ENCODERS AND FAULT DETECTION

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Abstract

Monitoring large-scale systems is of fundamental importance in modern infrastructures. Many of these large-scale systems are complex interconnections of sub-components which interact by means of communication channels with limited bandwidth. Therefore the information must be encoded in order to be transmitted. Given a class of encoders, the problem of detecting faults affecting some of these sub-components starting from the encoded information is studied, and a precise characterization of a class of faults which can be detected is given.

1 Introduction

We are interested in the problem of detecting faults which occur in systems of the form

$$x(k+1) = Ax(k) + Bu(k) + Mm(k)\ , \quad k \geq 0\ , \qquad (1)$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^m$ is a vector-valued and measured input signal and $m(\cdot) : \mathbb{Z}_+ \to \mathbb{R}$ is a fault signal. System (1) can be interpreted as a model of one of many sub-components which comprise a complex system. By fault it is meant a signal which is identically zero from $k = 0$ through the unknown time $\bar{k}-1$ and which becomes non zero for the first time at $\bar{k}$. The time behavior of $m(\cdot)$ is otherwise unknown. The (fault) vector M is nonzero to avoid triviality.

Were the measurements of system (1) available in the (non-quantized) form

$$y(k) = Cx(k)\ , \qquad (2)$$

the solution of the problem of detecting the occurrence of the fault would be a straightforward one. Indeed, in the hypothesis of (C, A) observable, the diagnostic filter

$$\begin{array}{rcl} \xi(k+1) & = & (A-GC)\xi(k) + Bu(k) + Gy(k) \\ r(k) & = & y(k) - C\xi(k)\ , \end{array} \qquad (3)$$

where $r(\cdot)$ is the diagnostic signal (*residual*) and G is the gain matrix for which all the eigenvalues of $A - GC$ lie within the unitary ball, solves the detection problem (see e.g. [11]). Namely, the residual $r(\cdot)$ is zero (or, more realistically, asymptotically converges to zero) when no fault is present, and becomes nonzero when the fault occurs. The proof of this result rests on showing that the device (3) gives rise to a law

$$r(\cdot) = \varphi(y|_{[0,\cdot]}, u|_{[0,\cdot]})$$

which cascaded to the system makes the fault-residual map injective[1], that is the (linear) map from $m(\cdot)$ to $r(\cdot)$ is one-to-one and as such non zero signals $m(\cdot)$ necessarily yield non zero signals $r(\cdot)$.

In the present paper, however, we are concerned with measurements which must be transmitted through communication channels in order to be evaluated. This means that signal $y(k)$ is not directly accessible by the diagnostic device (possibly located at a remote location) but must be encoded using a finite set $\mathcal{A}$ of $2^R + 1$ words or symbols, where R is a design parameter[2]. Control and estimation with quantized and/or encoded measurements has attracted a good deal of interest in the recent years (see, to cite a few, [8, 7, 6, 17, 18, 14, 2, 9, 16, 19, 20, 3, 1, 13] and references therein). Fault detection for quantized systems has been studied for instance in [10, 15]. Here we consider a purely deterministic framework in which the parameters of the quantizer on which the encoder is based can be adjusted on-line, following in this respect previous work (cf. e.g. [9, 17, 14]). In particular, of the many possible available approaches to tackle the problem, we adopt here the point of view of [17] which is particularly suitable to our purposes.

Encoding signal $y(k)$ is carried out by a device (*encoder*)

$$s(\cdot) = \varepsilon(y|_{[0,\cdot]}, u|_{[0,\cdot]})\ ,$$

which outputs $s(k) \in \mathcal{A}$, one of the $2^R + 1$ possible words. Therefore, in the case of quantized measurements, the detection device must rely on $s(\cdot)$ rather than $y(\cdot)$, and it will be better described by a law of the form

$$r(\cdot) = \varphi(s|_{[0,\cdot]})\ .$$

Before being able to provide such a detection device, we specify the class of encoders (the law ε which maps y and u into

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[1] In the case of linear continuous-time systems, an elegant proof of this fact, in weaker hypothesis, has been provided in [12]. In the case of discrete-time linear systems, this well-known result can be given an elementary and standard proof which is reported in the Appendix for the sake of completeness.

