22} & \overline{B}_2 \\ \hline & \overline{C}_2 & \overline{D} \end{array}\right],$$
$$\bar{\Delta} = \mathrm{diag}(\overline{\delta}_1 I_{r_1}, \ldots, \overline{\delta}_k I_{r_k}),$$

where $\overline{A}_{22}$, $\overline{B}_2$, and $\overline{C}_2$ are submatrices of the resulting normalized model

$$\overline{M} = \left[\begin{array}{c|c|c} I & \overline{A} & \overline{B} \\ \hline & \overline{C} & \overline{D} \end{array}\right] = \left[\begin{array}{cc|cc|c} I_{r_0} & 0 & \overline{A}_{11} & \overline{A}_{12} & \overline{B}_1 \\ 0 & I & \overline{A}_{21} & \overline{A}_{22} & \overline{B}_2 \\ \hline & & \overline{C}_1 & \overline{C}_2 & \overline{D} \end{array}\right], \qquad (15)$$
$$\bar{\Delta} = \mathrm{diag}(I_{r_0}, \overline{\delta}_1 I_{r_1}, \ldots, \overline{\delta}_k I_{r_k}). \qquad (16)$$

Proof: Follows easily by observing that as a consequence of the particular structure of Δ_{sl} in (12), the submatrices $\overline{A}_{11}, \overline{A}_{21}, \overline{C}_1$ in (15) are null. ■

An important aspect of building LFT realizations is that the normalization step is desirable to be performed at the end of the LFT realization. Otherwise, the resulting realizations can have orders larger than those resulting without normalization. Consider the simple example of an expanded normalized product

$$\begin{aligned}
\delta_1\delta_2 &= (\delta_{1_{nom}} + \overline{\delta}_1)(\delta_{2_{nom}} + \overline{\delta}_2) \\
&= \delta_{1_{nom}}\delta_{2_{nom}} + \delta_{1_{nom}}\overline{\delta}_2 + \overline{\delta}_1\delta_{2_{nom}} + \overline{\delta}_1\overline{\delta}_2
\end{aligned}$$

By using an object oriented symbolic realization approach, an LFT representation of order 4 (instead of 2) could result. Since the standard 1-D or n-D order reduction techniques [9] assume that the δ_1 and δ_2 (seen as operators) do not commute, (i.e., $\delta_1\delta_2 \neq \delta_2\delta_1$), there is in general no guarantee that an LFT representation of lower order can be found for a system with parametric uncertainties, where $\delta_1\delta_2 = \delta_2\delta_1$ (see [3] for such an example).

5 LFT-realization for linear parametric descriptor systems

Consider a linear parametric system in descriptor form

$$\begin{aligned}
E(\boldsymbol{\delta})\dot{x}(t) &= A(\boldsymbol{\delta})x(t) + B(\boldsymbol{\delta})u(t) \\
y(t) &= C(\boldsymbol{\delta})x(t) + D(\boldsymbol{\delta})u(t)
\end{aligned} \qquad (17)$$

with $u(t) \in \mathbf{R}^m, x(t) \in \mathbf{R}^n, y(t) \in \mathbf{R}^p$ for $t \geq 0$. We assume that $E(\boldsymbol{\delta})$, $A(\boldsymbol{\delta})$, $B(\boldsymbol{\delta})$, $C(\boldsymbol{\delta})$, $D(\boldsymbol{\delta})$ depend rationally on the components of the parameter vector $\boldsymbol{\delta}$. $E(\boldsymbol{\delta})$ and $A(\boldsymbol{\delta})$ are square matrices and $E(\boldsymbol{\delta})$ may be singular, but we assume it has constant rank for all $\boldsymbol{\delta} \in \Pi$.

The transfer function matrix $G(s,\boldsymbol{\delta})$ of the descriptor system (17) is given by

$$G(s,\boldsymbol{\delta}) = C(\boldsymbol{\delta})(sE(\boldsymbol{\delta}) - A(\boldsymbol{\delta}))^{-1}B(\boldsymbol{\delta}) + D(\boldsymbol{\delta}) \tag{18}$$

where the pencil $sE(\boldsymbol{\delta}) - A(\boldsymbol{\delta})$ is assumed to be regular for all values of $\boldsymbol{\delta} \in \Pi$.

