A LOWER BOUND ON ACHIEVED CLOSED-LOOP PERFORMANCE BASED ON FINITE DATA

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Abstract

A new lower bound is derived for the level of performance achieved by a given feedback compensator with a plant that is not known completely. The bound involves quantities reflecting the performance of the controller with a nominal model of the plant, quantities that can be computed from a finite number of frequency response samples of the unknown plant, and quantities related to the complexity (in the sense of Vinnicombe) of all systems involved.

Keywords: ν-gap metric, robust performance, controller validation

Notation

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. The symbol $\mathcal{L}_\infty$ is used to denote the space of all (possibly matrix valued) functions $F(z)$ that are essentially bounded on $\partial\mathbb{D}$ and have finite norm $\|F\|_{\mathcal{L}_\infty} := \text{ess}\sup_\omega \overline{\sigma}(F(e^{j\omega}))$, where $\overline{\sigma}(\cdot)$ represents the maximum singular value. The symbol $\mathcal{H}_\infty$ denotes the space of functions $F(z)$ that are analytic in $\mathbb{D}$ and have finite norm $\|F\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty$. Given a system transfer function $F(z)$, the transfer function of the adjoint system is denoted by $F^*(z) := F(\frac{1}{z})^T$, where the superscript 'T' denotes matrix transpose. Note that for a real rational transfer function $F^*(e^{j\omega}) = \overline{F(e^{j\omega})}^T$, where the overline denotes complex conjugate.

Recall that any linear, time invariant discrete-time system P that is stabilisable, can be expressed as $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ with

1. $G_P := \begin{bmatrix} N \\ M \end{bmatrix}$ inner (*i.e.* $G_P^* G_P = I$) and left invertible in $\mathcal{H}_\infty$; and
2. $\tilde{G}_P := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}$ co-inner (*i.e.* $\tilde{G}_P \tilde{G}_P^* = I$) and right invertible in $\mathcal{H}_\infty$.

G_P (resp. $\tilde{G}_P$) is called the normalised right (resp. normalised left) graph symbol of plant P.

1 Introduction

Feedback can reduce the sensitivity of a designed system to uncertainty arising, for example, from the inevitable mismatch between open-loop models used in the design process and the physical systems that these models represent. Indeed, significant research has been dedicated to understanding, both quantitatively and qualitatively, the uncertainty that the feedback mechanism can handle. For general linear systems, gap-like metrics are known to capture the difference between open-loop systems from the perspective of their behaviour in closed-loop [1, 2, 3, 4].

In this paper, the ν-gap metric [5] and an associated measure of system complexity [6] are used to obtain a new tractable bound on the level of performance achieved by a given feedback compensator C with a plant P_t that is not known completely. Here, closed-loop performance is taken to mean the generic performance/stability measure $b(P_t, C)$, which is central the $\mathcal{H}_\infty$ loop-shaping paradigm of McFarlane and Glover [7]. The *a priori* information required to compute the bound includes a nominal model P_m of the true plant, a model of the compensator, which is assumed to stabilise both the plant model and the true plant, and a bound on the complexity (in the sense of Vinnicombe [6]) of P_t. The *a posteriori* information is a finite set of frequency response samples of the true plant. Work that is related in terms of assessing closed-loop performance from measured-data/identified-sets, but distinct in terms of the approach taken, can be found in [8] and the references therein.

The paper is structured as follows. First some ν-gap metric robustness results are reviewed. Then in a subsequent section, the new lower bound on achieved performance is derived and discussed.

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2 Review of $\nu-$gap -metric robustness results

The $\nu-$gap between two linear time-invariant plants P_1 and P_2 is defined as

$$\begin{aligned}\delta_\nu(P_1,P_2) &= \inf_{Q,Q^{-1}\in\mathcal{L}_\infty} \| G_1 - G_2 Q \|_\infty \text{ if } I(P_1,P_2)=0 \\ &= \quad 1 \quad \text{otherwise} \qquad (1)\end{aligned}$$

where $I(P_1,P_2) :=$ wno det $(G^*_{P_2}G_{P_1})$ and wno (g) denotes the winding number of $g(z)$ evaluated on the standard Nyquist contour indented around any poles and zeros on $\partial\mathbb{D}$ [5]. For a real rational transfer matrix X satisfying $X, X^{-1} \in \mathcal{L}_\infty$ the winding number wno det $(X) = \eta(X^{-1}) - \eta(X)$ where $\eta(x)$ denotes the number of unstable poles of x. When the winding number condition is satisfied, $\delta_\nu(P_1,P_2)$ equals the $\mathcal{L}_2$ *-gap*:

