STABILITY OF A NETWORKED CONTROL SYSTEM USING LINEAR MATRIX INEQUALITIES

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Abstract

In this paper we study the stability of a networked control system using the stability theory of Lyapunov. The framework considered in this work is to allow in the discrete dynamics of the networked system unknown delays and some dropping in the samples between sensor and actuators. Solving a linear matrix inequality can demonstrate the stability of a system with time delays, and a bilinear matrix inequatily the stability with data dropping. As a case study, a simple integrator is used as a example.

1 Introduction

Computer-controlled systems started to emerge in the 1950s [14]. At the beginning stage, the potential of using digital computers as control system components was limited, since computers were too big, consumed too much power and were not highly reliable. The early implementation of computers in control systems operated in supervisor mode, either as an operator guide or as a set-point control. Ordinary analog-control equipment were needed in both cases. In the 1990s, the development of the microprocessor had a profound impact on the way computers are applied to control entire production plants. Furthermore, sensors and actuators could be equipped with network interfaces, and thus become independent nodes on a real-time control network

Feedback control systems wherein the control loops are closed through a data communication network are known as networked control systems (NCS) [7, 9, 12, 13, 14]. Sensors, controller and actuators are nodes of the data network, therefore all the signals in such a control system, reference input, plant output and control input, are exchanged as data packets using the shared communication network of Figure (1). Advantages of NCSs are low cost, simple installation, easy diagnosis and maintenance and high reliability. Nowadays NCSs are very common.

The insertation of the communication network in the feedback control loop makes the analysis and design of NCSs complex. Conventional control configurations use point-to-point links, in which the delays sensor-to-controller and controller-to-actuator are null. Data networks connect devices to an unreliable transmission path, this means that the information can suffer delays and even be lost. Delays are either constant or time varying and degrade the performance of control systems and can destabilize the system. There are two delays in the system: delay in any communication between sensor and controller, τ_k^{sc}, and delay in any communication between controller and actuator, τ_k^{ca}. Any computational delay can be absorbed into either τ_k^{sc} or τ_k^{ca} without lost of generality. Furthermore, Dropping network packets occasionally happens on an NCS due to node failures or message collisions.

Several previous authors have suggested different NCSs setups [5, 9, 14]. Sensor node is clock-driven, by this, we mean that the node samples with period h one o more variables and sends the information all together in one data packet to the controller. Controller and actuator are event-driven, this is, they start its activity when a data packet is received. The total delay between sensor and controller is less than the sampling period, $\tau_k = \tau_k^{sc} + \tau_k^{ca} < h$ as Figure (2) shows. The Lyapunov stability [3, 10, 14] concers the asymptotic behavior of the state of an autonomous dynamical system. The

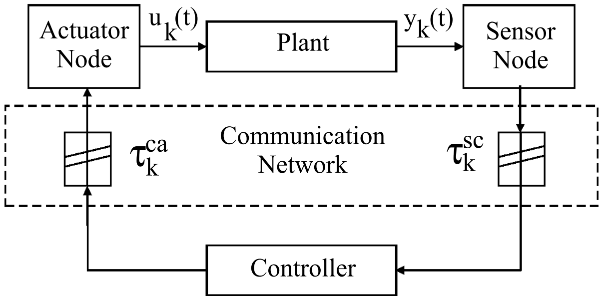

Figure 1: Network control system.

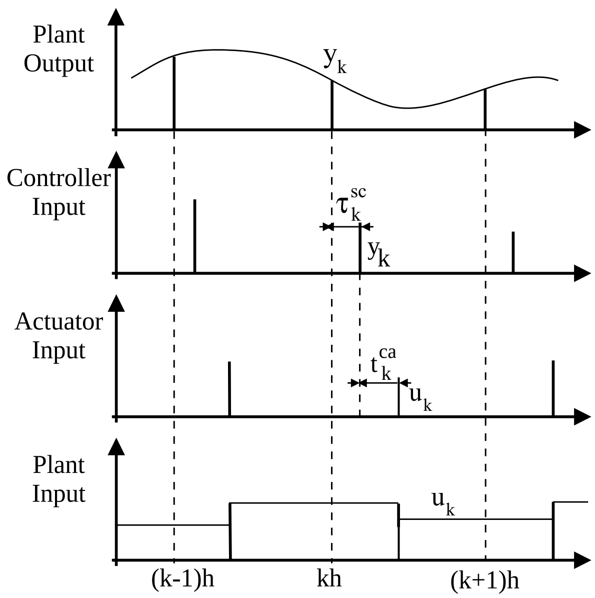

Figure 2: Delays in the feedback loop.