[2] Of course, if the communication channel has a limited bandwidth (BW bits/sec), then R and the available bandwidth of the channel are closely intertwined: The number $\log_2(2^R + 1)$ must be less than or equal to BW.

s) which is of interest in this paper. This is done in the next section. The definition of the fault detection problem with encoded full-state information is introduced in Section 3, where a solution is also proposed. The fault detection problem in the case of encoded partial-state information is dealt with in Section 4. Conclusions are drawn in Section 5.

2 Encoder

Assumption. First of all we introduce an assumption (which can be relaxed) concerning the dynamic matrix of system (1):

Assumption 1 *A is in Jordan canonical form and has all real eigenvalues.*

In other words, we consider henceforth the following form for A:

$$A = \text{block.diag}(J_1, \ldots, J_\nu)$$

where the J_i's are Jordan blocks, i.e.

$$J_i = \begin{pmatrix} \lambda_i & 1 & \ldots & 0 \\ 0 & \lambda_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}, \sum_{j=1}^{\nu} n_j = n \ .$$

Associated to A we also define the matrices

$$\bar{A} = \text{block.diag}(\bar{J}_1, \ldots, \bar{J}_\nu) \ ,$$

with

$$\bar{J}_i = \begin{pmatrix} |\lambda_i| & 1 & \ldots & 0 \\ 0 & |\lambda_i| & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & |\lambda_i| \end{pmatrix} \ ,$$

and

$$F = \text{block.diag}(F_1, \ldots, F_\nu) \ ,$$

$$F_i = \begin{pmatrix} 1/2^{R_i} & 0 & \ldots & 0 \\ 0 & 1/2^{R_i} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1/2^{R_i} \end{pmatrix} \in \mathbb{R}^{n_i \times n_i} \ .$$

The integers R_i, $i = 1, \ldots, \nu$, satisfy the inequality ([17])

$$R_i > \max\{0, \log_2 |\lambda_i|\} \ . \tag{4}$$

We now turn our attention to the encoder. Were we free to design an encoder only for monitoring purposes, the problem would be easily solvable using a simple set $\mathcal{A}$ with two words. For instance, in the case of full-state measurements, we might easily think of an encoder of the form

$$\begin{array}{rcl} \bar{x}(k+1) & = & A\bar{x}(k) + Bu(k) \ , \quad \bar{x}(0) = x(0) \\ s(k) & = & \left\{ \begin{array}{ll} 0 & \text{if } \ x(k) - \bar{x}(k) = 0 \\ 1 & \text{if } x(k) - \bar{x}(k) \neq 0 \end{array} \right. \ . \end{array} \tag{5}$$

A decision device would infer the occurrence of the fault by simply assessing the value of $s(k)$ ($s(k) = 1$ fault, $s(k) = 0$ no fault). Notice however that we do *not* design an encoder for fault detection. Our interest here is investigating the possibility of detecting faults using encoded information *given a specific encoder*. This point of view stems from the fact that an encoder must help for a wide range of purposes such as observation and control, and not only monitoring. Of course we might append device (5) to an existing encoder, but this would result not only in a more complex encoder but also in the need for an additional bit to encode the information generated by the "fault encoder", thus contradicting the principle of keeping the number of bits needed for a reliable transmission as small as possible. Thus, in the following we are going to introduce an encoder already presented in the literature and dissert on the possibility of inferring fault occurrence using only information encoded by this device.

