CHARACTERISTIC MODES OF TIME-VARIABLE LINEAR DIFFERENTIAL SYSTEMS

K. Zenger*, **R. Ylinen**†

* Helsinki University of Technology, Control Engineering Laboratory, P.O.Box 5400, FIN-02015 HUT, Finland.
Fax: +358 9 4515208, Email: kai.zenger@hut.fi
† University of Oulu, Systems Engineering Laboratory, P.O.Box 4300, 90014 Oulun yliopisto, Finland.
Email: raimo.ylinen@oulu.fi

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Abstract

The characteristic modes of time-variable linear SISO systems can be defined by using suitable linear transformations that change the realization of the system to a form, which generates the modes of the system. The stability of the system can be investigated by means of the transformation.

1 Introduction

Analysis and synthesis techniques for linear time-varying differential systems are not easily available when starting from the well-known classical theory of time-invariant systems. This is a fundamental consequence of the fact that the solution of the system equations cannot generally be solved in closed form, i.e. the state transition matrix cannot usually be expressed in terms of elementary functions. The fundamental difficulties related to stability and performance of the system are caused by the difficulty to define the concepts of poles and zeros that would have a similar relationship to system performance as in the case of time-invariant systems.

There have been several efforts to solve this problem, but none of these seems to have been generally accepted. The "natural" way of defining poles at each time instant from the "frozen" system matrix is known to be inadequate, because the resulting poles do not give enough information on system stability [8]. Another way to define poles (or more specifically pole sets) was used in [4], where factorizations of operator polynomials were used to define the pole sets. Based on this analysis conditions for the stability of the system were obtained. A similar method by using polynomial algebra was used in [1], and specifically in the time-varying case in [12].

Another approach to the problem would be to use state-space techniques and state transformations to study the stability of the system. The well-known theory of Lyapunov transformations [5] is a powerful tool in this respect, because of its stability preserving characteristics in the state transformation. In [13] it was shown that any time-varying system matrix of a continuous linear state-space representation can be changed into a constant matrix, but the needed state transformation depends on the state-transition matrix, which is generally impossible to solve analytically. Hence it is not possible to know, whether the transformation is a Lyapunov transformation or not. The topic has also been discussed e.g. in [3].

In this paper the state transformations have been elaborated further by defining the characteristic modes of the system. The theory is closely related to the concept of extended eigenvalues and extended eigenvectors discussed e.g. in [11] and [15].

The work to be presented has also connections to results reported in [4], [7], [14] and [6].

2 State transformations

Consider a *SISO* input-output differential system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are continuously differentiable matrix functions with suitable dimensions. The linear but possibly time-varying transformation

$$x(t) = P(t)s(t) \tag{2}$$

where $P(\cdot)$ is an invertible square matrix of the same dimension as $A(\cdot)$, is used to change the system representation (1) into the form

$$\begin{aligned} \dot{s}(t) &= E(t)s(t) + F(t)u(t) \\ y(t) &= G(t)s(t) + H(t)u(t) \end{aligned} \tag{3}$$

($s(t_0) = P^{-1}(t_0)x_0$) with

$$\begin{aligned} E(t) &= P^{-1}(t)[A(t)P(t) - \dot{P}(t)] \\ F(t) &= P^{-1}(t)B(t) \\ G(t) &= C(t)P(t) \\ H(t) &= D(t) \end{aligned} \tag{4}$$

It has been shown in [13] that the matrix $E(\cdot)$ of the target system can be chosen arbitrarily by choosing

$$P(t) = \Phi_A(t,t_0)P(t_0)\Phi_E^{-1}(t,t_0) \tag{5}$$

where $\Phi_A(\cdot,\cdot)$, $\Phi_E(\cdot,\cdot)$ are the state transition matrices related to $A(\cdot)$ and $E(\cdot)$, respectively.