main contribution of Lyapunov has been that stability of such systems can be verified in terms of existence of functions, called Lyapunov functions. For the general class of nonlinear systems there are no systematic procedures for finding such functions. However, for linear systems the problems of finding Lyapunov functions can be solved adequately as a feasibility test of a linear matrix inequality (LMI) [2, 4, 8, 10, 11]. LMIs problems can be efficienty solved using widely available software. We compute Lyapunov functions to prove some level of performance of the systems to study in the presence of delays between sensors and actuators. Sections 3 and 4, study the stability with constant and varying delays use Lyapunov functions.

Asynchronous dynamical systems (ADS) [6, 13, 14] incorporate continuous and discrete dynamics. In section 5, these systems successful modelate data dropout of samples in a NCS. It allows to analyze the rate at which the data should be transmitted to achieve the desired performance. Now compute a Lyapunov function for an ADS is cast to a optimization problem involving bilinear matrix inequalities (BMI).

2 NCS description

In Figure (1) the NCS is illustrated in a block diagram. The continuous controlled plant is assumed to be

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1}$$

and the linear feedback state controller is

$$u(t) = -Kx(t) \tag{2}$$

where $x(t) \in \mathcal{R}^n$ is the state of the plant, $u(t) \in \mathcal{R}^m$ the controlled input and $y(t) \in \mathcal{R}^n$ the output; A, B, C and K are matrix of appropiate sizes. We assume that the control system can be designed without having the network in mind. That is, the original continuous plant and the continuous state feedback controller without the network connection is stable or satisfies certain control specifications.

Even though the delay between sensor to controller can be known, the controller does not take it into account. Thus, the influence from the network is collected in the total delay between sensor and actuator, $\tau_k = \tau_k^{sc} + \tau_k^{ca}$. The variable τ_k is assumed to be less than the sampling interval h. If this condition is not satisfied, the data will be removed as in section 5. The communication network gives total delays between sensor and actuator that are totally random and the network messages are received in the order they were generated. Now Equations (1) and (2) are rewritten as

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \in [kh + \tau_k, (k+1)h + \tau_{k+1}],$$
$$y(t) = Cx(t),$$
$$u(t^+) = -Kx(t - \tau_k),\ t \in \{kh + \tau_k, k = 0, 1, 2, \ldots\} \tag{3}$$

Two control samples u_{k-1} and u_k will be applied during the kth sampling period as in Figure (2). Then, Equations (3) sampled with period h are [1]

$$x_{k+1} = \Phi x_k + \Gamma_0(\tau_k)u_k + \Gamma_1(\tau_k)u_{k-1},$$
$$y_k = Cx_k.$$

where

$$\Phi = e^{Ah},$$
$$\Gamma_0(\tau_k) = \int_0^{h-\tau_k} e^{As}Bds,$$
$$\Gamma_1(\tau_k) = e^{A(h-\tau_k)} \int_0^{\tau_k} e^{As}Bds.$$

If we define $z_k = [x_k^T, u_{k-1}^T]^T$ as the augmented state vector, the new state is

$$z_{k+1} = \bar{\Phi}(\tau_k)z_k + \bar{\Gamma}(\tau_k)u_k = \begin{pmatrix} \Phi & \Gamma_1(\tau_k) \\ 0 & 0 \end{pmatrix} z_k + \begin{pmatrix} \Gamma_0(\tau_k) \\ 1 \end{pmatrix} u_k, \tag{4}$$

or

$$z_{k+1} = \Phi_z(\tau_k) z_k \tag{5}$$

where

$$\Phi_z(\tau_k) = \begin{pmatrix} \Phi - \Gamma_0(\tau_k)K & \Gamma_1(\tau_k) \\ -K & 0 \end{pmatrix}. \tag{6}$$

We get a discrete time linear time variant system.

3 Stability with constant delays

The simplest model of the network delay is to model it as being constant for all the transfers in the communications network. Even if the network has varying delays, the worst case delay can be used in the analysis using buffers in the input of the plant.