Quantizer. The functioning of the encoder is heavily based on the quantizer. This is a device which at each time step divides the state space $\mathbb{R}^n$ into a finite set of (quantization) regions (in this case, $2^R + 1$ regions) and then associates to the state at that time the word in $\mathcal{A}$ representing the region to which the state belongs. In formula, a quantizer is simply a map

$$\mathcal{Q} : \mathbb{R}^n \to \mathcal{A} \ .$$

There are many ways to partition the state space into regions and many choices for the quantizer. As usual in this paper, we choose to follow the simplest way suggested in [17]. At each time step k a hyper-rectangle (quantization region) $\Omega(k) \subset \mathbb{R}^n$ is defined by specifying its centroid $CNT(k) \in \mathbb{R}^n$, range vector $L(k) \in \mathbb{R}^n$ ($L_i(k)$, the i-th component of vector $L(k)$, gives the length of the i-th side of the hyper-rectangle). The quantization region $\Omega(k)$ is then *uniformly* partitioned into 2^R subregions $\Omega_i(k)$, where $R = \sum_{i=1}^{\nu} n_i R_i$, and the values R_i are those required to satisfy the inequality (4). (Notice that, while CNT and L are time-varying quantities, the rate vector is taken constant for all the times.) In particular, for each $i = 1, \ldots, n$, the i-th side of the hyper-rectangle is uniformly divided into R_{ℓ_i+1} parts, where ℓ_i is a positive integer satisfying $0 \leq \ell_i \leq \nu - 1$ and such that the the index i is equal to $n_1 + \ldots + n_{\ell_i} + j$, with $n_0 = 0$ and $1 \leq j \leq n_{\ell_i+1}$. If the purpose is to encode the state $x(k)$ and this lies within the sub-region $\Omega_i(k)$ for some $i = 1, \ldots, 2^R$, then the encoder will generate a symbol $s(k)$ which is (for instance) the binary representation of the index i. On the other hand, if the state $x(k)$ does not lie within the quantization region $\Omega(k)$, that is no index $i = 1, \ldots, 2^R$ exists such that $x(k) \in \Omega_i(k)$, then an overflow symbol will be generated, that is $s(k)$ will be the binary representation of the index 0.

Encoder. The role of the encoder is that of updating the values of the quantities CNT, L at each time step ([14, 17]). Before writing the update laws, we introduce a new assumption which will be relaxed later on.

Assumption 2 $C = I_n$.

The update laws are as follows ([17]):

$$\begin{array}{rcl} CNT(k+1) & = & A\hat{x}(k)+Bu(k) \\ L(k+1) & = & \bar{A}FL(k) \ , \end{array} \qquad (6)$$

with initial conditions $CNT(0) = 0$ and $L_i(0) \geq 2|x_i(0)|$ for each $i = 1, 2, \ldots, n$. At each time $k \geq 0$, the quantity $\hat{x}(k)$ is the centroid of the sub-region $\Omega_j(k)$, for some $j = 1, 2, \ldots, 2^R$, in which the state $x(k)$ lies. Of course $\hat{x}(k)$ is well-defined only if the state $x(k)$ lies in the quantization region $\Omega(k)$. If this is actually the case, the i-th entry, with $i = 1, 2, \ldots, n$, of the difference $|x(k) - \hat{x}(k)|$ turns out to be less than or equal to $L_i(k)/2^{R_{\ell_i}+1}$ by construction, with index ℓ_i defined as before (cf. the description of the quantizer above).

3 Fault Detection

Having introduced the encoder, we are now ready to turn our attention to the problem of fault detection. We define more formally our goal:

Definition. Consider system (1) and encoder (6). The fault detection problem with encoded full-state information is said to be solvable with respect to a class $\mathcal{M}$ of faults if there exists a law

$$r(\cdot) = \varphi(s|_{[0,\cdot]}) \ ,$$

that, when cascaded to (1), (6), satisfies the properties:

(i) $r(\cdot) = 0$ if $m(\cdot) = 0$;

(ii) $r(\cdot) \neq 0$ if $m(\cdot) \in \mathcal{M}$ and $m(\cdot) \neq 0$.$\triangleleft$

We briefly explain the reason why the fault detection problem in the case of encoded measurements is expressed with respect to a specific class of faults. We have discussed in the introductory section how in the case of non-encoded measurements the observability property makes it possible to design a simple detection filter able to reveal *any fault* (at least theoretically) by creating a one-to-one map from the fault to the diagnostic signal. In the case of encoded measurements this is not possible anymore, and in particular there are faults which cannot be distinguished at all. This can be illustrated by means of the following simple example.