We develop a general method to determine an LFT representation (M, Δ) such that

$$G(s,\overline{\boldsymbol{\delta}}) = \mathcal{F}_u(M,\Delta),$$

with

$$M = \left[\begin{array}{c|c|c} E_M & A_M & B_M \\ \hline & C_M & D_M \end{array}\right],$$

$$\Delta = \mathrm{diag}(I_{r_0}, \frac{1}{s}I_{r_1}, \overline{\delta}_2 I_{r_2}, \ldots, \overline{\delta}_k I_{r_k}), \tag{19}$$

where $\overline{\delta}_i, i = 2,\ldots,k$, are the normalized parameters (i.e. $\delta_i = \delta_{i_{nom}} + \delta_{i_{sl}}\overline{\delta}$).

In this LFT-representation the integration operator $1/s$ (with s as the Laplace variable) is also included in Δ by defining $\delta_1 = 1/s$.

In [5, 6] an LFT-realization procedure for parametric descriptor systems was proposed. However, it was assumed that the system matrices depend polynomially on the components of the parameter vector $\boldsymbol{\delta}$. Furthermore, in [6] it was assumed, that $E(\boldsymbol{\delta})$ is invertible.

For the realization of $G(s,\overline{\boldsymbol{\delta}})$ as an LFT-representation, we can distinguish between two cases: (1) $E(\boldsymbol{\delta})$ general (possibly non-invertible); (2) $E(\boldsymbol{\delta})$ invertible. We discuss building of LFT-representations for these two cases.

5.1 $E(\boldsymbol{\delta})$ general

The LFT realization of $G(s,\overline{\boldsymbol{\delta}})$ can be built using the following steps:

1. Use the LFT-realization procedure of Section 3 and apply the normalization to determine normalized standard LFT representations for each system matrix of (17), i.e. realize

$$A(\overline{\boldsymbol{\delta}}) = \mathcal{F}_u\left(\left[\begin{array}{c|c|c} I_{n_a} & A_A & B_A \\ \hline & C_A & D_A \end{array}\right], \Delta_A\right),$$

$$B(\overline{\boldsymbol{\delta}}) = \mathcal{F}_u\left(\left[\begin{array}{c|c|c} I_{n_b} & A_B & B_B \\ \hline & C_B & D_B \end{array}\right], \Delta_B\right)$$

   and the same for $C(\overline{\boldsymbol{\delta}}), D(\overline{\boldsymbol{\delta}}), E(\overline{\boldsymbol{\delta}})$. Since these matrices do not depend on s, the size of I_{r_1} within $\Delta_A, \ldots, \Delta_E$ is zero.

2. Construct a LFT representation $G(s,\overline{\boldsymbol{\delta}}) = \mathcal{F}_u(M(s), \tilde{\Delta})$ with $\tilde{\Delta} = \mathrm{diag}(\Delta_E, \Delta_A, \Delta_B, \Delta_C, \Delta_D)$

$$M(s) = \left[\begin{array}{c|c|c} I & M_{11}(s) & M_{12}(s) \\ \hline & M_{21}(s) & M_{22}(s) \end{array}\right],$$

   and

$$\widetilde{M}(s) = \left[\begin{array}{c|c} M_{11}(s) & M_{12}(s) \\ \hline M_{21}(s) & M_{22}(s) \end{array}\right] = \left[\begin{array}{ccccc|c} A_E & 0 & 0 & 0 & 0 & 0 \\ 0 & A_A & 0 & 0 & 0 & 0 \\ 0 & 0 & A_B & 0 & 0 & B_B \\ 0 & 0 & 0 & A_C & 0 & 0 \\ 0 & 0 & 0 & 0 & A_D & B_D \\ \hline 0 & 0 & 0 & C_C & C_D & D_D \end{array}\right]$$

$$- \left[\begin{array}{c} sB_E \\ -B_A \\ 0 \\ B_C \\ 0 \\ \hline D_C \end{array}\right] (sD_E - D_A)^{-1} \left[\begin{array}{ccccc|c} C_E & C_A & -C_B & 0 & 0 & -D_B \end{array}\right].$$

3. Compute a minimal order descriptor realization for the rational matrix $\widetilde{M}(s)$ (e.g. using the methods of [10], followed by the elimination of non-dynamic modes [11]), as

$$\widetilde{M}(s) = C'(sE' - A')^{-1}B' + D',$$

   with $E' = \mathrm{diag}(0_{r_0}, I_{r_1})$ and build the corresponding descriptor LFT-representation, i.e.

$$\tilde{M}(s) = \mathcal{F}_u(M', \Delta') = C'\Delta'(E' - A'\Delta')^{-1}B' + D',$$

   with

$$M' = \left[\begin{array}{c|c|c} E' & A' & B' \\ \hline & C' & D' \end{array}\right], \Delta' = \mathrm{diag}(I_{r_0}, \frac{I_{r_1}}{s}). \tag{20}$$

4. Apply *(vii)* of Lemma 2.1 to obtain

$$G(s,\overline{\boldsymbol{\delta}}) = \mathcal{F}_u(M,\Delta). \tag{21}$$

   with $\Delta = \mathrm{diag}(\Delta', \tilde{\Delta})$.