When the delay is constant $\tau = \tau_k, \forall k$, Equations (5) and (6) are

$$z_{k+1} = \Phi_z(\tau) z_k, \tag{7}$$

$$\Phi_z(\tau) = \begin{pmatrix} \Phi - \Gamma_0(\tau)K & \Gamma_1(\tau) \\ -K & 0 \end{pmatrix}, \tag{8}$$

all the coefficients of $\Phi_z(\tau)$ are constant and therefore the system is still time invariant and the analysis is simplifyed.

The stability triangle [1] can be used to explicitly calculate the relations between τ and h. If the system is complex, it may be analytically infeasible to calculate the exact stability region. However, stability regions can be determined by simulation and the Lyapunov stability.

A discrete time linear time invariant (DTLTI) system as in Equations (7) and (8) is said stable in the sense of Lyapunov if there exists a Lyapunov function $V(z_k)$ such that $V(z_{k+1}) - V(z_k) < 0,\ \forall z_k$ with $z_0 \neq \Theta$. If a DTLTI system is asymptotically stable, there exist a quadratic Lyapunov function $V(z_k) := z_k^T P z_k > 0,\ P \in S^n,\ \forall z \neq \Theta$. This can be rewriting as an LMI:

$$V(z_{k+1}) - V(z_k) = z_{k+1}^T A^T P A z_k - z_k^T P z_k =$$

$$= z_k^T (A^T P A - P) z_k < 0,\ \forall z_k \neq \Theta$$

equivalent to

$$A^T P A - P \prec \Theta \tag{9}$$

The stability region is plotted by incrementally increasing the delay, τ, and testing the existence of a Lyapunov matrix, P. The Matlab's LMI control toolbox [4] is used to to find P and demostrate que system is stable. A point is marked in the location of the stability region.

If we considerer the following integrator example

$$\dot{x}(t) = u(t),\ u(t) = -x(t),$$

Equation (8) is

$$\Phi_z(\tau) = \begin{pmatrix} 1 - (h-\tau)K & \tau \\ -K & 0 \end{pmatrix}, \tag{10}$$

and its stability region, points that satisfyes $\Phi_z(\tau)^T P \Phi_z(\tau) - P \prec \Theta$, is plotted in Figure (3).

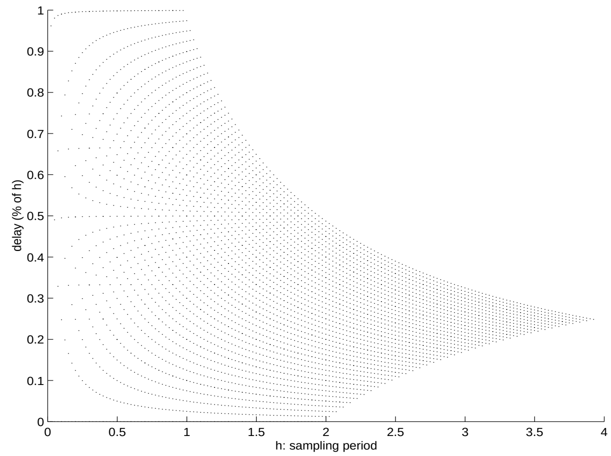

Figure 3: Stability region of the integrator

4 Stability with varying delays

Normally, network delays are usually random due to several reasons: waiting for the network to be idle, transmision and waiting for queued messages, and retransmisions because of errors and collisions. If the delays are varying, the NCS represented by Equation (5) and (6), is a DTLTV system.

If an DTLTV system can be formulated as

$$z_{k+1} = A z_k,\ z_k \in \mathcal{R}^n,\ A \in \mathcal{A} \tag{11}$$

where $\mathcal{A}$ is an closed convex set, $\mathcal{A} = \text{co}[A_1, \ldots, A_N]$, this system is robustly stable if (11) is asymptotically stable for all $A \in \mathcal{A}$.

Lyapunov quadratic stability can be applied. The following statements are equivalent:

a) The system in Equation (11) where $A \in \text{co}[A_1, \ldots, A_N]$ is quadratically stable

b) There exists $P \in S^n$ such that the LMI

$$A_i^T P A_i - P \prec \Theta \tag{12}$$

is feasible.

c) There exists $P \in S^n$ such that the LMI

$$\begin{bmatrix} P & A_i^T P \\ P A_i & P \end{bmatrix} \succ \Theta \tag{13}$$

is feasible.

Furthermore, if the system of Equation (11) is quadratically stable then it is also robustly stable.