Example. Consider the scalar system

$$x(k+1) = ax(k) + m(k) \ ,$$

where $a = 2^\lambda$, $0 < \lambda < 1$ and $x(0) > 0$. Assume that starting from the time $\bar{k}$ at which the fault occurs the time behavior of the fault satisfies the following inequality:

$$-\frac{x(0)}{2^{(1-\lambda)k}} \leq \sum_{j=\bar{k}}^{k-1} a^{k-1-j} m(j) < 0 \ , \quad k > \bar{k} \ . \qquad (7)$$

There are several non-zero fault signals with this property. Encode the information $x(k)$ by means of the encoder (6), for which $L(0) = 2x(0)$ and $R = 1$. In particular any time the information lies in the interval $[CNT,\ CNT + L/2]$, it is encoded using the same symbol. Now consider the case in which no fault occurs in the system. In this *fault-free* case the dynamics of the system are of course those described by the equation

$$x_{ff}(k+1) = ax_{ff}(k) \ .$$

Choose $x_{ff}(0) = x(0)$ and use (6) to encode $x_{ff}(k)$ as well. It is not difficult to see that the sequences of words generated during the two (respectively, faulty and fault-free) experiments are *exactly* the same. This is readily derived from the equalities

$$CNT(k) + L(k)/2 = x_{ff}(k) \text{ for all } k \geq 0 \ ,$$

and

$$x(k) = x_{ff}(k) + \sum_{j=\bar{k}}^{k-1} a^{k-1-j} m(j) \text{ for } k > \bar{k} \ ,$$

and

$$L(k) = 2^{(\lambda-1)k+1} x(0) \ . \triangleleft$$

The previous result suggests that not all the faults can be detected, especially those whose magnitude is too small. In this section a result is stated which characterizes a class $\mathcal{M}$ of faults with respect to which the fault detection problem with encoded full-state information is solvable.

Proposition 1 *Let Assumptions 1 and 2 hold. There exists a positive real number $\bar{m}$ for which the law*

$$r(k) = \varphi(s(k)) = \begin{cases} 0 & \textit{if } \ s(k) \neq 0 \\ 1 & \textit{if } \ s(k) = 0 \end{cases}$$

solves the fault detection problem with encoded full-state information with respect to the class

$$\begin{array}{rl} \mathcal{M} = & \{m(\cdot) : \mathbb{Z}_+ \to \mathbb{R},\ \exists \bar{k} \in \mathbb{Z}_+ \ \bar{k} \neq 0 \text{ s.t.} \\ & |m(\bar{k})| \geq \bar{m} \text{ and } m(\bar{k}) = 0 \ \forall 0 \leq k < \bar{k}\} \ . \end{array}$$

Proof. We show first that property (i) in the definition of the fault detection problem with encoded full-state information holds[3]. To see this, notice that if $m(\cdot) = 0$, then the dynamics of the system becomes

$$x(k+1) = Ax(k) + Bu(k) \ . \qquad (8)$$

Also observe that by construction at time $k = 0$ the state lies within the quantization region $\Omega(0)$. By induction this is true for each $k \geq 0$. Indeed, let the following hold for some k:

$$|x_i(k) - CNT_i(k)| \leq L_i(k)/2 \ , \quad \forall i = 1, \ldots, n \ ,$$

[3]The first part of the proof is basically taken from [17]. We report it here for the sake of completeness and to introduce relations which will be used to prove that also property (ii) in the definition of the fault detection problem with encoded full-state information holds.

and consider the difference $|x_i(k+1) - CNT_i(k+1)|$. This satisfies (Assumption 1 is used here)

$$
\begin{aligned}
&|x_i(k+1) - CNT_i(k+1)| = |\sum_{j=1}^{n} a_{ij}(x_j(k) - \hat{x}_j(k))| \\
&\leq \sum_{j=1}^{n} \bar{a}_{ij}|x_j(k) - \hat{x}_j(k)| \ .
\end{aligned} \tag{9}
$$

