STABILITY OF LIMIT CYCLES WITH CHATTERING IN RELAY FEEDBACK SYSTEMS

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Abstract

A relay feedback linear system with the transfer function of the pole excess equal to two can have a sliding mode in the set of co-dimension two and chattering mode around it. The chattering can be a part of a limit cycle in the relay feedback system and this limit cycle can be unique and globally stable. It was shown before that a limit cycle exists for a special class of systems with stable denominator and unstable numerator. In this paper, it is proved that the cycle is locally stable under the same conditions. An approach for global stability analysis was developed and illustrative results are shown.

1 Introduction

Relay feedback systems were shown to be useful for system identification and PID controller design [1, 2, 3, 11]. Recently a new type of system behaviour that contains fast switches was studied [5, 9] and called chattering. A smooth envelope for chattering variables were obtained in [10]. The envelope together with sliding mode for smooth variables represents a simple approximation to the complicated trajectory with unbounded number of switchings.

In [10] an existence of a limit cycle with chattering was established for a class of relay feedback systems with sufficiently small zeros of the transfer function. Local stability of this limit cycle was not proved, and it is presented in this paper. The problem of global stability of relay feedback systems was not solved analytically. A numerical approach proposed in [7] is based on the LMI technique and deals with the simple limit cycle only. In this paper a new approach for global stability analysis is proposed for the same class of chattering systems as in [10].

Consider a linear time-invariant system with relay feedback. The linear system is described by the scalar transfer function

$$G(s) = b(s)/a(s)$$

of relative degree two:

$$\begin{aligned} a(s) &= s^n + a_1 s^{n-1} + \cdots + a_n, \\ b(s) &= s^{n-2} + b_1 s^{n-3} + \cdots + b_{n-2}. \end{aligned}$$

The relay feedback is defined by

$$u = -\operatorname{sgn} y \in \begin{cases} \{-1\}, & y > 0, \\ [-1, 1], & y = 0, \\ \{1\}, & y < 0. \end{cases} \tag{1}$$

Choose the state-space representation in the standard Frobenius form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2}$$

with $x \in \mathbf{R}^n$ and

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

The *switch plane* is denoted $\mathcal{S} = \{x : Cx = 0\}$. A *limit cycle* $\mathcal{L} \subset \mathbf{R}^n$ in this paper denotes the set of values attained by a periodic trajectory, which is isolated and not an equilibrium. A limit cycle $\mathcal{L}$ is stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|d_{\mathcal{L}}(x(0))| < \delta$ implies that $d_{\mathcal{L}}\big(x(t)\big) < \epsilon$ for all $t > 0$, where $d_{\mathcal{L}}(x)$ is the Euclidean distance from a point x to $\mathcal{L}$.

A *sliding mode* is the part of a trajectory that belongs to the switch plane: $x(t)$ is a sliding mode for $t \in (t_1, t_2)$ with $t_1 > 0$ if $Cx(t) = 0$ for all $t \in (t_1, t_2)$. Sliding modes are treated thoroughly in [4].

Since $CB = 0$, $CAB > 0$, any sliding mode belongs to the set $\mathcal{S}_2 := \{x : Cx = CAx = 0, |CA^2x| \le 1\}$ which is called the second-order sliding set. Any trajectory close to the set $\mathcal{S}_2$ will give fast relay switches [9], and it is called *chattering mode*. Due to the state-space parameterization, the fast behaviour takes place in the variables x_1 and x_2. They are therefore called the *chattering variables*. The components $x_3, \dots, x_n$ are called the *smooth variables* of the chattering mode.

2 Local Stability of Limit Cycles

The denominator $a(s)$ of the linear part transfer function $G(s)$ is the characteristic polynomial of the "smooth" part of the relay system where $u = \text{const}$ while the numerator $b(s)$ of $G(s)$

is the characteristic polynomial of the sliding mode. If $a(s)$ is Hurwitz and $b(s)$ is not Hurwitz then any trajectory containing "smooth" and "chattering" parts has stable and unstable behaviour. This gives rise to limit cycles which contain a smooth part determined by a stable linear system and a chattering part close to the set $\mathcal{S}_2$ determined by the unstable sliding mode.