$$\delta_{\mathcal{L}_2}(P_1,P_2) := \|\tilde{G}_{P_2}G_{P_1}\|_\infty = \sup_\omega \kappa(P_1,P_2)(e^{j\omega}), \qquad (2)$$

where $\kappa(P_1,P_2)(e^{j\omega})$ is the pointwise *chordal* distance between the stereographic projections of the frequency responses of P_1 and P_2 onto the Riemann sphere, as defined by

$$\kappa(P_1,P_2)(j\omega) := \overline{\sigma}\Big((I+P_2P_2^*)^{-\frac{1}{2}}(P_2-P_1)(I+P_1^*P_1)^{-\frac{1}{2}}\Big)(j\omega)$$

Note that, $\kappa(\cdot,\cdot)$ can be computed from purely frequency response data.

Given a controller C and a (possibly frequency weighted) plant P_i, a useful measure of closed-loop performance, which is central to the $\mathcal{H}_\infty$ loop-shaping paradigm for design, is

$$\begin{aligned}b(P_i,C) &= \|H(P_i,C)\|_\infty^{-1} \\ &= \inf_\omega \rho(P_i,C)(e^{j\omega}) \text{ if } C \text{ stabilises } P \\ &= 0 \quad \text{otherwise.} \qquad (3)\end{aligned}$$

where

$$\rho(P_i,C)(e^{j\omega}) := \underline{\sigma}(H(P_i,C))(e^{j\omega}),$$

$\underline{\sigma}(\cdot)$ denotes the minimum singular value and the closed-loop transfer function $H(P_i,C)$ is defined by

$$H(P_i,C) := \begin{bmatrix} P_i \\ I \end{bmatrix} (I-CP_i)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

It is known that any controller that stabilises a plant P_1 and achieves $b(P_1,C) > \beta$, also stabilises all plants in the set $\{P_2 : \delta_\nu(P_1,P_2) \le \beta\}$ [5]. Furthermore, the difference between the level of closed-loop performance achieved by a feedback compensator C with a nominal plant P_1 and with a perturbed plant P_2 can be quantified in terms of $\delta_\nu(P_1,P_2)$:

$$b(P_2,C) \ge b(P_1,C) - \delta_\nu(P_1,P_2). \qquad (4)$$

A pointwise-in-frequency version of this performance bound also holds:

$$\rho(P_2,C)(e^{j\omega}) \ge \rho(P_1,C)(e^{j\omega}) - \kappa(P_1,P_2)(e^{j\omega}). \qquad (5)$$

Indeed (4) follows from this since

$$\inf_\omega \Big(\rho(P_1,C) - \kappa(P_1,P_2)\Big) \ge \inf_\omega \rho(P_1,C) - \sup_\omega \kappa(P_1,P_2).$$

Here, the argument $(e^{j\omega})$ is omitted for brevity. Note, however, that the pointwise bound in (5) is useful only if C is known to stabilise both P_1 and P_2. To this end the following result is easily inferred from the development in [5] (see also [9]):

Lemma 1 *Suppose, a true plant P_t, a model P_m and a nominal controller C_n satisfy the following conditions:*

1. *$H(P_t,C_n)$ and $H(P_m,C_n)$ are stable;*
2. *$\kappa(P_t,P_m)(e^{j\omega}) < \rho(P_m,C_n)(e^{j\omega}) \;\forall\; \omega$.*

Then any other controller C which stabilises P_m and satisfies $\kappa(P_t,P_m)(e^{j\omega}) < \rho(P_m,C)(e^{j\omega}) \;\forall\; \omega$ is guaranteed to stabilise P_t with closed-loop performance

$$b(P_t,C) \ge \inf_\omega \Big(\rho(P_m,C)(e^{j\omega}) - \kappa(P_m,P_t)(e^{j\omega})\Big). \qquad (6)$$

Note that the lower bound in (6) is tighter than that in (4). However, no computationally tractable characterisation for this is known.