Equivalently, the transformation matrix $P(t)$ can be solved from the equations

$$\dot{P}(t) = A(t)P(t) - P(t)E(t) \tag{6}$$

The relationship of the two state-transition matrices becomes accordingly

$$\Phi_A(t,\tau) = P(t)\Phi_E(t,\tau)P^{-1}(\tau) \tag{7}$$

or

$$\Phi_E(t,\tau) = P^{-1}(t)\Phi_A(t,\tau)P(\tau) \tag{8}$$

By using these relationships it follows that the weighting functions of the input-output systems become

$$p'_A(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau) \tag{9}$$

$$\begin{aligned} p'_E(t,\tau) &= G(t)\Phi_E(t,\tau)F(\tau) \\ &= C(t)P(t)\Phi_E(t,\tau)P^{-1}(\tau)B(\tau) \\ &= C(t)\Phi_A(t,\tau)B(\tau) \end{aligned} \tag{10}$$

where it is assumed that the systems are strictly proper so that $D(t) \equiv 0$ and $H(t) \equiv 0$. The result shows that the weighting functions and impulse responses of the original and transformed systems are the same.

As for controllability and observability, consider the controllability gramian

$$W_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A(t_0,t)B(t)B^T(t)\Phi_A^T(t_0,t)\,dt \tag{11}$$

which for the transformed system becomes

$$\begin{aligned} W_E(t_0,t_1) &= \int_{t_0}^{t_1} \Phi_E(t_0,t)F(t)F^T(t)\Phi_E^T(t_0,t)\,dt \\ &= \int_{t_0}^{t_1} \{P^{-1}(t_0)\Phi_A(t_0,t)P(t)\cdot \\ &\quad P^{-1}(t)B(t)B^T(t)(P^T(t))^{-1}P^T(t)\cdot \\ &\quad \Phi_A^T(t_0,t)(P^T(t_0))^{-1}\}\ dt \\ &= P^{-1}(t_0)W_A(t_0,t_1)(P^T(t_0))^{-1} \end{aligned} \tag{12}$$

Because the matrix $P(t_0)$ has full rank, the definiteness of the gramians W_A and W_E is the same. Thus, controllability remains invariant in the transformation. A similar calculation shows that the observability gramian

$$M_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A^T(t,t_0)C^T(t)C(t)\Phi_A(t,t_0)\,dt \tag{13}$$

changes into the form

$$M_E(t_0,t_1) = P^T(t_0)M_A(t_0,t_1)P(t_0) \tag{14}$$

so that observability is also invariant in the transformation.

To investigate the preservation of stability, the important concept of a *Lyapunov transformation* can be used. Results related to this theory can be found here and there in the literature, see e.g. [5], [8], [3].

A definition used in [8] is: An $n \times n$ matrix $P(t)$ that is continuously differentiable and invertible at each t is called a Lyapunov transformation if there exist finite positive constants ρ and η such that for all t

$$\|P(t)\| \le \rho, \quad |\det P(t)| \ge \eta \tag{15}$$

which is equivalent to finding a finite positive constant ρ such that

$$\|P(t)\| \le \rho, \quad \left\|P^{-1}(t)\right\| \le \rho \tag{16}$$

If a system matrix is changed into another one by a Lyapunov transformation, the stability properties of the original and target representations are the same. The key issue is then to determine, whether the matrix

$$P(t) = \Phi_A(t,t_0)P_0\Phi_E^{-1}(t,t_0) \tag{17}$$

is a Lyapunov-transformation matrix or not. As long as the transition matrices $\Phi_A(\cdot,\cdot)$ and $\Phi_E(\cdot,\cdot)$ are not known, there seems to be no general procedure to determine this.

The concept *reducibility* is defined to imply that a system matrix can be changed into a constant form by using a Lyapunov-transformation. More generally, two representations which are equivalent through a Lyapunov transformation are called *kinematically similar* in [3]; the term *topologically equivalent realizations* is used in [9]. A well-known result in classical literature is that periodic systems are always reducible; see e.g. [2], [8].

The above issues concerning stability, controllability and observability constitute the background and motivation for the use of time-variable state transformations in control design. If Lyapunov transformations are used, the original and target representations are structurally very similar. If the target system is easier to deal with, it is then reasonable to use it as a starting point in solving analysis and synthesis problems.