Sufficient conditions for existence of a limit cycle containing smooth and chattering parts are given in [10, Theorem 3]. The next theorem states that under the same conditions the limit cycle is locally stable.

Theorem 1. *Consider the system (1)–(2) with $n \geq 4$ and let $b(s)/a(s) = C(sI-A)^{-1}B_2$. Assume $b(s) = \epsilon^{n-2}\bar{b}(s/\epsilon)$ with $\bar{b}(s) = s^{n-2}+\bar{b}_1 s^{n-3}+\ldots+\bar{b}_{n-2}$ and let $\bar{F}_2$ be the Frobenius matrix for the polynomial $\bar{b}$. If*

1. *the matrix A is Hurwitz and the eigenvalue of A with the largest real part is unique;*
2. *$\bar{b}_{n-2} > 0$;*
3. *the solution of*
$$\dot{\bar{w}}(t) = \bar{F}_2\bar{w}(t), \qquad \bar{w}(0) = (1, \bar{b}_1, \ldots, \bar{b}_{n-3})^T$$
reaches the hyperplane $\bar{w}_1 = -1$ at $t = \bar{\tau} > 0$, it holds that $|\bar{w}_1(t)| < 1$ for $t \in (0, \bar{\tau})$, and $\bar{w}_2(\bar{\tau}) < -\bar{b}_1$; and
4. *$e_1^T e^{At} e_4 > 0$ for all $t > 0$, where $e_1 = (1, 0, \ldots, 0)^T$ and $e_4 = (0, 0, 0, 1, 0, \ldots, 0)^T$;*

then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the system (1)–(2) has a symmetric stable limit cycle with chattering.

Proof is given in Appendix.

Chattering mode does not exist for two-dimensional systems. If $n = 3$ then a chattering mode can not be a part of a limit cycle. Hence, $n = 4$ is the smallest dimension for the existence of smooth loops in the limit cycle which are useful in system identification.

Under the conditions of Theorem 1 the limit cycle consists of two parts in a half period. The smooth part is proportional to ϵ and located around the points $x_3 = \pm 1$ and all other variables are zero. Denote the segment between these points by I_0.

The chattering part winds around the segment of sliding mode which is close to the segment I_0. The trajectory tends to I_0 as $\epsilon \to 0$. The plots of the limit cycle for the transfer function $G(s)=(s-\epsilon)^2/(s+1)^4$ with $\epsilon=0.04$ are shown in Figure 1.

A limit cycle with chattering exists not only for very small values of ϵ. In Figure 2 the limit cycle is shown for the same transfer function with $\epsilon = 0.2$. The magnified part of the chattering variable x_2 is shown in Figure 3.

3 Global Stability of Limit Cycles

Sufficient conditions for global stability of the limit cycle with chattering are derived in this section for systems which satisfy assumptions of Theorem 1.

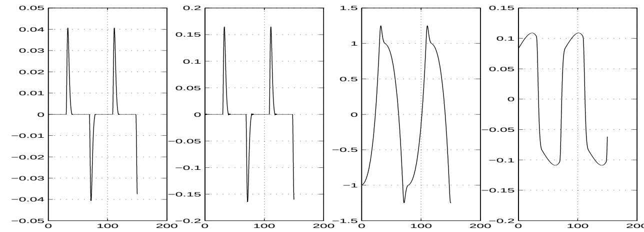

Figure 1: Limit cycle consisting of smooth and chattering parts. $G(s) = (s-0.04)^2/(s+1)^4$. Variables x_1, x_2, x_3, x_4.

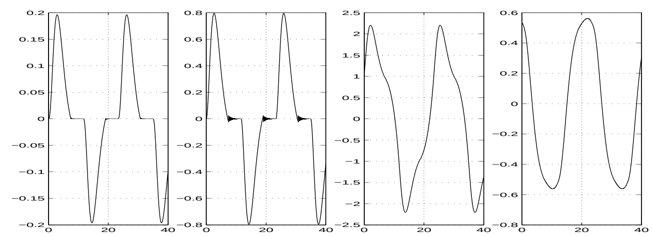

Figure 2: Chattering part of the limit cycle becomes shorter for $G(s) = (s-0.2)^2/(s+1)^4$. Variables x_1, x_2, x_3, x_4.

