COMPONENTWISE STABILIZABILITY AND DETECTABILITY OF LINEAR SYSTEMS

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Abstract

The componentwise asymptotic stability (CWAS) and componentwise exponential asymptotic stability (CWEAS), introduced and characterized in previous works, are used to define componentwise stabilizability / detectability and componentwise exponential stabilizability / detectability for both discrete- and continuous-time linear systems. The approach brings a substantial refinement to the classical concepts of stabilizability / detectability, by ensuring an individual monitoring of each state variable. Necessary and sufficient conditions are formulated for the existence of componentwise exponential stabilizability / detectability with guaranteed performance. These conditions represent the theoretical background for the synthesis procedures of CWEAS regulators / observers, which are developed as constrained nonlinear optimization problems, so as to ensure the computational tractability.

1. Introduction

Consider the linear dynamical system:

$$\begin{aligned} &\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \\ &\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), \\ &\boldsymbol{A} \in \boldsymbol{R}^{n\times n}, \boldsymbol{B} \in \boldsymbol{R}^{n\times m}, \boldsymbol{C} \in \boldsymbol{R}^{p\times n}, \boldsymbol{D} \in \boldsymbol{R}^{p\times m}, \\ &m \le n,\ p \le n \end{aligned} \tag{1}$$

where $t \in \mathbf{T}$ denotes the independent variable with *discrete-time* (DT) meaning $\mathbf{T} = \mathbf{Z}_+$, or *continuous-time* (CT) meaning $\mathbf{T} = \mathbf{R}_+$, and the action of the operator ( )' is defined by:

$$\boldsymbol{x}'(t) = \begin{cases} \boldsymbol{x}(t+1) & \text{for the DT case } t \in \mathbf{T} = \mathbf{Z}_+; \\ \dot{\boldsymbol{x}}(t) & \text{for the CT case } t \in \mathbf{T} = \mathbf{R}_+. \end{cases} \tag{2}$$

The stabilizability of system (1) refers to the dynamics:

$$(\boldsymbol{x}^{\mathrm{S}}(t))' = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}^{\mathrm{S}}(t),\quad \boldsymbol{x}^{\mathrm{S}}(t_0) = \boldsymbol{x}(t_0), \tag{3}$$

where $\boldsymbol{x}^{\mathrm{S}}(t)$ denotes the state space vector of the closed-loop system resulting from system (1) controlled by the regulator:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t),\quad \boldsymbol{K} \in \mathbf{R}^{m\times n}. \tag{4}$$

By definition, system (1) is stabilizable if there exists a matrix $\boldsymbol{K} \in \mathbf{R}^{m\times n}$ such that the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ is (*i*) Schur stable (DT case) or (*ii*) Hurwitz stable (CT case).

The detectability of system (1) refers to the dynamics:

$$(\boldsymbol{x}^{\mathrm{D}}(t))' = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{x}^{\mathrm{D}}(t),\quad \boldsymbol{x}^{\mathrm{D}}(t_0) = \boldsymbol{x}(t_0), \tag{5}$$

where $\boldsymbol{x}^{\mathrm{D}}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$ denotes the error vector resulting from the estimation of the states of system (1) by the observer

$$\begin{aligned} &\dot{\hat{\boldsymbol{x}}}(t) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{\boldsymbol{x}}(t) + (\boldsymbol{B} - \boldsymbol{L}\boldsymbol{D})\boldsymbol{u}(t) + \boldsymbol{L}\boldsymbol{y}(t), \\ &\boldsymbol{L} \in \mathbf{R}^{n\times p},\quad \hat{\boldsymbol{x}}(t_0) = 0. \end{aligned} \tag{6}$$

By definition, system (1) is detectable if there exists a matrix $\boldsymbol{L} \in \mathbf{R}^{n\times p}$ such that the matrix $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ is (*i*) Schur stable (DT case) or (*ii*) Hurwitz stable (CT case).

In the current paper, we are interested in a refinement of these two properties along the lines of the *componentwise asymptotic stability* (CWAS) and *componentwise exponential asymptotic stability* (CWEAS) introduced and characterized in [16], [17} for CT linear systems, as special types of *asymptotic stability* (AS). Further works [8], [10], [12] extended the CWAS and CWEAS concepts for the DT case too.

Using notation (2), let us remind the definition of CWAS for the linear system:

$$\boldsymbol{z}'(t) = \boldsymbol{F}\boldsymbol{z}(t), \boldsymbol{F} \in \mathbf{R}^{n\times n}, \boldsymbol{z}(t_0) = \boldsymbol{z}_0, \tag{7}$$

by incorporating both DT and CT cases:

Definition 1. System (7) is called CWAS if there exists a vector function $\boldsymbol{h}(t): \boldsymbol{T} \to \mathbf{R}^n$, $h_i(t) > 0$, $i = 1,\ldots,n$, $\lim_{t\to\infty} \boldsymbol{h}(t) = 0$ and continuously differentiable in the CT case, such that

$$\begin{aligned} &\forall t_0, t \in \mathbf{T},\ t_0 \le t: |z_i(t_0)| \le h_i(t_0) \Rightarrow \\ &|z_i(t)| \le h_i(t),\ i = 1, \ldots, n, \end{aligned} \tag{8}$$

where $z_i(t), i = 1,\ldots,n$, denote the state variables of system (7). ■

The usage of CWAS particularized to a vector function $\boldsymbol{h}(t)$ of exponential type yields the following:

Definition 2. (*i*) In the DT case, system (7) is called CWEAS if there exist a vector with positive entries $\boldsymbol{\alpha} = [\alpha_1\ \ldots\ \alpha_n]^T \in \mathbf{R}^n$, $\alpha_i > 0, i = 1,\ldots,n$, and a positive, subunitary constant 0<*r*<1, such that:

$$\forall t_0, t \in \mathbf{T} = \mathbf{Z}_+, t_0 \le t: \ |z_i(t_0)| \le \alpha_i \Rightarrow$$
$$|z_i(t)| \le \alpha_i r^{(t-t_0)}, \ i = 1,\ldots,n; \tag{9a}$$

(*ii*) In the CT case, system (7) is called CWEAS if there exist a vector with positive entries $\boldsymbol{\alpha} = [\alpha_1 \ \ldots \ \alpha_n]^\tau \in \mathbf{R}^n$, $\alpha_i > 0$, $i = 1,\ldots,n$, and a negative constant $r<0$, such that:

$$\forall t_0, t \in \mathbf{T} = \mathbf{R}_+, t_0 \le t: \ |z_i(t_0)| \le \alpha_i \Rightarrow$$
$$|z_i(t)| \le \alpha_i e^{r(t-t_0)}, \ i = 1,\ldots,n. \tag{9b}$$ ■

Unlike the standard AS giving global information, in terms of norms, for the state vector approaching the equilibrium point of system (7), CWAS and CWEAS allow the individual monitoring of each state variable. Therefore the purpose of this paper is to study the stabilizability and detectability of system (1) in the light of CWAS and CWEAS, instead of the standard AS.

2. Defining the new concepts

We define the componentwise stabilizability / detectability of system (1) in a very natural manner, which preserves the essence of the classical notion, but considers the stability of the matrix $\boldsymbol{A} - \boldsymbol{BK}$ / $\boldsymbol{A} - \boldsymbol{LC}$ in the stronger sense of CWAS.

Definition 3. System (1) is called *componentwise stabilizable* if there exists a matrix $\boldsymbol{K} \in \mathbf{R}^{m \times n}$ so that system (3) is CWAS. Regulator (4) equipped with such a matrix $\boldsymbol{K}$ is called a CWAS regulator. ■

Definition 4. System (1) is called *componentwise detectable* if there exists a matrix $\boldsymbol{L} \in \mathbf{R}^{n \times p}$ so that system (5) is CWAS. Observer (6) equipped with such a matrix $\boldsymbol{L}$ is called a CWAS observer. ■

Now let us apply the CWEAS concept (Definition 2) to linear system (3) / (5). Let $\mathbf{X}_0 \subset \mathbf{R}^n$ be the bounded set of all initial states $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ of system (1). Denote by

$$\mathbf{X}_0^{\mathrm{S}} = \{\boldsymbol{v}^{\mathrm{S}} \in \mathbf{R}^n \mid -\alpha_i^{\mathrm{S}} \le v_i^{\mathrm{S}} \le \alpha_i^{\mathrm{S}}, i = 1,\cdots,n\} \ /$$
$$\mathbf{X}_0^{\mathrm{D}} = \{\boldsymbol{w}^{\mathrm{D}} \in \mathbf{R}^n \mid -\alpha_i^{\mathrm{D}} \le w_i^{\mathrm{D}} \le \alpha_i^{\mathrm{D}}, i = 1,\cdots,n\}, \tag{10}$$

a rectangular box in the state space of system (3) / (5), where the positive constants $\alpha_i^{\mathrm{S}} > 0, i = 1,\cdots,n$ / $\alpha_i^{\mathrm{D}} > 0, i = 1,\cdots,n$, are adequately selected (according to practical reasons) such that $\mathbf{X}_0 \subseteq \mathbf{X}_0^{\mathrm{S}}$ / $\mathbf{X}_0 \subseteq \mathbf{X}_0^{\mathrm{D}}$, i.e. to ensure the *componentwise boundedness* of the initial states of system (3) / (5):

$$-\alpha_i^{\mathrm{S}} \le x_i^{\mathrm{S}}(t_0) = x_i(t_0) \le \alpha_i^{\mathrm{S}}, \ \ i = 1,\cdots,n \backslash$$
$$-\alpha_i^{\mathrm{D}} \le x_i^{\mathrm{D}}(t_0) = x_i(t_0) \le \alpha_i^{\mathrm{D}}, \ \ i = 1,\cdots,n, \tag{11}$$

for arbitrary $t_0 \in \mathbf{T}$. Aiming to a compact writing for further usage, we introduce the vector notations