Suppose, though, that the true plant frequency response $P_t(e^{j\omega_i})$ is known to lie within a known chordal distance from the frequency response of the model $P_m(e^{j\omega_i})$ at a finite number of frequencies $\{\omega_0,\omega_1,\omega_2,\ldots,\omega_m\}$. Then provided the frequency responses of P_m, P_t and C_n are 'sufficiently smooth' and the frequency grid is 'sufficiently dense', the infimum over the continuum of frequencies in the lower bound of (6) may be approximated by an minimum over the finite set $\{\omega_i\}$:

$$\inf_\omega (\rho(P_m,C) - \kappa(P_t,P_m)) \approx \min_i (\rho_i(P_m,C) - \kappa_i(P_t,P_m))$$

where $\rho_i(P_m,C) = \rho(P_m,C)(e^{j\omega_i})$ and $\kappa_i(P_t,P_m) = \kappa(P_t,P_m)(e^{j\omega_i})$. In the next section, a lower bound on the right-hand side of (6) is derived by using an appropriate notion of smoothness and a finite set of frequency response data. The quantity used to capture the smoothness of frequency responses, in terms of the variation in chordal distance, is defined by

$$V_P = \|\tilde{G}_P G'_P\|_\infty, \qquad (7)$$

where $G'_P = \frac{dG_P}{dz}$. As shown in [6], given two frequencies ω_1 and ω_2,

$$\kappa(P(e^{j\omega_1}),\, P(e^{j\omega_2})) \le V_P|\omega_1 - \omega_2|.$$

Given P, it is not difficult to compute V_P. First, $G_P, \tilde{G}_P$ can be computed using by well-known techniques which are available in standard software such as MATLAB. Then given $G_P = H(zI - A)^{-1}F + D$ (where $\{A, F, H, D\}$ is any state space realisation of G_P), it follows that $G'_P = S_1 S_2$ where $S_1 = -H(zI - A)^{-1}I$ and $S_2 = I(zI - A)^{-1}F$. That is, V_P is simply the infinity norm of a product of three transfer functions all of which may be easily derived from P.

3 A bound on achieved performance

Below, a lower bound on $\rho(P_t, C)$ at any intermediate frequency $\omega_\star \in [\omega_i, \omega_{i+1}]$ is obtained using the definition of V_P.

Proposition 1 *Let C be a controller that stabilises both the true plant P_t and the model P_m. Then for any $\omega_\star \in [\omega_i, \omega_{i+1}]$ the following inequality holds:*

$$\rho(P_t, C)(e^{j\omega_\star}) \geq \left(\max \left\{ b(P_m, C), \sqrt{1 - x_i^2} \right\} - y_i \right) \tag{8}$$

where

$$\begin{aligned} x_i \;&:= \min \left\{ \kappa(P_m, -C^*)(e^{j\omega_i}), \kappa(P_m, -C^*)(e^{j\omega_{i+1}}) \right\} \\ &\quad + (V_{P_m} + V_{-C^*})|\omega_{i+1} - \omega_i|, \qquad (9) \\ y_i \;&:= \min \left\{ \kappa(P_m, P_t)(e^{j\omega_i}), \kappa(P_m, P_t)(e^{j\omega_{i+1}}) \right\} \\ &\quad + (V_{P_t} + V_{P_m})|\omega_{i+1} - \omega_i| \qquad (10) \end{aligned}$$

and V_P is as defined in (7).

Proof : Given P, it can be shown that

$$\rho(P, C)(e^{j\omega}) = \sqrt{1 - \kappa^2(P, -C^*)(e^{j\omega})} \tag{11}$$

at all frequencies [5]. Hence, it follows from (5) that

$$\rho(P_t, C)(e^{j\omega_\star}) \geq \sqrt{1 - \kappa^2(P_m, -C^*)(e^{j\omega_\star})} - \kappa(P_t, P_m)(e^{j\omega_\star})$$

The lower bound claimed follows by bounding, from above, each of the two chordal distances terms shown. Since $\kappa(\cdot, \cdot)$ is a metric [3], it follows by the triangle inequality,

$$\begin{aligned} &\kappa(P_m, P_t)(e^{j\omega_\star}) \leq \kappa(P_m(e^{j\omega_\star}), P_m(e^{j\omega_i})) \\ &+ \kappa(P_m(e^{j\omega_i}), P_t(e^{j\omega_i})) + \kappa(P_t(e^{j\omega_i}), P_t(e^{j\omega_\star})) \\ &\leq \kappa(P_m(e^{j\omega_i}), P_t(e^{j\omega_i})) + (V_{P_m} + V_{P_t})\, |\omega_\star - \omega_i|. \qquad (12) \end{aligned}$$