The chattering part of the limit cycle described in Theorem 1 tends to the segment I_0 as $\epsilon \to 0$. Therefore, the first question is whether the segment I_0 is globally stable for $\epsilon = 0$. This will be proved by the Kalman-Yakubovich lemma. As a result, this segment for $\epsilon = 0$ is quadratically stable and there exists a quadratic Lyapunov function.

For small $\epsilon > 0$ the domain of attraction is close to the segment I_0 because small variations destroy the quadratic Lyapunov function in a small neighbourhood of the limit set. But in the small neighbourhood of the chattering mode the Brauwer fixed point theorem can be applied for the analysis of the Poincare mapping, as it was shown in the proof of Theorem 3 in [10]. It was proved there that this mapping is a contraction, that implies stability of the limit cycle in this neighbourhood.

Consider the case $\epsilon = 0$ and hence, $G(s) = s^{n-2}/a(s)$. It is easy to see that the segment between the points $z_- = (0, 0, -1, 0, \ldots, 0)$ and $z_+ = (0, 0, +1, 0, \ldots, 0)$ consists of stationary points only. It is required to find conditions under which this segment is globally stable.

Our approach is based on the absolute stability theory developed by V.M.Popov, V.A.Yakubovich, R.Kalman. In particular,

Proposition 1. [6] *The linear system*
$$\dot{x} = Ax + B\xi,$$
is globally stable if it satisfies the integral quadratic constraint
$$\exists\gamma, (T_k) \to \infty : \quad \int_0^{T_k} F(x(t), \xi(t))\, dt \leq \gamma$$
with some quadratic form F and if the frequency condition
$$F(i\omega I - A)^{-1}B\tilde{\xi}, \tilde{\xi}) > 0$$

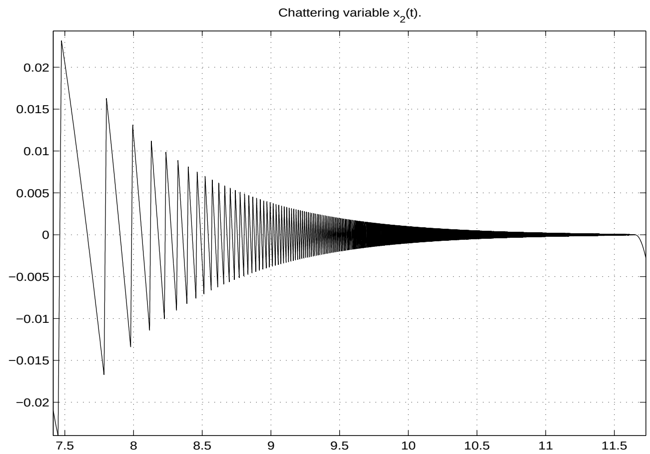

Figure 3: Magnified value of x_2 in the chattering mode. $G(s) = (s - 0.2)^2/(s+1)^4$.

holds for all complex $\tilde{\xi} \neq 0$ *and all real* $\omega \geq 0$.

There are two standard integral quadratic constraints (IQC's) for the relay feedback. First, the quadratic function $F_1(x,u) = yu = -|y|$ is obviously negative. The second function is $F_2(x,u) = \pm\dot{y}u = \pm(x_2 - a_{n-1}x_1)u$. If t_k, t_{k+1} are two successive switching instants then $u(t) = u_k = -\operatorname{sgn}(y(t))$ is constant between them and

$$\int_{t_k}^{t_{k+1}} \dot{y}(t)u(t)\,dt = u_k(y(t_{k+1}) - y(t_k)) = 0.$$

But the frequency condition does not hold for these IQC's or their linear combinations for the general case. For instance, it does not hold for the example $s^2/(s+1)^4$. Therefore other IQC's are required.