$$\boldsymbol{\alpha}^{\mathrm{S}} = [\alpha_1^{\mathrm{S}},\ldots,\alpha_n^{\mathrm{S}}]^\tau, \alpha_i^{\mathrm{S}} > 0, i = 1,\ldots,n \ \ /$$
$$\boldsymbol{\alpha}^{\mathrm{D}} = [\alpha_1^{\mathrm{D}},\ldots,\alpha_n^{\mathrm{D}}]^\tau, \alpha_i^{\mathrm{D}} > 0, i = 1,\ldots,n, \tag{12a}$$

where the symbol τ means the transposition and the superscript $^{\mathrm{S}}$ / $^{\mathrm{D}}$ abbreviates the type of problem, namely stabilizability / detectability. With the same signification for the notation, we also introduce the following two constants:

$$0 < r^{\mathrm{S}} < 1 \text{ (DT case)}, \ r^{\mathrm{S}} < 0 \ \text{ (CT case)} \ /$$
$$0 < r^{\mathrm{D}} < 1 \text{ (DT case)}, \ r^{\mathrm{D}} < 0 \ \text{ (CT case)}. \tag{12b}$$

Definition 5. a) System (1) is called *componentwise exponentially stabilizable* if there exists a matrix $\boldsymbol{K} \in \mathbf{R}^{m \times n}$ so that system (3) is CWEAS. Regulator (4) equipped with such a matrix $\boldsymbol{K}$ is called a CWEAS regulator. **b)** If, for a positive vector $\boldsymbol{\alpha}^S$ and a constant r^S *a priori* given, system (3) is CWEAS, then system (1) is called *componentwise exponentially stabilizable* with guaranteed $(\boldsymbol{\alpha}^S, r^S)$. **c)** If, for a positive vector $\boldsymbol{\alpha}^S$ *a priori* given, system (3) is CWEAS, then system (1) is called *componentwise exponentially stabilizable* with guaranteed $\boldsymbol{\alpha}^S$. **d)** If, for a constant r^S *a priori* given, system (3) is CWEAS, then system (1) is called *componentwise exponentially stabilizable* with guaranteed r^S. ■

Definition 6. a) System (1) is called *componentwise exponentially detectable* if there exists a matrix $\boldsymbol{L} \in \mathbf{R}^{n \times p}$ so that system (5) is CWEAS. Observer (6) equipped with such a matrix $\boldsymbol{L}$ is called a CWEAS observer. **b)** If, for a positive vector $\boldsymbol{\alpha}^{\mathrm{D}}$ and a constant r^{D} *a priori* given, system (5) is CWEAS, then system (1) is called *componentwise exponentially detectable* with guaranteed $(\boldsymbol{\alpha}^{\mathrm{D}}, r^{\mathrm{D}})$. **c)** If, for a positive vector $\boldsymbol{\alpha}^{\mathrm{D}}$ *a priori* given, system (5) is CWEAS, then system (1) is called *componentwise exponentially detectable* with guaranteed $\boldsymbol{\alpha}^{\mathrm{D}}$. **d)** If, for a constant r^{D} *a priori* given, system (5) is CWEAS, then system (1) is called *componentwise exponentially detectable* with guaranteed r^{D}. ■

Voicu in [19] introduced and characterized the concepts of CWEAS detectability and CWEAS observer in a slightly different form than formulated above by Definition 6. However the design procedure for CWEAS observers proposed in [19] had a totally different orientation than our current work (whose theoretical results are easy to convert into computationally tractable forms and allow a parallel study of stabilizability / detectability in the (DT) and (CT) cases).

CWAS and CWEAS (as special types of AS) result from the theory of time-dependent invariant sets of rectangular form, for which the dependence of time is a priori known and explicitly defined. Also called flow invariance, this theory is equally applicable to linear and nonlinear systems as shown by [18]. During the last decade, a large body of work was devoted to the study of polyhedral invariant sets and their application in control - see, for instance, the remarkable survey paper [5] and the papers cited therein. For most of these researches, the polyhedral invariant sets do not depend on time, or if they do, the time-dependence is understood as a contraction of exponential type, operating uniformly on the constraints of the initial conditions. Therefore, such researches (focusing on the generality of the polyhedrons, but neglecting the generality of the time dependence) did not realize the connections with the componentwise analysis of system dynamics, revealed by rectangular sets with general time-dependence.