3 Extended eigenvalues and extended eigenvectors

Consider the autonomous system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \tag{18}$$

where the dimensions of the matrix $A(t)$ and vector $x(t)$ are $n \times n$ and $n \times 1$, respectively. When searching for modal solutions (*characteristic modes*) of the form

$$x(t) = e(t) \cdot \exp\left(\int_{t_0}^{t} \lambda(\tau)d\tau\right) \tag{19}$$

where $e(t)$ is an $n \times 1$ vector and λ a scalar function, it is easy to verify that the solution fulfils equation

$$\dot{e}(t) = [A(t) - \lambda(t)I]\,e(t) \tag{20}$$

which resembles the classical result of time-invariant systems. The difference is that the eigenvalues λ and eigenvectors e are now time-variable. Therefore they are sometimes called *extended eigenvalues* and *extended eigenvectors* [11]. In [14] the extended eigenvectors $e(t)$ were expected to have constant values in order to make the stability analysis easier; however, in the current paper that assumption is relaxed to allow the eigenvectors to be time-varying.

Consider now the system matrix

$$E(t) = \operatorname{diag}(\lambda_i(t)),\ i = 1, 2, \ldots, n$$

of the target representation. It is important to understand that this representation can always be formed from the original matrix $A(t)$ by using a time variable state transformation according to (5) or (6). Then (6) is written as

$$\dot{e}_i(t) = A(t)e_i(t) - \lambda_i(t)e_i(t) \tag{21}$$

where the terms $e_i(t)$ are the column vectors of matrix $P(t)$. It follows that the extended eigenvalues are the elements on the main diagonal of $E(t)$ and the extended eigenvectors are the column vectors of $P(t)$.

Also, it is easy to show that the stability of the original system can be characterized by the *mode-vectors* associated to each eigenpair by

$$m_i(t) = e^{\left(\int_{t_0}^{t} \lambda_i(\tau)d\tau\right)} e_i(t) \tag{22}$$

The state transition matrix of the target system is clearly

$$\Phi_E(t, t_0) = \operatorname{diag}\left(e^{\int_{t_0}^{t} \lambda_i(\tau)d\tau}\right) \tag{23}$$

so that the transition matrix of the original system becomes

$$\begin{aligned} &\Phi_A(t, t_0) = P(t)\Phi_E(t, t_0)P(t_0)^{-1} \\ &= \left[e_1(t)e^{\int_{t_0}^{t} \lambda_1(\tau)d\tau} \ \ldots \ e_n(t)e^{\int_{t_0}^{t} \lambda_n(\tau)d\tau}\right] P(t_0)^{-1} \\ &= [m_1(t)\ m_2(t)\ \ldots\ m_n(t)]\, P(t_0)^{-1} \end{aligned} \tag{24}$$

It is then evident that the original system is stable, if the norm of every mode-vector is bounded, and asymptotically stable, if, additionally, the norm of every mode-vector converges to zero as time approaches infinity [11], [13]. Also, it should be noted that the above solution can be written in the form

$$\Phi_A(t, t_0) = \sum_{i=1}^{n} m_i(t) q_i^H(t_0) \tag{25}$$

with

$$P(t_0)^{-1} = Q(t_0) = \left[\begin{array}{cccc} q_1(t_0) & q_2(t_0) & \cdots & q_n(t_0) \end{array}\right]^H$$

The result is seen to be be analogous to that of time-invariant systems, for which λ_i and q_i denote the eigenvalues and (left) eigenvectors of the system matrix.

The problem in the above approach is that there is much freedom in the selection of the extended eigenvalues and extended eigenvectors. In fact, the extended eigenvalues can be chosen freely, whereafter the extended eigenvectors are determined by the transformation equations; the same holds vice versa also. If the transformation matrix $P(t)$ is chosen to be a Lyapunov transformation, the stability properties of the original and target systems are the same, but the system matrix of the target system is not necessarily diagonal. Consequently, if the target system is chosen to be diagonal, it is difficult to give conditions for $P(t)$ to be a Lyapunov transformation.

Because of these difficulties it is instructive to consider the case of triangular target systems.