The next assertion follows from Proposition 1 and Theorem 1.

Theorem 2. *Assume conditions of Theorem 1 hold and there exist* $\beta \geq 0$, α *and a symmetric matrix* P *such that*

1. The function

$$\int_0^T (-\beta|y(t)| + x(t)^* P x(t))\,dt$$

is upper bounded on the solutions of the equation $\dot{x} = Ax + Bu$ *with* $|u| \leq 1$ *and* $x(0) = 0$.

2. Let $\tilde{x}(s) = (sI - A)^{-1}B$. *The function*

$$W(s) = \alpha\frac{s^{n-1}}{a(s)} + \beta\frac{s^{n-2}}{a(s)} + \tilde{x}(s)^* P \tilde{x}(s)$$

is positive real, that is, $\operatorname{Re} W(i\omega) > 0$ *for all* $\omega \geq 0$.

Then there exists $\epsilon_0 > 0$ *such that for every* $\epsilon \in (0, \epsilon_0)$ *the system (1)–(2) has a symmetric limit cycle with chattering which is globally stable.*

4 Example

The relay feedback system with the plant transfer function $(s-\epsilon)^2/(s+1)^4$ was studied in [10] and exhibits the chattering behavior of trajectories. The existence of a limit cycle was proved in [10]. It follows from Theorem 1 that this limit cycle is locally stable.

The analysis of global stability by Theorem 2 is based on a choice of the appropriate quadratic form F for the system with $\epsilon = 0$. The system has a segment of equilibrium points between $(0,0,-1,0)$ and $(0,0,1,0)$.

Define $\alpha = 1/4$, $\beta = 0$ and

$$x^* P x = x_1^2 - (x_2 - 4x_1)^2 = |y|^2 - |\dot{y}|^2.$$

Then $\tilde{y}(s) = s^2/(s+1)^4$ and it is easy to find that

$$\operatorname{Re} W(s) = \operatorname{Re}\frac{s^3}{4(s+1)^4} + \frac{|s|^4}{|s+1|^8} - \frac{|s|^6}{|s+1|^8} = 0$$

where $s = i\omega$. The conclusion of Theorem 2 holds for this case of nonnegative real function if the integral in the first condition is negative, that is, if for some sequence $T_k \to \infty$ it holds

$$\int_{T_k}^{T_{k+1}} y^2(t)\,dt < \int_{T_k}^{T_{k+1}} \dot{y}^2(t)\,dt$$

for all $k = 1, 2, \ldots$

All trajectories of the system are obviously bounded. Then it can be shown that there exists a domain of attraction D such that the interval between successive instants in this domain $t_{k+1} - t_k$ is less than 3 or the trajectory tends to one of the point $(0,0,-1,1)$ or $(0,0,1,0)$ directly or with one segment of chattering mode.

Assume the distance between switching instants is less than 3. Then the following Virtinger inequality [8] can be used:

$$\int_a^b f(x)^2\,dx \leq \frac{4(b-a)^2}{\pi^2}\int_a^b f'(x)^2\,dx$$

whenever the differentiable function f has a zero in $[a,b]$.

Apply this inequality for half intervals $[t_k, (t_{k+1}+t_k)/2]$ and $[(t_{k+1}+t_k)/2, t_{k+1}]$. According to the conclusion of Theorem 2 there exists a limit cycle with chattering for the system under consideration with any small $\epsilon > 0$. This cycle is globally stable. For the cases $\epsilon = 0.04$ and $\epsilon = 0.2$ these limit cycles are depicted in Figures 1 and 2.

5 Sensitivity analysis

Proof of Theorem 1 is based on the variational analysis near the trajectory of the limit cycle. If the Poincare mapping is a contraction then the limit cycle is stable. This approach was developed by K.J.Astrom for stability analysis of relay systems [1]. A straightforward computation of the Jacobian of the Poincare mapping produces a full solution but it is very complicated to calculate a product of an unbounded number of matrices.