Since by the inductive hypothesis $x(k) \in \Omega(k)$, we have

$$
|x_j(k) - \hat{x}_j(k)| \leq L_j(k)/2^{R_{\ell_j}+1} \ , \quad \forall j = 1, \ldots, n \ . \tag{10}
$$

Therefore

$$
\begin{aligned}
|x_i(k+1) - CNT_i(k+1)| &\leq \sum_{j=1}^{n} \bar{a}_{ij} \cdot L_j(k)/2^{R_{\ell_j}+1} \\
&= (\bar{A}FL(k)/2)_i \\
&= L_i(k+1)/2 \ ,
\end{aligned} \tag{11}
$$

that is $x(k+1)$ lies within the quantization region. Hence, so far as $m(\cdot) = 0$, no overflow word is generated by the encoder, that is $s(k) \neq 0$ for all $k \geq 0$, and as a consequence, $r(k) = 0$ for all $k \geq 0$.
We now turn to the proof of property (ii). Let $\bar{k} > 0$ be an integer for which $m(\bar{k}) = \bar{m} \neq 0$, with

$$
\bar{m} > \min_{\substack{1 \leq i \leq n, \\ M_i \neq 0}} \ \sup_{0<k<\infty} \left\{ \frac{L_i(k)}{|M_i|} \right\} , \tag{12}
$$

and $m(k) = 0$ for $0 \leq k < \bar{k}$. Now

$$
\begin{aligned}
&|x_i(\bar{k}+1) - CNT_i(\bar{k}+1)| \geq |M_i m(\bar{k})| - \\
&|\sum_{j=1}^{n} a_{ij}(x_j(\bar{k}) - \hat{x}_j(\bar{k}))| \ , \quad \forall i = 1, \ldots, n \ .
\end{aligned} \tag{13}
$$

Since at time $\bar{k}$ no fault is present, from the inequality in (16) and from (10) and (11), we have

$$
|\sum_{j=1}^{n} a_{ij}(x_j(\bar{k}) - \hat{x}_j(\bar{k}))| \leq L_i(\bar{k}+1)/2 \ .
$$

On the other hand $|m(\bar{k})| \geq \bar{m}$ and hence

$$
|x_{i^*}(\bar{k}+1) - CNT_{i^*}(\bar{k}+1)| > L_{i^*}(\bar{k}+1)/2 \ ,
$$

where i^* is the index at which minimum is achieved in (12). This means that at time $\bar{k}+1$ the system state is driven outside the quantization region and the encoder generates the overflow symbol, that is $s(\bar{k}+1) = 0$ and the fault is detected at this time. $\triangleleft$

Remark. The class $\mathcal{M}$ with respect to which the fault detection problem is solvable is not void. As a matter of fact, from the inequality (12), we see that $\bar{m}$ is finite and more specifically bounded from below by a function which is in turn bounded for all $k \geq 0$ and actually converges to zero as k tends to infinity. The same inequality along with inequality (13) point out how a fault which occurs later on is more likely to be detected. The rate of decay of the lower bound on $\bar{m}$ can be made faster by increasing the transmission rate, thus confirming that a higher transmission rate may improve the detection capability embedded in the encoder without modifying the structure of the encoder itself. $\triangleleft$

The example and the proposition in this section illustrate two limit situations. In the first one, the fault is such that the state of the system remains in the same quantization sub-region that it would have occupied if no fault have occurred. In the second situation, the fault steers the state of the system outside the quantization region and the occurrence of the fault becomes evident. The most intriguing situation is the intermediate one when the fault causes the state to jump into quantization sub-regions which are adjacent or otherwise close to the one the state would lie in the un-faulty situation. We do not address further this issue here and it will be the object of investigation in a different paper.