4 A triangularization procedure

According to the idea in [6] consider an autonomous system $\dot{x}(t) = A_k(t)x(t)$, where $A_k(t)$ is a $k \times k$ matrix

$$A_k(t) = \begin{bmatrix} D_{k-1}(t) & b_{k-1}(t) \\ c_{k-1}^T(t) & d_{k-1}(t) \end{bmatrix} \tag{26}$$

The dimensions of the above submatrices $D_{k-1}(t)$, $b_{k-1}(t)$, $c_{k-1}^T(t)$, $d_{k-1}(t)$ are $(k-1) \times (k-1)$, $(k-1) \times 1$, $1 \times (k-1)$ and 1×1, respectively. By using the transformation matrix [6]

$$P_k(t) = \begin{bmatrix} I_{k-1} & 0 \\ p_{k-1}^T(t) & 1 \end{bmatrix} \tag{27}$$

where the dimensions of the submatrices are in accordance to those in $A_k(t)$, the system matrix changes into

$$E_k(t) = \begin{bmatrix} A_{k-1}(t) & b_{k-1}(t) \\ 0 & \lambda_k(t) \end{bmatrix} \tag{28}$$

by the transformation

$$\dot{P}_k(t) = A_k(t)P_k(t) - P_k(t)E_k(t) \tag{29}$$

In the new system matrix

$$A_{k-1}(t) = D_{k-1}(t) + b_{k-1}(t)p_{k-1}^T(t) \tag{30}$$

and

$$\begin{aligned} \dot{p}_{k-1}^T(t) &= c_{k-1}^T(t) + d_{k-1}(t)p_{k-1}^T(t) - p_{k-1}^T(t) \cdot \\ &\quad \left(D_{k-1}(t) + b_{k-1}(t)p_{k-1}^T(t)\right) \\ &= c_{k-1}^T(t) + d_{k-1}(t)p_{k-1}^T(t) \\ &\quad - p_{k-1}^T(t)D_{k-1}(t) \ - p_{k-1}^T(t)b_{k-1}(t)p_{k-1}^T(t) \end{aligned} \tag{31}$$

$$\lambda_k(t) = d_{k-1}(t) - p_{k-1}^T(t)b_{k-1}(t) \tag{32}$$

The procedure can be generalized in order to change the original system matrix to a triangular form. To this end, take

$$A_n^{(k)}(t) = \begin{bmatrix} A_k(t) & X(t) \\ 0 & \Lambda(t) \end{bmatrix} \tag{33}$$

where the dimensions of the matrices $A_k(t)$,$X(t)$ and $\Lambda(t)$ are $k \times k$, $k \times (n-k)$ and $(n-k) \times (n-k)$, respectively. The matrix is assumed to be partly triangular such that $\Lambda(t)$ is block diagonal with the elements

$$(\lambda_{n-k}(t), \lambda_{n-k-1}(t), \ldots, \lambda_1(t))$$

in the main diagonal. Take the transformation matrix

$$P_n^{(k)}(t) = \left[\begin{array}{cc} P_k(t) & 0 \\ 0 & I_{n-k} \end{array} \right] \tag{34}$$

where $P_k(t)$ is a $k \times k$-dimensional transformation matrix (as described above) and I_{n-k} is a $(n-k) \times (n-k)$-dimensional identity matrix. The system matrix changes to the form

$$E_n^{(k)}(t) = \left[\begin{array}{cc} E_{11}(t) & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{array} \right] \tag{35}$$

with

$$\dot{P}_k(t) = A_k(t)P_k(t) - P_k(t)E_{11}(t) \tag{36}$$

$$E_{12}(t) = P_k^{-1}(t)X(t) \tag{37}$$

$$E_{21}(t) = 0, \quad E_{22}(t) = \Lambda(t) \tag{38}$$

It is then obvious that any $n \times n$-dimensional system matrix $A(t)$ can be triangularized by the transformation $x(t) = P(t)s(t)$, in which

$$P(t) = P_n^{(2)}(t)P_n^{(3)}(t) \cdots P_n^{(n)}(t) \tag{39}$$

Note that for all k

$$\det P_k(t) = \det P_n^{(k)}(t) = 1 \tag{40}$$

such that also

$$\det P(t) = 1 \tag{41}$$

The matrix $P(t)$ is then a Lyapunov transformation provided that all its elements are bounded.