Similarly,

$$\begin{aligned} \kappa(P_m, P_t)(e^{j\omega_\star}) \;&\leq\; \kappa(P_m(e^{j\omega_{i+1}}), P_t(e^{j\omega_{i+1}})) \\ &\quad + (V_{P_m} + V_{P_t})\, |\omega_\star - \omega_{i+1}|. \qquad (13) \end{aligned}$$

Note that the upper bound y_i on $\kappa(P_m, P_t)(e^{j\omega_\star})$ in (10) is simply a consequence of equations (12-13). The upper bound x_i on $\kappa(P_m, -C^*)(e^{j\omega_\star})$ in (9) follows in a similar manner. The result now follows by using equation (11) and the fact that both $b(P_m, C)$ and $\sqrt{1 - x_i^2}$ are lower bounds on $\rho(P_m, C)(e^{j\omega_\star})$. ■

When the right hand side in (8) is positive for all i, proposition 1 further suggests that

$$\begin{aligned} b(P_t, C) \;&:=\; \inf_\omega \rho(P_t, C)(e^{j\omega}) \\ &=\; \min_i \inf_{\omega \in [\omega_i, \omega_{i+1}]} \rho(P_t, C)(e^{j\omega}) \\ &\geq\; \min_i \left(\max \left\{ b(P_m, C), \sqrt{1 - x_i^2} \right\} - y_i \right). \qquad (14) \end{aligned}$$

Note that the right-hand side of (14) is a lower bound for the right-hand side of (6), as discussed at the end of the previous section. Furthermore, observe that all terms in this lower bound (except V_{P_t}) are either known or can be computed from the measured data. Although the complexity of P_t is unlikely to be known exactly, one may incorporate an 'educated guess' into the *a priori* information; *e.g.* P_t may be allowed to be *at most twice as complex* as its model P_m (*i.e.* $V_{P_t} \leq 2V_{P_m}$). In addition to being computationally tractable, the effect of complexity on achieved worst case performance is clearly visible in this new bound. In particular,

- The frequency separation $|\omega_{i+1} - \omega_i|$ should be small wherever $\kappa(P_m, P_t)(e^{j\omega_i})$ is large and/or $\rho(P_m, C)$ is small. This will reduce the effect of complexity terms on the bound.
- An increase in the complexity of plant or model or controller worsens the lower bound on achieved performance. If this lower bound is poor, one may consider re-designing a controller with lower complexity (*e.g.* using the technique suggested in [6]) or obtaining a plant model P_m with lower complexity.

To conclude, it is interesting to consider more closely the term V_{-C^*} in (9), which was simply considered to be a measure of controller complexity, without explanation. Note that it is not clear that $V_{-C^*} = V_C$ in general. However, by considering the behaviour of frequency-domain symbols on the unit circle only (which is all that is done in this paper), and since in this case $\tilde{G}_C^*$ and G_C^* can be considered to be normalised graph symbols (now left and right invertible in $\mathcal{L}_\infty$) of $-C^*$, it follows that

$$\begin{aligned} V_{-C^*} &= \|\tilde{G}_{-C^*}(G_{-C^*})'\|_\infty = \|G_C^*(\tilde{G}_C^*)'\|_\infty \\ &= \|\tilde{G}_C' G_C\|_\infty =: \tilde{V}_C, \end{aligned}$$

where the norms here all correspond to the one on $\mathcal{L}_\infty$, and the third equality holds because given an $X \in \mathcal{RL}_\infty$, $-z^2(X^*)' = (X')^*$ and $\|\frac{-1}{z^2}X\|_\infty = \|X\|_\infty = \|X^*\|_\infty$. Hence, mimicking the proof of Lemma 5.4 in [3], it also follows that

$$\kappa(C(e^{j\omega_1}), C(e^{j\omega_2})) \leq \tilde{V}_C |\omega_1 - \omega_2|.$$

That is, $\tilde{V}_C$ can be thought of as measuring the complexity of C in the same way as V_C. In fact, for a single-input single-output C or a diagonal multiple-input multiple-output C, it may be easily seen that $\tilde{V}_C = V_C$.

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