Denote the time instants of switches by $t_0 = 0$, t_1, …, t_N, so that $x(t_N) = x(0)$. Obviously, the number N is even. The function $u(t)$ is constant: $u(t) = u_k$ on each interval (t_k, t_{k+1}) and changes sign in the points t_k. Denote the length of the intervals by $\ell_k = t_{k+1} - t_k$ for $k = 0, 1, \ldots, N-1$.

Consider a variation $\delta x(0)$ of the initial state $x(0)$. According to the system equation this variation implies a variation $\delta x(t_N)$ of the state after N switches. We assume $C\delta x(t_N) = 0$, that is, the end point lies always in the switch plane. If the mapping $\delta x(0) \to \delta x(t_N)$ is a contraction with an appropriate metric then the limit cycle is stable.

Lemma 1. *Define the $n \times n$-matrix H by*

$$H = \prod_{k=0}^{N-1} \left(e^{A\ell_k} - \frac{e^{A\ell_k}(Ax(t_k) + Bu_k)Ce^{A\ell_k}}{Ce^{A\ell_k}(Ax(t_k) + Bu_k)} \right),$$

where the order of the matrix product is from the right to the left.

If all eigenvalues of H are inside the unit circle then the limit cycle is stable. If the matrix H has an eigenvalue outside the closed unit circle then the limit cycle is unstable.

Proof. Consider an interval between two successive switches: $[t_k, t_{k+1}]$. It holds $Cx(t_k) = 0$, $Cx(t_{k+1}) = 0$ and

$$x(t_{k+1}) = e^{A\ell_k}x(t_k) + (e^{A\ell_k} - I)A^{-1}Bu_k$$

where I is the identity matrix of the order n. The variation gives

$$\delta x(t_{k+1}) = e^{A\ell_k}\delta x(t_k) + e^{A\ell_k}(Ax(t_k) + Bu_k)\delta\ell_k.$$

The condition $C\delta x(t_k) = 0$ implies

$$\delta\ell_k = -\frac{Ce^{A\ell_k}\delta x(t_k)}{Ce^{A\ell_k}(Ax(t_k) + Bu_k)}.$$

Hence,

$$\delta x(t_{k+1}) = \left(e^{A\ell_k} - \frac{e^{A\ell_k}(Ax(t_k) + Bu_k)Ce^{A\ell_k}}{Ce^{A\ell_k}(Ax(t_k) + Bu_k)} \right) \delta x(t_k)$$

for $k = 0, 1, \ldots, N-1$. It remains to obtain by induction the variation through the period of the limit cycle: $\delta x(t_N) = H\delta x(0)$. The assertion of Lemma 1 follows from the stability conditions of this linear mapping.

If a limit cycle contains a small number of switches then the direct matrix multiplication and analysis of the eigenvalues of H show whether the limit cycle is stable or not. The number of switches increases in the chattering mode and the length ℓ_k of each interval tends to zero according to Theorem 1 as the chattering variable x_2 tends to zero. In this case, each multiplier in the matrix H tends to the identity matrix while the number of the multipliers tends to infinity. The asymptotic behaviour of the product is described in the following assertion.

Theorem 3. *Consider the sliding mode on the interval $[T_0, T_1]$ with zero chattering variables and the smooth variables $w(t) = (x_3(t), \ldots, x_n(t))^T$ satisfying the equation $\dot{w}(t) = F_2 w(t)$ with $|w_1(t)| < 1$ for all $T_0 \le t \le T_1$.*

Denote by $x(t)$ the chattering mode on the same interval $[T_0, T_1]$ with the initial state vector $x(T_0) = (0, x_2(T_0), w_1(T_0), \ldots, w_{n-2}(T_0))^T$ with sufficiently small $x_2(T_0)$. Denote the switch instants of $x(t)$ by $t_0 = T_0$, t_1, …, t_M.