5. Reorder (M,Δ) such that Δ is of the form as given in (19).

5.2 $E(\boldsymbol{\delta})$ invertible

In the case of an invertible $E(\boldsymbol{\delta})$ we can derive a simpler procedure:

1. Construct a descriptor LFT representation, such that

$$\left[\begin{array}{cc|cc} A(\boldsymbol{\delta}) & B(\boldsymbol{\delta}) & E(\boldsymbol{\delta}) & 0 \\ C(\boldsymbol{\delta}) & D(\boldsymbol{\delta}) & 0 & I_p \end{array}\right] = \left[\begin{array}{c|c} N(\boldsymbol{\delta}) & D(\boldsymbol{\delta}) \end{array}\right]$$
$$= \mathcal{F}_u\left(\left[\begin{array}{c|c|cc} \widetilde{E} & \widetilde{A} & \widetilde{B}_N & \widetilde{B}_D \\ \hline & \widetilde{C} & \widetilde{D}_N & \widetilde{D}_D \end{array}\right], \Delta\right).$$

2. Apply (9) and perform the normalization step to obtain the standard LFT-representation

$$\left[\begin{array}{cc} (E(\overline{\boldsymbol{\delta}}))^{-1}A(\overline{\boldsymbol{\delta}}) & (E(\overline{\boldsymbol{\delta}}))^{-1}B(\overline{\boldsymbol{\delta}}) \\ C(\overline{\boldsymbol{\delta}}) & D(\overline{\boldsymbol{\delta}}) \end{array}\right]$$
$$= \mathcal{F}_u\left(\left[\begin{array}{c|c|cc} I & A' & B_1' & B_2' \\ \hline & C_1' & D_{11}' & D_{12}' \\ & C_2' & D_{21}' & D_{22}' \end{array}\right], \overline{\Delta}\right).$$

3. Construct $G(s,\overline{\boldsymbol{\delta}})$ as

$$
\begin{aligned}
G(s,\overline{\boldsymbol{\delta}}) &= \mathcal{F}_u(M,\Delta) \\
&= \mathcal{F}_u\left(\left[\begin{array}{cc|cc|c} I_{r_1} & 0 & D'_{11} & C'_1 & D'_{12} \\ 0 & I & B'_1 & A' & B'_2 \\ \hline & & D'_{21} & C'_2 & D'_{22} \end{array}\right], \left[\begin{array}{cc} \frac{I_{r_1}}{s} & 0 \\ 0 & \Delta \end{array}\right]\right).
\end{aligned}
$$

4. Reorder (M,Δ) such that Δ is of the form as given in (19).

The main advantage of this simpler LFT-realization procedure is, that we can apply the symbolic preprocessing techniques of [12, 13] to the concatenated symbolic matrix $[N(\boldsymbol{\delta})|D(\boldsymbol{\delta})]$ (see step 1), which contains all the system matrices. Hence, it is expected that the resulting LFT-realization is of lower order than an LFT-representation, which is realized using the more general procedure of subsection 5.1, where each system matrix is realized separately.

In [14] we successfully applied the proposed, generalized LFT-realization method to build a minimal order LFT-representation of a vehicle model.

6 Conclusion

We proposed a general descriptor system representation based LFT realization technique for rational parametric matrices. With this approach, we can completely avoid the normalization of the parameters as a preliminary step of the LFT realization. Therefore, it is generally expected that the resulting LFT representations are of lower order than equivalent representations generated with standard LFT based realization methods. Since the proposed overall realization method is based on elementary LFT manipulations it can easily be automated.

In addition, the descriptor system based LFT realization approach allows to directly derive LFT representations from linear parametric state space systems in descriptor form, which is a usual representation for physical systems. In the proposed procedure, no preliminary symbolic matrix manipulation, like explicit inversion of $E(\boldsymbol{\delta})$ is necessary and even systems with non-invertible $E(\boldsymbol{\delta})$ can be easily handled.

The existing MATLAB *LFR-toolbox* [3] for the realization of standard LFT representations can in principle be extended to handle also descriptor LFT-representations. Together with reliable numerical tools for handling descriptor systems available in the MATLAB *Descriptor System Toolbox* [15] and with symbolic preprocessing techniques for parametric system matrices of [12, 13], we have a very promising approach to efficiently generate low order LFT representations of uncertain physical systems.

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