The Schur complement lemma converts a class of convex nonlinear inequalities as in Equation (12) to an LMI as in Equation (13). The Schur complement lemma says

$$\begin{matrix} R \succ \Theta \\ Q - S R^{-1} S \succ \Theta \end{matrix} \Leftrightarrow \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ \Theta$$

thus, an equivalent alternative form for $P \succ \Theta$, $A^T P A - P \prec \Theta$ is

$$\begin{bmatrix} P & A^T P \\ P A^T & P \end{bmatrix} \succ \Theta$$

Therefore, it can be demostrated that if Equation (12) is satisfyed for $\{A_1, \ldots, A_N\}$, also is satisfyed for $A \in \text{co}[A_1, \ldots, A_2]$. A can be written as

$$A = \sum_{i=1}^{N} \lambda_i A_i$$

where $\sum_{i=1}^{N} \lambda_i = 1$, and multiplying each inequality by λ_i

$$\begin{bmatrix} \lambda_i P & (\lambda_i A_i^T) P \\ P(\lambda_i A_i^T) & \lambda_i P \end{bmatrix} \succ \Theta, \ \forall i = 1, \ldots, N. \quad (14)$$

and summing up all the above inequalities provides

$$\begin{bmatrix} \sum_{i=1}^{N} \lambda_i P & \sum_{i=1}^{N} (\lambda_i A_i^T) P \\ P \sum_{i=1}^{N} (\lambda_i A_i) & \sum_{i=1}^{N} \lambda_i P \end{bmatrix} = \begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} \succ \Theta \quad (15)$$

which implies that A is stable.

Returning to the integrator example, the coefficients of its 2x2 matrix, $\Phi_z(\tau_k)$, are depicted in Figure (4). This is a very simple case, where $\Phi_z(\tau_k)$ changes as a straight line. The integrator system with varying delays is rewritten as

$$\Phi_z(\tau_k) = \lambda \Phi_z(0) + (1 - \lambda) \Phi_z(h), \ \lambda \in [0, 1]$$

and the quadratic stability condition is

$$P \succ \Theta$$

$$\Phi_z^T(0) P \Phi_z(0) - P \prec \Theta$$

$$\Phi_z^T(h) P \Phi_z(h) - P \prec \Theta$$

If such P exists, it exists for the convex set formed by $\Phi_z(0)$ and $\Phi_z(h)$ and the integrator with varying delays is stable irrespective of how fast the time variations of $\Phi_z(\tau_k)$ take place. In fact, $V(z) := z^T P z$ serves as a Lyapunov function for this time-varying system.

Conventionally, a faster sampling rate is desirable in sampled-data systems so the discrete-time control design and performance can aproximate that of the continuous system. But in NCS, a faster sampling rate means a increasing in the network load, which in turn results in a longer delay of signals. We must find the sampling rate that can both tolerate the network-induced delay and achieve the best performance.

After some tests, we found that the maximum sampling rate, h_{max}, which allows stability with varying delays between 0 and h, is 0.9 s. The LMIs are feasible for this value of sampling rate. The P of Lyapunov found is

$$P = \begin{pmatrix} 1.0000 & 0.4600 \\ 0.4600 & 0.8219 \end{pmatrix}$$

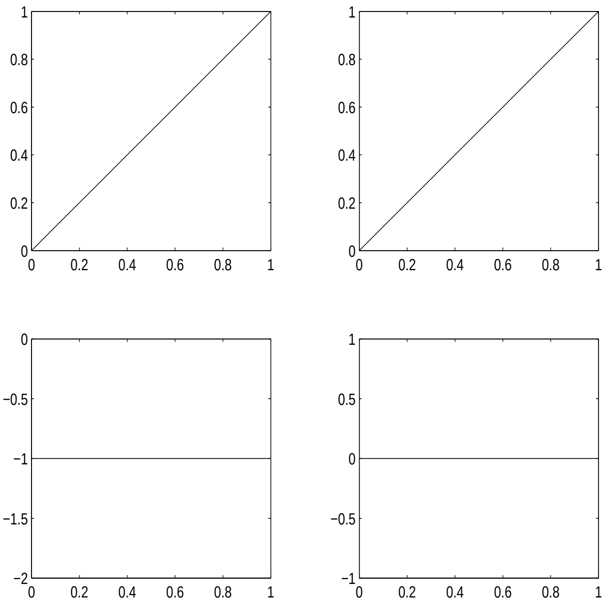

Figure 4: Network control system.