4 Fault Detection from Output Measurements

In this section we consider the same problem as in the previous section, except that we replace the knowledge of the full state $x(k)$ with the knowledge of the output $y(k) = Cx(k)$. In this case the problem is defined as the problem of fault detection with encoded partial-state information. To deal with this case, another class of encoders must be considered. These encoders present the following additional dynamics ([17]):

$$
\bar{x}(k+1) = A\bar{x}(k) + Bu(k) + G(y(k) - C\bar{x}(k)) \ , \tag{14}
$$

with G chosen so that $A - GC$ is asymptotically stable (we are assuming that (C, A) is observable). The state $\bar{x}$, which asymptotically estimates x, feeds the encoder (6), in the sense that now $\hat{x}$ is the centroid of the region $\Omega_i(k)$ where $\bar{x}(k)$ (and not $x(k)$) lies. Another modification of the encoder is needed for accommodating the case of partial-state information. Let $\mu < 1, \ell$ be positive numbers for which the inequality

$$
|GC(x(k) - \bar{x}(k))| \leq \ell\mu^k
$$

is satisfied for all $k \geq 0$. Then replace the second difference equation in (6) with the following ([17])

$$
L(k+1) = \bar{A}FL(k) + 2\ell\mu^k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \ .
$$

The overall structure of the encoder in this case then becomes

$$
\begin{aligned}
CNT(k+1) &= A\hat{x}(k) + Bu(k) \\
\bar{x}(k+1) &= (A - GC)\bar{x}(k) + Bu(k) + Gy(k) \\
L(k+1) &= \bar{A}FL(k) + 2\ell\mu^k \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T \ ,
\end{aligned} \tag{15}
$$

with initial conditions $CNT(0) = 0$ and $L_i(0) \geq 2|\bar{x}_i(0)|$, for each $i = 1, 2, \ldots, n$, and where $\hat{x}(k)$ is the centroid of the sub-region $\Omega_i(k)$ in which $\bar{x}(k)$ lies. If $\bar{x}(k) \in \Omega_i(k)$, the word $s(k)$ generated by the encoder is the binary representation of the index i, or it is the binary representation of 0 in the case

$\bar{x}(k) \notin \Omega(k)$, where $\Omega(k)$ is the quantization region at time k, that is the hyper-rectangle with centroid $CNT(k)$ and range vector $L(k)$ defined by equations (15).

A result substantially analogous to the case of full-state measurements holds.

Proposition 2 *Consider system (1) with partial-state measurements $y(k) = Cx(k)$ and encoder (15). Let Assumption 1 hold and assume the pair (C, A) to be observable. There exists a positive real number $\bar{m}$ for which the law*

$$r(k) = \varphi(s(k)) = \begin{cases} 0 & \text{if } s(k) \neq 0 \\ 1 & \text{if } s(k) = 0 \end{cases}$$

solves the fault detection problem with encoded partial-state information with respect to the class

$$\begin{aligned} \mathcal{M} = \ & \{m(\cdot) : \mathbb{Z}_+ \to \mathbb{R},\ \exists \bar{k} \in \mathbb{Z}_+\ \bar{k} \neq 0 \text{ s.t.} \\ & |m(\bar{k})| \geq \bar{m} \text{ and } m(\bar{k}) = 0\ \forall 0 \leq k < \bar{k}\}\ . \end{aligned}$$

Proof. It is easily derived from the proof of Proposition 1. In particular, if [4] $m(\cdot) = 0$, then it is proven by induction that $\bar{x}(k)$ lies within the quantization region at each time k. The argument by induction assumes that the thesis holds true at time k, that is

$$|\bar{x}_j(k) - \hat{x}_j(k)| \leq L_j(k)/2^{R\ell_j + 1}$$

for all $j = 1, \ldots, n$, and shows that the same holds true at time $k + 1$, as it is seen from the following inequality, which holds for each $i = 1, \ldots, n$:

$$\begin{aligned} & |\bar{x}_i(k+1) - CNT_i(k+1)| = |\textstyle\sum_{j=1}^n a_{ij}(\bar{x}_j(k) - \hat{x}_j(k)) \\ & + GC(x(k) - \bar{x}(k))| \leq (\bar{A}FL(k)/2)_i + \ell\mu^k = L_i(k+1)/2\ . \end{aligned} \tag{16}$$