Then the asymptotic behaviour of the variations $\delta x(t_k)$ as $x(T_0) \to 0$ is given by

$$\begin{aligned}
\delta x_2(t_k) &= (-1)^k \delta x_2(T_0) e^{-\frac{2}{3}(a_1 - b_1)(t_k - T_0)} \times \\
&\quad \left(\frac{1 - w_1^2(t_k)}{1 - w_1^2(T_0)} \right)^{\frac{2}{3}} + \mathcal{O}_2(x_2(T_0))\|\delta x(T_0)\|, \\
\delta x_j(t_k) &= \frac{1}{x_2(T_0)} \left\{ \int_{T_0}^{t_k} e^{-\frac{1}{3}(a_1 - b_1)(t_k - T_0)} \times \right. \\
&\quad \left. \left(\frac{1 - w_1^2(t_k)}{1 - w_1^2(T_0)} \right)^{\frac{1}{3}} \dot{w}_{j-2}(t)\, dt\, \delta x_2(T_0) \right\} \\
&\quad + o_j(\|\delta x(T_0)\|),
\end{aligned}$$

with $3 \le j \le n$, where $\mathcal{O}_2(x_2(T_0))/x_2(T_0)$ is bounded and $o_j(\|\delta x(T_0)\|)/\|\delta x(T_0)\| \to 0$ as $x_2(T_0) \to 0$ uniformly for $1 \le k \le M$ and $3 \le j \le n$.

Proof of Theorem 3. The chattering mode occurs when $|x_3| < 1$ and $|x_2| \ll 1 - x_3^2$ in the switch point. This is true under the conditions of Theorem 3 because $x_3(t) \approx w_1(t)$ and there exists $\gamma > 0$ such that $|w_1(t)| < 1 - \gamma$ for $T_0 \le t \le T_1$ while $x_2(t) \to 0$ as $x_2(T_0) \to 0$. According to the proof of Theorem 1 in [10] the lengths $\ell_k = t_{k+1} - t_k$ of the intervals between successive switches are proportional to $x_2(T_0)$. Therefore the Taylor expansion can be implemented for the asymptotic analysis.

Fix the switch instant t_k with $0 \le k \le M-1$ and denote

$$\begin{aligned}
\alpha_1 &= C(Ax(t_k) + Bu_k) = x_2(t_k), \\
\alpha_2 &= CA(Ax(t_k) + Bu_k) = x_3(t_k) + u_k - a_1\alpha_1, \\
\alpha_3 &= CA^2(Ax(t_k) + Bu_k) = x_4(t_k) + b_1 u_k - a_1\alpha_2 - a_2\alpha_1.
\end{aligned}$$

The length $\ell_k = t_{k+1} - t_k$ can be directly estimated [10]:

$$\ell_k = -\frac{2\alpha_1}{\alpha_2} - \frac{4}{3}\frac{\alpha_1^2\alpha_3}{\alpha_2^3} + \mathcal{O}(\alpha_1^3).$$

The matrix Taylor expansion for $e^{A\ell_k}$ gives

$$\begin{aligned}
&Ce^{A\ell_k}\delta x(t_k) = (\ell_k CA + \ell_k^2/2CA^2)\delta x(t_k) + \mathcal{O}(\alpha_1^3)\|\delta x(t_k)\| \\
&= \ell_k(\delta x_2(t_k) + \frac{\alpha_1}{\alpha_2}(a_1\delta x_2(t_k) - \delta x_3(t_k)) + \mathcal{O}(\alpha_1^3)\|\delta x(t_k)\|,
\end{aligned}$$

and

$$Ce^{A\ell_k}(Ax(t_k) + Bu_k) = CAx(t_k) + \ell_k CA(Ax(t_k) + Bu_k)$$
$$+\alpha_3\ell_k^2 + \mathcal{O}(\alpha_1^3)\|\delta x(t_k)\| = -\alpha_1 + \frac{2}{3}\frac{\alpha_1^2\alpha_3}{\alpha_2^2} + \mathcal{O}(\alpha_1^3)\|\delta x(t_k)\|.$$