5 Stability with data dropout

Not all the networks can garantee that the delay will be less than a known value. If the delay is longer than one sampling period ($\tau_k > h$), it may be advantageous to discard the old sample and wait for the next one. In this case, only one control input ($u_k = u_{k-1}$) will be applied during the kth sampling period as Figure (5) shows. Furthermore, network are unreliable data transmision paths, where data are dropped occasionally due to packet collision or network node failure. Feedback controlled plants can tolerate a certain amount of data loss, thus it is valuable to determine whether the system is stable when only receiving the data at a certain rate.

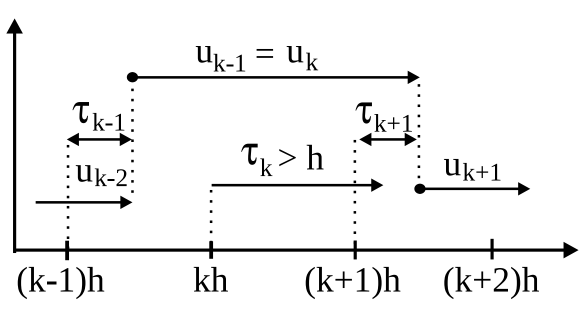

Figure 5: Data dropout on the input signal.

An NCS with data dropout can be modeled as an asynchronous dynamical system (ADS) with rate constraints on events. ADSs, like hibrid systems, are systems that incorporate continuous and discrete dynamics. The continuous dynamics are governed by differential or difference equations, and the discrete dynamics are governed by a finite automata that are driven asynchronously by external discrete events with fixed rates [6].

Data dropout can be easily represented in an ADS system with two events as in Figure (6). The switch closes at a certain rate r_1 (event E_1, position s_1) and data is transmitted and arrives on time ($\tau_k < h$). The switch is open at a rate $r_2 = 1 - r_1$ (event E_2, position s_1) and data arrives on late ($\tau_k > h$). In this last case, the output of the switch is held at the previous value and the data is lost.

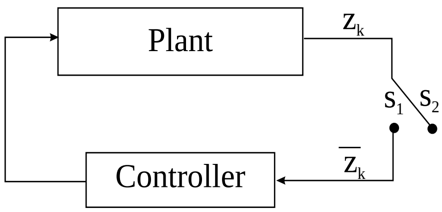

Figure 6: Dropout in input signal.

If the continuous dynamics of the ADS is governed by difference equations, the system is exponentially stable if

$$\lim_{k\to\infty} \alpha^k \|x_k\| = 0 \tag{16}$$

for some $\alpha > 0$. The largest α is referred to as the decay rate of the system. In what follows it is shown a Lyapunov argument to compute bounds on the decay rate of an ADS.

Suppose a Lyapunov function $V : R^n \to R_+$ continuosly differentiable and

$$\beta_1\|x_k\|^2 \leq V(x) \leq \beta_2\|x_k\|^2 \tag{17}$$

where $\beta_{1,2} > 0$. If the evolution of the state is given by two difference equations, $x_{k+1} = f_s(x_k)$ where $s \in \{1, 2\}$, and r_1 and r_2 are its rates, a sufficient condition for exponential stability of the ADS is the existence of an V as in (17) and two scalars $\alpha_{1,2} > 0$ satisfying

$$V(x_{k+1}) - V(x_k) \leq (\alpha_s^{-2} - 1)V(x_k),\ s \in \{1, 2\} \tag{18}$$

and

$$\alpha_1^{r_1}\alpha_2^{r_2} > \alpha > 1$$

This last equation can be equivalently written as

$$r_1 \log\alpha_1 + r_2 \log\alpha_2 > \log\alpha > 0 \tag{19}$$

Under these conditions the ADS satisfyes Equation (16) and the system is exponentially stable on the average. On the average means that it is not require that every difference equation of the ADS have to decrease monotonically at some rate α, but rather it guarantees the ADS will be stable on the whole.