3. Exploring componentwise stabilizability

Given system (1), consider an arbitrary matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$. Denote by $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ the matrix constructed from the matrix $\boldsymbol{A}-\boldsymbol{BK}$ of system (3), as follows:
(*i*) for the DT case:

$$(\overline{\boldsymbol{A}-\boldsymbol{BK}})_{ij}=|(\boldsymbol{A}-\boldsymbol{BK})_{ij}|,\ \ i,j=1,\cdots,n\,; \tag{13a}$$

(*ii*) for the CT case:

$$\begin{aligned}&(\overline{\boldsymbol{A}-\boldsymbol{BK}})_{ii}=(\boldsymbol{A}-\boldsymbol{BK})_{ii},\ i=1,\cdots,n,\\&(\overline{\boldsymbol{A}-\boldsymbol{BK}})_{ij}=|(\boldsymbol{A}-\boldsymbol{BK})_{ij}|,\ i\neq j, i,j=1,\cdots,n.\end{aligned} \tag{13b}$$

Theorem 1. a) Linear system (1) is componentwise stabilizable if and only if there exists a matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$ such that the matrix $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ is (*i*) Schur stable (DT case) or (*ii*) Hurwitz stable (CT case). **b)** The set of vector functions $\boldsymbol{h}(t):\mathbf{T}\to\mathbf{R}^n$ ensuring CWAS for system (3) is the set of solutions of the (*i*) difference inequality (DT case) or (*ii*) differential inequality (CT case):

$$\boldsymbol{h}'(t)\geq\overline{\boldsymbol{A}-\boldsymbol{BK}}\boldsymbol{h}(t),\ t\in\mathbf{T}\,. \tag{14}$$

Proof: It results from Theorem 1 in [17] for the CT case and from Theorem 5 in [10] for the DT case. ■

There are two aspects in exploring the componentwise stabilizability that deserve a special attention from the theoretical point of view, namely its link with the stabilizability taken in the classical sense and its preservation under small variations of the matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$.

The fact that componentwise stabilizability is stronger than the standard concept of stabilizability results directly from the relation between CWAS and AS discussed in the previous section. However this aspect can be also addressed in terms of matrix eigenvalues. Denote by $\lambda_i(\boldsymbol{\Theta})$, $i=1,\cdots,n$, the n eigenvalues of an n-th order square matrix $\boldsymbol{\Theta}$.

Theorem 2. Given system (1), consider an arbitrary matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$. Let $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ be the matrix constructed from $\boldsymbol{A}-\boldsymbol{BK}$ in accordance with (13). **a)** (*i*) In the DT case, if $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ is built according to (13a), then it has a real nonnegative eigenvalue (simple or multiple) denoted by $\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}})$, which fulfills the dominance condition (spectral radius):

$$|\lambda_i(\overline{\boldsymbol{A}-\boldsymbol{BK}})|\leq\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}}),\ \ i=1,\cdots,n\,; \tag{15a}$$

(*ii*) In the CT case, if $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ is built according to (13b), then it has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}})$, which fulfills the dominance condition (spectral abscissa):

$$\mathrm{Re}[\lambda_i(\overline{\boldsymbol{A}-\boldsymbol{BK}})]\leq\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}}),\ \ i=1,\cdots,n\,. \tag{15b}$$

b) The dominant eigenvalue $\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}})$ also dominates the whole spectrum of the matrix $\boldsymbol{A}-\boldsymbol{BK}$, i.e.
(*i*) In the DT case,

$$|\lambda_i(\boldsymbol{A}-\boldsymbol{BK})|\leq\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}}),\ \ i=1,\cdots,n\,. \tag{16a}$$

(*ii*) In the CT case,

$$\mathrm{Re}[\lambda_i(\boldsymbol{A}-\boldsymbol{BK})]\leq\lambda_{\max}(\overline{\boldsymbol{A}-\boldsymbol{BK}}),\ \ i=1,\cdots,n\,. \tag{16b}$$

Proof: It results from Theorems 4 and 6 in [10]. ■

Remark 1. According to Theorem 2, it is obvious that the stabilizability in the standard sense represents only a necessary condition for the componentwise stabilizability. The possibility that, under some supplementary hypotheses, the standard stabilizability involves componentwise stabilizability needs further investigation. For instance, the stronger condition of state controllability for system (1) cannot ensure its componentwise stabilizability, as simply shown by the CT case of a single input system in the *controllable canonical form*. ■

The above discussion points out that there exists no simple connection between the eigenvalues of the matrices $\boldsymbol{A}-\boldsymbol{BK}$ and $\overline{\boldsymbol{A}-\boldsymbol{BK}}$. However, an important connection can be found at the level of the region of the complex plane where the eigenvalues are located, as suggested in [16], [18], [19].

Theorem 3. Given system (1), consider an arbitrary matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$. Let $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ be the matrix constructed from $\boldsymbol{A}-\boldsymbol{BK}$ in accordance with (13). The generalized Gershgorin disks $|\lambda-c_i|<r_i$, $|\lambda-\overline{c}_i|<\overline{r}_i$, $i=1,\ldots,n$, of the matrix $\boldsymbol{A}-\boldsymbol{BK}$ and $\overline{\boldsymbol{A}-\boldsymbol{BK}}$, respectively, have identical radii $r_i=\overline{r}_i$, $i=1,\ldots,n$, and (*i*) in the DT case, the centers are identical or differ by a sign $|c_i|=\overline{c}_i$, $i=1,\ldots,n$, (*ii*) in the CT case, the centers are identical $c_i=\overline{c}_i$, $i=1,\ldots,n$.