The j-th entry of the vector $e^{A\ell_k}\delta x$ is equal to

$$\begin{aligned}(e^{A\ell_k}\delta x)_j &= \delta x_j(t_k) + \ell_k \delta x_{j+1}(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\| \\ &= \delta x_j(t_k) - \frac{2\alpha_1}{\alpha_2}\delta x_{j+1}(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\|\end{aligned}$$

for $2 \le j \le n$. It was proved in Lemma 1 that

$$\delta x(t_{k+1}) = \left(e^{A\ell_k} - \frac{e^{A\ell_k}(Ax(t_k)+Bu_k)Ce^{A\ell_k}}{Ce^{A\ell_k}(Ax(t_k)+Bu_k)}\right)\delta x(t_k).$$

After some algebra we get

$$\begin{aligned}\delta x_2(t_{k+1}) &= -\delta x_2(t_k) + \frac{4}{3}\frac{\alpha_1\alpha_3}{\alpha_2^2}\delta x_2(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\| \\ &= -(1+\frac{2}{3}\frac{\alpha_3}{\alpha_2}\ell_k)\delta x_2(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\| \\ &= -(1+\frac{1}{3}\frac{\alpha_3}{\alpha_2}\ell_k)^2\delta x_2(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\| \\ &= -\left(\frac{x_2(t_{k+1})}{x_2(t_k)}\right)^2\delta x_2(t_k) + \mathcal{O}(\alpha_1^2)\|\delta x(t_k)\|.\end{aligned}$$

It follows by induction that

$$\delta x_2(t_k) = (-1)^k\frac{x_2^2(t_k)}{x_2^2(T_0)}\delta x_2(T_0) + \mathcal{O}(\alpha_1)\|\delta x(t_k)\|.$$

The first assertion of Theorem 3 follows from this equation and from the following explicit expression of $x_2(t_k)$ given in [10, Theorem 1]:

$$\begin{aligned}x_2(t_k) &= (-1)^k x_2(t_0)\exp[-\frac{1}{3}(a_1-b_1)(t_k-t_0)] \\ &\quad \times\left(\frac{1-x_3^2(t_k)}{1-x_3^2(t_0)}\right)^{1/3} + o(x_2(t_0);t_k)\end{aligned}$$

where $o(x_2(t_0);t_k)/x_2(t_0) \to 0$ as $x_2(t_0) \to 0$ uniformly for all $1 \le k \le M$.

Let $3 \le j \le n$. Direct computation leads to the equation

$$\begin{aligned}\delta x_j(t_{k+1}) &= \delta x_j(t_k) - \frac{2(x_{j+1}+b_{j-2}u_k)}{\alpha_2}\delta x_2(t_k) \\ &+ \mathcal{O}(\alpha_1)\|\delta x(t_k)\|\end{aligned}$$

for $0 \le k \le M-1$.

Consider two successive intervals $[t_k, t_{k+1}]$ and $[t_{k+1}, t_{k+2}]$. The increment of $\delta x_j(t_k)$ on the two intervals is proportional to

$$\begin{aligned}&\frac{x_{j+1}(t_k)+b_{j-2}u_k}{x_3(t_k)+u_k} - \frac{x_{j+1}(t_{k+1})+b_{j-2}u_{k+1}}{x_3(t_{k+1})+u_{k+1}} \\ =\;& \frac{x_{j+1}(t_k)+b_{j-2}u_k}{x_3(t_k)+u_k} - \frac{x_{j+1}(t_k)-b_{j-2}u_k}{x_3(t_k)-u_k} + \mathcal{O}(x_2(t_k)) \\ =\;& \frac{x_{j+1}(t_k)-b_{j-2}x_3(t_k)}{1-x_3^2(t_k)}2u_k + \mathcal{O}(x_2(t_k)).\end{aligned}$$