The proof is as follows. From any sampling instant, t_k, until the following one, t_{k+1}, only one event occurs, either E_1 or E_2, then from Equation (18) we get

$$\frac{V(x_{k+1})}{V(x_k)} \leq (\alpha_s^{-2} - 1),\ s \in \{1, 2\}$$

or

$$\log V(x_{k+1}) - \log V(x_k) \leq -2\log\alpha_s \tag{20}$$

Notice that whenever an event E_i occurs we have a contributing term α_i at the right hand side of Equation (20). Hence, summing up these inequialities for $k = 1, 2, \ldots, K - 1$ gives

$$\log V(x_k) - \log V(x_0) \leq -2\log\alpha_1 T_1 - 2\log\alpha_2 T_2$$

where T_i is the number of times E_i occurs. In the limit, T_i is equal to $r_i k$ as $K \to \infty$. Therefore,

$$\log V(x_k) - \log V(x_0) \leq -2\log\alpha_1 r_1 k - 2\log\alpha_2 r_2 k$$

or by Equation (18)

$$\log V(x_k) - \log V(x_0) \leq -2\log\alpha k$$

so that

$$V(x_k) \leq e^{-2\log\alpha k} V(x_0)$$

Now using Equation(17) we get $V(x_0) \leq \beta_2\|x_0\|^2$ and $\beta_1\|x_k\|^2 \leq V(x_k)$, or

$$\alpha^k\|x_k\| \leq \sqrt{\frac{\beta_2}{\beta 1}}\|x_0\|$$

or

$$\lim_{k\to\infty} \alpha^k\|x_k\| = 0$$

Let the state of the ADS, $w_k = [z_k^T,\ \bar{z}_k^T]^T$, be the input and the output of the switch of Figure (6), and its dynamics $w_{k+1} = \Phi_{w,i}\ w_k$. From Equation (4), when the switch is in position s_1

$$\Phi_{w,1}(\tau_k) = \begin{bmatrix} \bar\Phi(\tau_k) & -\bar\Gamma(\tau_k)K \\ \bar\Phi(\tau_k) & -\bar\Gamma(\tau_k)K \end{bmatrix}$$

whereas if the position is s_2

$$\Phi_{w,2}(\tau_k) = \begin{bmatrix} \bar\Phi(\tau_k) & -\bar\Gamma(\tau_k)K \\ 0 & I \end{bmatrix}$$

For the integrator example, its dynamics when the switch is in position s_1 is

$$\Phi_{w,1}(\tau_k) = \lambda_i\Phi_{w,1}(0) + \lambda_j\Phi_{w,1}(h) =$$

$$= \lambda_i \begin{bmatrix} \bar\Phi(0) & -\bar\Gamma(0)K \\ \bar\Phi(0) & -\bar\Gamma(0)K \end{bmatrix} + \lambda_j \begin{bmatrix} \bar\Phi(h) & -\bar\Gamma(h)K \\ \bar\Phi(h) & -\bar\Gamma(h)K \end{bmatrix} \tag{21}$$

for any λ_1, λ_2 such $\lambda_1 + \lambda_2 = 1$. If the switch is in s_2

$$\Phi_{w,2} = \begin{bmatrix} \bar\Phi(h) & -\bar\Gamma(h)K \\ 0 & I \end{bmatrix} \tag{22}$$

The stability condition by Equations (18), (19), (21) and (22) is

$$\alpha_1^{r_1}\alpha_2^{r_2} > 1$$

$$\Phi_{w,1}^T(0) P \Phi_{w,1}(0) \leq \alpha_1^{-2} P$$

$$\Phi_{w,1}^T(h)P\Phi_{w,1}(h) \leq \alpha_1^{-2}P$$
$$\Phi_{w,2}^T P\Phi_{w,2} \leq \alpha_2^{-2}P \quad (23)$$

The integrator is found stable on the average for $h = 0.5$ and a dropout rate (r_2) of 20% with the following solutions to Equations (23)

$$\alpha_1 = 0.7852,\ \alpha_2 = 2.6307$$

$$P = \begin{pmatrix} 1.0000 & 0.2356 & 1.2565 & 0.9874 \\ 0.2356 & 0.6983 & 2.5845 & 1.4223 \\ 1.2565 & 2.5845 & 1.1764 & 3.1209 \\ 0.9874 & 1.4223 & 3.1209 & 1.2236 \end{pmatrix}$$

6 Conclusions

This paper studied the two main issues in network control systems. The first one is the network-induced delay when transmiting sensor and control data. The delay can be either constant or time varying. If the delay is constant the relationship between the sampling rate h and the network-induced delay τ was showed using a stability plot. If the delay is time varying, an LMI has been used to demostrate the quadratic stability. Then we modeled the packet dropout as an asynchronous dynamical system. We determine whether the NCS is stable at a certain rate of data loss.

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