Proof: For an arbitrary set of positive constants $p_j>0$, $j=1,\ldots,n$, it is obvious that:

$$\begin{aligned}&|c_i|=|(\boldsymbol{A}-\boldsymbol{BK})_{ii}|=\overline{c}_i,\\&r_i=\frac{1}{p_i}\sum_{\substack{i=1\\j\neq i}}^{n}|(\boldsymbol{A}-\boldsymbol{BK})_{ij}|p_j=\overline{r}_i,\quad i=1,\ldots,n.\end{aligned} \quad ■(17)$$

Remark 2. If, for a set of positive constants $p_j>0$, $j=1,\ldots,n$, a regulator of type (4) places the generalized Gershgorin disks associated with system (3) (i.e. of the matrix $\boldsymbol{A}-\boldsymbol{BK}$) in the stability region of the complex plane, then the standard stabilizability and the componentwise stabilizability of system (1) are ensured concomitantly. ■

The exploitation of the Gershgorin disks naturally leads to a necessary and sufficient condition for the componentwise stabilizability of system (1), equivalent to Theorem 1 part **a)**, which does not involve the matrix $\overline{\boldsymbol{A}-\boldsymbol{BK}}$ constructed according to (13). Consider a diagonal matrix.

$$\boldsymbol{P}=\mathrm{diag}\{p_1,\ldots,p_n\},\ p_j>0,\ j=1,\ldots,n\,. \tag{18}$$

Theorem 4. System (1) is componentwise stabilizable if and only if there exist a matrix $\boldsymbol{K}\in\mathbf{R}^{m\times n}$ and a diagonal matrix $\boldsymbol{P}$ such that
(*i*) in the DT case,

$$\|\boldsymbol{P}^{-1}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{P}\|_\infty<1\,; \tag{19a}$$

(*ii*) in the DT case,

$$\lim_{\xi\to 0+}\frac{\|\boldsymbol{I}_n+\xi\boldsymbol{P}^{-1}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{P}\|_\infty-1}{\xi}<0\,, \tag{19b}$$

where $\boldsymbol{I}_n$ denotes the identity matrix of order *n*.

Proof: (*i*) Inequality (19a) is equivalent to the placement of the generalized Gershgorin disks associated with the matrix $\overline{A-BK}$ within the unit circle of the complex plane. (*ii*) Inequality (19b) is equivalent to the placement of the generalized Gershgorin disks associated with the matrix $\overline{A-BK}$ within the left half plane of the complex plane. ■

Remark 3. The left hand sides of inequalities (19a) and (19b) refer to different mathematical entities, namely a matrix norm induced by a vector norm (in the DT case) and a matrix measure e.g. [9] (in the CT case). Actually the left hand side of (19b) can be expressed in a simpler manner, which avoids the formal writing as a limit

$$\lim_{\xi\to 0+}\frac{\| I_n+\xi P^{-1}(A-BK)P\|_\infty -1}{\xi}= \\ =\| \sigma I_n+P^{-1}(A-BK)P\|_\infty -\sigma, \tag{20}$$

where $\sigma = 1/\xi >> 0$ is a big positive constant (arbitrarily taken) which ensures the positiveness for the diagonal elements of the matrix $\sigma I_n+\xi P^{-1}(A-BK)P$. ■

Along the same lines, we can also characterize the dominant eigenvalue $\lambda_{\max}(\overline{A-BK})$.

Theorem 5. Given system (1), consider an arbitrary matrix $K\in R^{m\times n}$. Let $\overline{A-BK}$ be the matrix constructed from $A-BK$ in accordance with (13).

(*i*) In the DT case,

$$\lambda_{\max}(\overline{A-BK})=\inf_P \| P^{-1}(A-BK)P\|_\infty . \tag{21a}$$

(*ii*) In the CT case,

$$\lambda_{\max}(\overline{A-BK})=\lim_{\xi\to 0+}\inf_P\frac{\| I_n+\xi P^{-1}(A-BK)P\|_\infty -1}{\xi}. \tag{21b}$$

Proof: (*i*) $\| P^{-1}(A-BK)P\|_\infty=\| P^{-1}(\overline{A-BK})P\|_\infty$ and $\lambda_{\max}(\overline{A-BK})=\inf_P \| P^{-1}(\overline{A-BK})P\|_\infty$ since all the elements of the matrix $P^{-1}(\overline{A-BK})P$ are nonnegative [13]. (*ii*) It is similar to the DT case, by ensuring the nonegativeness of the diagonal elements. ■

It is a straightforward task to show that, for a given diagonal matrix P (18), whenever inequalities (19a) and (19b) are satisfied, they define a convex set of matrices $K\in \mathbf{R}^{m\times n}$, since their left-hand sides are convex functions with respect to K. This motivates us to get more accurate information about the preservation of the componentwise stabilizability of system (1) under small variations of the matrix $K\in R^{m\times n}$. Assume that the matrix $K_0\in \mathbf{R}^{m\times n}$ satisfies the condition:

(*i*) in the DT case,

$$0\le\lambda_{\max}(\overline{A-BK_0})<1; \tag{22a}$$

(*ii*) in the CT case,

$$\lambda_{\max}(\overline{A-BK_0})<0. \tag{22b}$$

We intend to prove that inequalities (22) remain valid when, instead of $K_0\in \mathrm{R}^{m\times n}$, an arbitrary matrix $K\in\mathbf{R}^{m\times n}$ is selected from the symmetrical matrix interval

$$K_0-D_K\le K\le K_0+D_K,\ D_K\ge 0, \tag{23}$$

where the nonnegative matrix $D_K\ge 0$ can be characterized in quantitative terms with respect to the known matrices A and B.