It should be noted that transformations for diagonalization and triangularization of a given system matrix always exist according to equations (5) or (6). In the case of a diagonalizable constant system matrix the diagonalizing transformation P coincides with the well-known similarity transformation. Also, the equation (5) guarantees that in the triangularization procedure the nonlinear Riccati type equations (31) have unique solutions.

5 Input-Output systems

Consider a *SISO* input-output differential system

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{j=0}^{n} b_j(t)\frac{d^j u(t)}{dt^j} \tag{42}$$

where it is assumed that for all time instants $a_n(t) \equiv 1$ and the functions $a_i(\cdot)$ and $b_j(\cdot)$ are differentiable at least $n-1$ times. The system has the realization (1) [10] with

$$A(t) = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & \cdots & \cdots & -a_{n-1}(t) \end{array} \right]$$

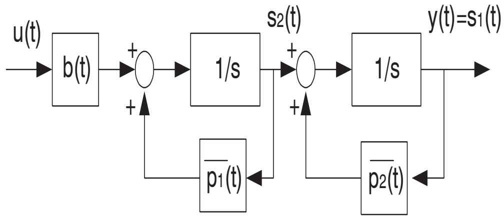

Figure 1: Realization by the characteristic modes

$$B(t) = \left[\begin{array}{c} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{array} \right]$$

$$C(t) = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array} \right] \quad D(t) = \gamma_0(t)$$

and

$$\begin{cases} \gamma_0(t) = b_n(t), \\ \gamma_i(t) = b_{n-i}(t) \\ - \sum\limits_{k=0}^{i-1} \sum\limits_{j=0}^{i-k} \frac{(n+j-i)!}{j!(n-i)!} a_{n-i+k+j}(t) \frac{d^j \gamma_k(t)}{dt^j} \end{cases}$$

$(i = 1, 2, \cdots, n)$. Consider now the two-dimensional system

$$\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = b(t)u(t) \tag{43}$$

which has a realization

$$\begin{array}{l} A(t) = \left[\begin{array}{cc} 0 & 1 \\ -a_0(t) & -a_1(t) \end{array} \right], \quad B(t) = \left[\begin{array}{c} 0 \\ b(t) \end{array} \right] \\ C(t) = \left[\begin{array}{cc} 1 & 0 \end{array} \right], \quad D(t) = 0 \end{array} \tag{44}$$

The system matrix of the target representation will have the form

$$E(t) = \left[\begin{array}{cc} \bar{p}_2(t) & 1 \\ 0 & \bar{p}_1(t) \end{array} \right] \tag{45}$$

The transformation matrix is

$$P(t) = \left[\begin{array}{cc} 1 & 0 \\ p_2(t) & 1 \end{array} \right] \tag{46}$$

which gives the input-output representation

$$\begin{array}{l} E(t) = \left[\begin{array}{cc} \bar{p}_2(t) & 1 \\ 0 & \bar{p}_1(t) \end{array} \right], \quad F(t) = \left[\begin{array}{c} 0 \\ b(t) \end{array} \right] \\ G(t) = \left[\begin{array}{cc} 1 & 0 \end{array} \right], \quad H(t) = 0 \end{array} \tag{47}$$

where

$$\begin{array}{l} \bar{p}_2(t) = p_2(t) \\ \bar{p}_1(t) = -p_2(t) - a_1(t) \end{array} \tag{48}$$

and

$$-p_2^2(t) - a_0(t) - a_1(t)p_2(t) - \dot{p}_2(t) = 0 \tag{49}$$

using an arbitrary initial condition.

The system structure in Fig. 1 is seen to be analogous to that used in [4]. It corresponds to the polynomial factorization

$$\begin{array}{l} [p^2 + a_1(t)p + a_0(t)]y(t) \\ = [p - \bar{p}_1(t)]\{[p - \bar{p}_2(t)]y(t)\} \end{array} \tag{50}$$

where $\bar{p}_2(t)$ is the *right pole* of the system. It is worth noting that if $p_2(t)$ is a bounded function, then $P(t)$ is a *Lyapunov transformation*.