Therefore, if $m(\cdot) = 0$, then $s(\cdot) = 0$ and also $r(\cdot) = 0$.
In the case $m(\cdot) \neq 0$, denote as before by $\bar{k} > 0$ the smallest integer for which $\bar{m} := m(\bar{k})$ is different from zero and let

$$\bar{m} > \min_{\substack{1 \leq i \leq n, \\ M_i \neq 0}} \ \sup_{0<k<\infty} \left\{ \frac{L_i(k)}{|M_i|} \right\} .$$

Analogously to what has been done in the proof of Proposition 1 one can see that, denoted by i^* the index at which the minimum in the latter expression is achieved, then

$$|\bar{x}_{i^*}(\bar{k}+1) - CNT_{i^*}(\bar{k}+1)| > L_{i^*}(\bar{k}+1)/2\ ,$$

which shows how at time $\bar{k}+1$ the state $\bar{x}(\bar{k}+1)$ does not belong to the quantization region and therefore an overflow symbol is generated revealing the occurrence of the fault. $\triangleleft$

In particular the expression for $\bar{m}$ is exactly the same as that in (12). However, because now the value of $L_i(k)$ is larger than before due to the presence of the forcing term

$$2\ell\mu^k \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T ,$$

$\bar{m}$ in the case of partial-state measurements will be larger than in the case of full-state measurements. That is, the use of partial-state rather than full-state measurements reduces the class $\mathcal{M}$ of detectable faults.

5 Conclusions

In this paper we have considered the problem of detecting faults starting from encoded measurements. A class of fault signals whose occurrence can be inferred from this kind of information has been clearly characterized. To ease the treatment we have considered discrete-time systems with a dynamic matrix in Jordan form. The latter assumption can be relaxed and the results can be adapted to the case of continuous-time systems following the approach in [9]. Extensions to more complex (nonlinear) classes of systems are also possible combining the nonlinear fault detection techniques of [4] with the approach of [9]. Modifications of existing encoders and/or decoders can enlarge the classes of faults detectable through quantized measurements. Furthermore, they can allow to deal with disturbances and noise in an “optimal” way, by discriminating between faults (we want to detect) and disturbances (we want to reject) and reducing the influence of noise in the detection process (cf. [5]). Using the methods of [11] and [4] it is also possible to deal with the case of multiple faults. We have considered here only the so-called ([17]) primitive quantizers and Class 1 encoders. Analogous considerations can be extended to different quantizers and encoders which have been presented in the literature. We have not extensively discussed the role of transmission rate in the process of detecting faults for a given encoder: One can easily envision the case in which a larger transmission rate can affect (improve) the fault detection process.

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A Appendix

Let $\xi(0) = x(0)$. Easy computations yield the following expression for $r(k)$, $k \geq 0$:

$$\begin{aligned} r(k) &= \sum_{j=0}^{k-1} CA^{k-1-j}Bm(j) \\ &= C\sum_{j=0}^{k-1} A^{k-1-j}Bm(j) \ . \end{aligned}$$

We aim to prove that $r(\cdot) = 0$ if and only if $m(\cdot) = 0$. The "if" part of the implication is trivial. As far as the "only if" part is concerned observe that, if $r(\cdot) = 0$ then

$$\phi(k) := \sum_{j=0}^{k-1} A^{k-1-j}Bm(j) \in \ker\{C\}$$

for all $k \geq 1$. Vector $\phi(k)$ belongs to the A-invariant subspace

$$\mathrm{span}\{B, AB, \ldots, A^{n-1}B\}$$

for all $k \geq 1$. On the other hand, because of the observability assumption, the largest A-invariant subspace contained in $\ker\{C\}$ is the origin $\{0\}$. Therefore, one concludes that

$$\phi(k) = \sum_{j=0}^{k-1} A^{k-1-j}Bm(j) = 0$$

for all $k \geq 1$. For $k = 1$, this yields $Bm(0) = 0$ and therefore $m(0) = 0$ (B is a nonzero vector). For $k = 2$, and keeping in mind that $m(0) = 0$, one obtains $Bm(1) = 0$, that is $m(1) = 0$. Iterating these arguments for all k, one obtains that $m(k) = 0$ for all $k \geq 0$.