Theorem 6. Consider the matrix $K_0\in \mathrm{R}^{m\times n}$ which ensures the componentwise stabilizability of system (1) and take a constant $\mu\in\mathbf{R}$ such that:

(i) $\lambda_{\max}(\overline{A-BK_0})<\mu\le 1$ - in the DT case,

(ii) $\lambda_{\max}(\overline{A-BK_0})<\mu\le 0$ - in the CT case.

a) If, for an arbitrary matrix norm $\|\ \|$, matrix $D_K\ge 0$ in (23) meets the condition:

$$\| D_K\|<1/\left\| \left(\mu I_n-\overline{A-BK_0}\right)^{-1}|B|\right\|, \tag{24}$$

then any matrix $K\in\mathbf{R}^{m\times n}$ belonging to matrix interval (23) ensures the componentwise stabilizability of system (1).

b) If matrix $D_K\ge 0$ in (23) is defined as $D_K=\gamma\Gamma$ with $\Gamma\ge 0$ a known nonnegative matrix and $\gamma>0$ a positive constant that, for an arbitrary matrix norm, meets the condition

$$\gamma<1/\left\| \left(\mu I_n-\overline{A-BK_0}\right)^{-1}|B|\Gamma\right\|, \tag{25}$$

then any matrix $K\in\mathbf{R}^{m\times n}$ belonging to matrix interval (23) ensures the componentwise stabilizability of system (1).

Proof: It results from Theorems 6 and 7 in [11]. ■

4. Componentwise exponential stabilizability

An important applicability of the componentwise stabilizability concept refers to constraining the state-space trajectories of system (3) by time-dependent hyperrectangles with exponential decay. Therefore our attention focuses on the characterization of the componentwise exponential stabilizability (introduced by Definition 5).

Theorem 7.. System (1) is componentwise exponentially stabilizable if and only if the following algebraic inequalities have solutions $K\in\mathbf{R}^{m\times n}$, $\alpha^S\in\mathbf{R}^n$, $r^S\in\mathbf{R}$:

(*i*) for the DT case:

$$\overline{A-BK}\alpha^S\le r^S\alpha^S,\ \overline{A-BK}\ \text{defined by (13a)}, \tag{26a}$$

$$0<\alpha^S, \tag{27a}$$

$$0\le r^S<1; \tag{28a}$$

(*ii*) for the CT case:

$$\overline{A-BK}\alpha^S\le r^S\alpha^S,\ \overline{A-BK}\ \text{defined by (13b)}: \tag{26b}$$

$$0<\alpha^S, \tag{27b}$$

$$r^S<0. \tag{28b}$$

Proof: It results from Theorem 1 part b), by taking $h(t)=\alpha^S(r^S)^t$ / $h(t)=\alpha^S e^{r^S t}$ for the DT / CT case. ■

An equivalent formulation for Theorem 7 can be given if the elements of the positive vector $\alpha^S\in R^n$, $0<\alpha^S$ are regarded as defining the diagonal matrix:

$$S=\mathrm{diag}\{\alpha_1^S,\ldots,\alpha_n^S\},\ \alpha_i^S>0,\ i=1,\ldots,n. \tag{29}$$

Lemma 1. (*i*) In the DT case, condition (26a) is equivalent to:

$$\| S^{-1}(A-BK)S\|_\infty\le r^S; \tag{30a}$$

(*ii*) In the CT case, condition (26b) is equivalent to:

$$\lim_{\xi \to 0+} \frac{\| \boldsymbol{I}_n + \xi \boldsymbol{S}^{-1} (\boldsymbol{A} - \boldsymbol{BK}) \boldsymbol{S} \|_\infty - 1}{\xi} \le r^{\mathrm{S}} . \qquad (30b)$$

Proof: It results from (26), if the diagonal matrix $\boldsymbol{S}$ is used. ■

Theorem 8.. System (1) is componentwise exponentially stabilizable if and only if the following algebraic inequalities have solutions $\boldsymbol{K} \in \mathbf{R}^{m \times n}$, $\alpha^{\mathrm{S}} \in \mathbf{R}^n$, $r^{\mathrm{S}} \in \mathbf{R}$: (*i*) for the DT case: (30a), (27a), (28a); (*ii*) for the CT case: (30b), (27b), (28b).