Based on Fig. 1 it is easy to see how the functions $\bar{p}_1$ and $\bar{p}_2$ characterize the internal behaviour of the system. In a way, they can also be called the characteristic modes of the system, although this definition differs from that used in Section 3. To understand the meaning of the right pole, try

$$y(t) = Ce^{\int_{t_0}^{t} p_2(\nu)d\nu} \tag{51}$$

where C is a constant, as a solution to

$$\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = 0 \tag{52}$$

It is seen that the equation holds if

$$Ce^{\int_{t_0}^{t} p_2(\nu)d\nu} \left[p_2^2(t) + \dot{p}_2(t) + a_1(t)p_2(t) + a_0(t)\right] = 0 \tag{53}$$

But this equation really holds according to (49). More important, the solution for $y(t)$ was written as a constant multiplied by an exponential term giving a right justification for the concept of a mode. Note also that

$$\dot{y}(t) = Ce^{\int_{t_0}^{t} p_2(\nu)d\nu} p_2(t) = y(t)p_2(t) \tag{54}$$

which means that if the state component $x_1 = y$ is bounded, then x_2 is also bounded if P is a Lyapunov transformation. The initial conditions for the differential equation are given by $y(t_0)$ and $\dot{y}(t_0)$, meaning that the constant C and the initial condition for (49) are fixed.

The integral in the exponential term of the mode must approach minus infinity for asymptotic stability, or stay bounded for stability. In order the stability result to hold for the original system, the function p_2 must be bounded. If it is not (that can happen), the matrix P is not a Lyapunov transformation. In this case the analysis of the characteristic modes through the triangularization procedure does not give any benefit when compared to the traditional approach (modes determined by diagonalization).

It is instructive to consider an example case where both coefficients a_0 and a_1 are constant. For example, let $a_1 = 0, a_0 = 1$. The Riccati equation now has both a constant and a dynamic solution. In the former case the right pole can have values $+i$ or $-i$ which is bounded and indicates a stable oscillation, as expected. In the latter case, the solution for the right pole is not bounded so that it does not give information about stability.

The above results are in accordance or even a consequence of the triangularization procedure described earlier. Therefore they are valid for systems for any dimension. For example, if $n = 3$ in (42) then the system matrix of the target system becomes

$$E(t) = \begin{bmatrix} p_1(t) & 1 & 0 \\ 0 & p_{2b}(t) & 1 \\ 0 & 0 & -a_2(t) - p_{2b}(t) \end{bmatrix} \tag{55}$$

where

$$\begin{aligned} \dot{p}_{2a}(t) &= -a_0(t) - a_2(t)p_{2a}(t) - p_{2a}(t)p_{2b}(t) \\ \dot{p}_{2b}(t) &= -a_1(t) - a_2(t)p_{2b}(t) - p_{2a}(t) - p_{2b}^2(t) \end{aligned} \tag{56}$$

and for the right pole

$$\dot{p}_1(t) = p_{2a}(t) + p_{2b}(t)p_1(t) - p_1^2(t) \tag{57}$$

The right pole can also be computed from

$$\begin{aligned} \ddot{p}_1(t) = &-a_2(t)\dot{p}_1(t) - p_1^3(t) - 3p_1(t)\dot{p}_1(t) \\ &-a_2(t)p_1^2(t) - a_1(t)p_1(t) - a_0(t) \end{aligned} \tag{58}$$

which can also be found from [4].

In practice, the triangularization procedure involves solving systems of nonlinear Riccati type equations, which in this case corresponds to solving the state transformation matrix of a time-varying systems. An analytic solution is seldom accessible.

6 Conclusion

The relationship of system modes to diagonal and triangular forms of the system matrix have been established in the paper. A time-varying state transformation was shown to be a powerful tool in the analysis of different realizations and stability issues of the system. The approach was also extended to the case of input-state-output systems in the SISO case, making it possible to define the system poles by using the modal structure discussed.

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