Proof: It results from Theorem 7 and Lemma 1. ■

Remark 4. One can immediately notice the similarity between the left hand side of the first inequality in (30a), (30b) and the left hand side of inequality (19a), (19b), respectively, showing that the matrix $\boldsymbol{S}$ defined by (29) plays the same role as the matrix $\boldsymbol{P}$ defined by (18). Thus, we reveal the complete meaning of generalized Gershgorin disks (17) placed for system (3) by regulator (4), by concluding that the positive constants $p_j > 0, j = 1, \ldots, n$, used in (17) represent the elements of the positive vector $\alpha^{\mathrm{S}} \in \mathbf{R}^n$, $0 < \alpha^{\mathrm{S}}$ involved in the CWEAS analysis of system (3). ■

Remark 5. Inequalities (*i*) for the DT case: (30a), (27a), (28a) and (*ii*) for the CT case: (30b), (27b), (28b) can be obtained if $V(\boldsymbol{x}) = \| \boldsymbol{S}^{-1} \boldsymbol{x} \|_\infty$ is used as a Lyapunov function for system (3), according to [9]. However, when these inequalities are compatible, the exploitation of the results in the cited paper [9] yields the conclusion that system (3) is AS in the classical sense and the time-independent set $\mathbf{X}_0^{\mathrm{S}} \in \mathbf{R}^n$ defined by (10) is an invariant set for system (3), which is evidently weaker than CWEAS of system (3) ensured by Theorem 8. There are several noticeable papers, such as [14], [15], [2], [7], [6], [1], addressing the linear constrained regulation problem, which give methods for designing regulator (4), not necessarily constant, that ensures the standard AS of system (3), concomitantly with the existence of a time-independent invariant set for system (3). Paper [6] deserves a special comment because it explores the link between the state constraints and the ($\boldsymbol{A}$,$\boldsymbol{B}$)-invariant subspaces of system (1), but the result is restrictive and cannot be applied to n symetrical constraints (as requested by the CWEAS approach). The regulation problem with time-dependent constraints is formulated in [3], [4], but the resulting controller has a variable structure. It should be also mentioned that in these papers the time dependence of exponential form (regarded as a contraction of the initial set) appears as the only possibility to handle the time dependence of the constraints, unlike the definition of CWEAS which is derived from the arbitrary time-dependence of CWAS. ■

5. Componentwise exponential stabilizability with guaranteed performance

The aim of this subsection consists in providing necessary and sufficient conditions for the characterization of the componentwise exponential stabilizability with different guaranteed performance, in accordance with the three cases formulated in Definition 5. Each characterization is given in two equivalent forms, resulting from Theorem 7 and Theorem 8, respectively.

Theorem 9. Assume that the positive vector $\alpha^{\mathrm{S}} \in \mathbf{R}^n$, $0 < \alpha^{\mathrm{S}}$ and the constant r^{S} ($0 < r^{\mathrm{S}} < 1$ for the DT case, $r^{\mathrm{S}} < 0$ for the CT case) are *a priori* given. System (1) is componentwise exponentially stabilizable with guaranteed (α^{S}, r^{S}) if and only if the following inequalities have solutions $\boldsymbol{K} \in \mathbf{R}^{m \times n}$: (*i*) for the DT case: (26a), equivalent to (30a); (*ii*) for the CT case: (26b), equivalent to (30b).

Proof: It results from Theorems 7 and 8 when conditions (27a), (28a) - DT case and (27b), (28b) - CT case are fulfilled. ■

Theorem 10. Assume that the positive vector $\alpha^{\mathrm{S}} \in \mathbf{R}^n$, $0 < \alpha^{\mathrm{S}}$ is *a priori* given. System (1) is componentwise exponentially stabilizable with guaranteed $\boldsymbol{\alpha}^{\mathrm{S}}$ if and only if the following inequalities have solutions $\boldsymbol{K} \in \mathbf{R}^{m \times n}$, $r^{\mathrm{S}} \in \mathbf{R}$: (*i*) for the DT case: (26a), (28a), equivalent to (30a), (28a); (*ii*) for the CT case: (26b), (28b) equivalent to (30b), (28b).

Proof: It results from Theorems 7 and 8 when conditions (27a) - DT case and (27b) - CT case are fulfilled. ■

Theorem 11. Assume that the constant r^{S} ($0 < r^{\mathrm{S}} < 1$ for the DT case, $r^{\mathrm{S}} < 0$ for the CT case) is *a priori* given. System (1) is componentwise exponentially stabilizable with guaranteed r^{S} if and only if the following inequalities have solutions $\boldsymbol{K} \in \mathbf{R}^{m \times n}$, $\alpha^{\mathrm{S}} \in \mathbf{R}^n$: (*i*) for the DT case: (26a), (27a), equivalent to (30a), (27a); (*ii*) for the CT case: (26b), (27b), equivalent to (30b), (27b).

Proof: It results from Theorems 7 and 8 when conditions (28a) - DT case and (28b) - CT case are fulfilled. ■

Theorems 9, 10, 11 can be exploited for computational purposes in order to find CWEAS regulators with guaranteed performance. The nature of the theoretical results suggests their exploitation as nonlinear minimization problems derived from the aforementioned theorems, by using adequate objective functions defined in terms of $\| \ \|_\infty$. We have to take into account the fact that, in practice, the gain factors of regulator (4) should be correlated with the admissible magnitude of the signals fed back to the control input $\boldsymbol{u}(t)$ of system (1). Consequently, one may consider a set of appropriate constraints for the matrix $\boldsymbol{K} \in \mathbf{R}^{m \times n}$ used by regulator (4). When dealing with a CWEAS regulator with guaranteed r^{S}, one may also consider a set of appropriate constraints for the positive vector $\alpha^{\mathrm{S}} \in \mathbf{R}^n$.

Remark 6. None of these computational approaches is equivalent to the theoretical result it relies on, and, therefore, the solutions returned by the numerical optimizer need a correct interpretation with regard to the influence of the initial guesses, solutions far from the global minimum, or even spurious solutions. Moreover, the objective functions defined in terms of the infinity norm present points where they are not differentiable, requiring high accuracy for the numerical derivatives. ■

6. Componentwise (exponential) detectability

In order to keep the text of the paper within the requested limits and also to maintain a reasonable balance between theoretical information and its applicability, the analysis of the componentwise (exponential) detectability is just briefly commented, but it will be illustrated by a numerical example. Such a brief discussion is justified by the similarity with the componentwise (exponential) stabilizability; this similarity results from the fact that CWAS and CWEAS analysis for system (3) / (5) operates with a family of linearly parameterized matrices, namely ***A-BK*** / ***A-LC***. Consequently, the theoretical background developed by the previous sections, as well as the proposed computational approaches can be *mutatis mutandis* reformulated as dual problems.

Example: Consider system (1) in the CT case with

$$\mathbf{A}=\begin{bmatrix}1 & 4 & 0\\ 1 & -3 & 1\\ 1 & -2 & -4\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},$$

taken as in [19]. For this system, let us design a CWEAS observer with guaranteed $\boldsymbol{\alpha}^{\mathrm{D}}=[2\ 1\ 1]^{\tau}$. Using the cost function $J:\mathbf{R}^{3\times 2}\to\mathbf{R}$, $J(\boldsymbol{L})=\|\sigma\boldsymbol{I}_3+\boldsymbol{S}^{-1}(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{S}\|_{\infty}-\sigma$, with $\boldsymbol{S}=\mathrm{diag}\{2,1,1\}$, $\sigma>>0$ and [-4, 4] constraining the entries of the matrix $\boldsymbol{L}\in\mathbf{R}^{3\times 2}$, the optimizer **fmincon** from *MATLAB Optimization Toolbox* returns the minimum -3, for several initial guesses. A family of CWEAS observers can be built with the decaying rate $r^{\mathrm{D}}=-3$, different matrices $\boldsymbol{L}$ resulting for different initial guesses. If the adjustment of the elements composing the vector $\boldsymbol{\alpha}^{\mathrm{D}}$ is permitted within a range of ±10% around the values $[2\ 1\ 1]^{\tau}$ taken initially, then one can use the cost function $J:\mathbf{R}^{3\times 2}\times\mathbf{R}^{3}\to\mathbf{R}$, $J(\boldsymbol{L},\alpha^{\mathrm{D}})=\|\sigma\boldsymbol{I}_3+\boldsymbol{S}^{-1}(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{S}\|_{\infty}-\sigma$, with $\boldsymbol{S}=\mathrm{diag}\{\alpha_1^{\mathrm{D}},\alpha_2^{\mathrm{D}},\alpha_3^{\mathrm{D}}\}$, $\sigma>>0$, that considers the components of $\boldsymbol{\alpha}^{\mathrm{D}}$ as variables subject to the constraints $[1.8\quad 0.9\quad 0.9]^{\tau}\le\boldsymbol{\alpha}^{\mathrm{D}}\le[2.2\quad 1.1\quad 1.1]^{\tau}$. Various initial guesses yield the minimum -4, meaning a better decaying rate $r^{\mathrm{D}}=-4$ for the components of the estimation error given by system (5). ■

7. Conclusions

The paper introduces the new concepts of componentwise (exponential) stabilizability / detectability, by replacing the classical requirement of AS for system (3) / (5) with the stronger one of CWAS (CWEAS). These new concepts are completely characterized by necessary and sufficient conditions formulated in algebraic terms. For the synthesis of CWEAS regulators / observers with guaranteed performance, computational methods are derived from the theoretical background, as nonlinear optimization problems with adequate constraints. Instead of the classical scenario placing the closed-loop eigenvalues within the stability region of the complex plane, the componentwise (exponential) stabilizability / detectability requires the placement of the generalized Gershgorin disks in the stability region. An important question that has not received a proper answer and remains still open for future researches is the existence of more intimate connections between the three types of componentwise exponential stabilizability / detectability with guaranteed performance and the structural properties of state controllability / observability.

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