The length of the double interval is equal to

$$\begin{aligned}\ell_k+\ell_{k+1} &= -\frac{2x_2(t_k)}{x_3(t_k)+u_k} - \frac{2x_2(t_{k+1})}{x_3(t_{k+1})+u_{k+1}} \\ &= -\frac{2x_2(t_k)}{x_3(t_k)+u_k} + \frac{2x_2(t_k)}{x_3(t_k)-u_k} + \mathcal{O}(x_2(t_k)) \\ &= -\frac{4x_2(t_k)u_k}{1-x_3^2(t_k)} + \mathcal{O}(x_2(t_k)),\end{aligned}$$

and the increment of the variation is

$$\begin{aligned}\delta x_j(t_{k+2}) &= \delta x_j(t_k) - (x_{j+1}(t_k) - b_{j-2}x_3(t_k))\times \\ &\quad \frac{4u_k}{1-x_3^2(t_k)}\delta x_2(t_k) + \mathcal{O}(x_2(t_k))\|\delta x(t_k)\| \\ &= \delta x_j(t_k) + (x_{j+1}(t_k) - b_{j-2}x_3(t_k))\times \\ &\quad (\ell_k+\ell_{k+1})\frac{\delta x_2(t_k)}{x_2(t_k)} + \mathcal{O}(x_2(t_k))\|\delta x(t_k)\|.\end{aligned}$$

The smooth variables $x_j(t)$ are close to that of the sliding mode,

$$x_j(t) = w_{j-2}(t) + \mathcal{O}(x_2(T_0))$$

for $j \ge 3$. Notice that

$$\begin{aligned}x_{j+1}(t_k) - b_{j-2}x_3(t_k) &= w_{j-1}(t_k) - b_{j-2}w_1(t_k) \\ &\quad +\mathcal{O}(x_2(T_0)) \\ &= \dot{w}_{j-2}(t_k) + \mathcal{O}(x_2(T_0))\end{aligned}$$

by the equations of the sliding mode.

Let $\bar{x}_2(t)$ be the smooth envelope of the values $|x_2(t_k)|$. It follows from the explicit expression of $x_2(t_k)$ that

$$\bar{x}_2(t) = e^{-\frac{1}{3}(a_1-b_1)(t-T_0)}\left(\frac{1-w_1^2(t)}{1-w_1^2(T_0)}\right)^{\frac{1}{3}}.$$

The next sum of the increments can be regarded as an integral sum:

$$\begin{aligned}\delta x_j(t_k) &= \delta x_j(T_0) + \sum_{i=0}^{k/2-1}(x_{j+1}(t_{2i}) - b_{j-2}x_3(t_{2i}))\times \\ &\quad (\ell_{2i}+\ell_{2i+1})\frac{\delta x_2(t_{2i})}{x_2(t_{2i})} + \mathcal{O}(x_2(t_{2i}))\|\delta x(t_{2i})\| \\ &= \delta x_j(T_0) + \int_{T_0}^{t_k}(w_{j-1}(t) - b_{j-2}w_1(t))\times \\ &\quad \frac{\delta x_2(T_0)\bar{x}_2(t)}{x_2^2(T_0)}\,dt + o_j(\|\delta x(T_0)\|) \\ &= \frac{1}{x_2(T_0)}\left\{\int_{T_0}^{t_k} e^{-\frac{1}{3}(a_1-b_1)(t_k-T_0)}\times\right. \\ &\quad \left.\left(\frac{1-w_1^2(t_k)}{1-w_1^2(T_0)}\right)^{\frac{1}{3}}\dot{w}_{j-2}(t)\,dt\,\delta x_2(T_0)\right\} \\ &\quad +o_j(\|\delta x(T_0)\|),\end{aligned}$$

that completes the proof of Theorem 3.

6 Appendix

Proof of Theorem 1. The existence of a limit cycle was shown in [10]. The limit cycle has one smooth part and one chattering part on each half-period. The time of the smooth part tends to infinity as $\epsilon \to 0$. The variation $\delta x(0)$ in the beginning of the smooth trajectory is multiplied by $\mathcal{O}(\epsilon^{n-2})$ in the point where the chattering part starts. Let it happens for $t = t_1$. The value of $x_2(t_1)$ is proportional to ϵ^{n-2} too.

Denote the point where the chattering mode is changed by the smooth mode by t_N. Then it follows from Theorem 3 that

$$\delta x_j(t_N) = \|\delta x(0)\|\mathcal{O}(I_\epsilon) + o(1),$$

where $o(1) \to 0$ as $\epsilon \to 0$ and

$$I_\epsilon = \int_{t_0}^{t_N} e^{-\frac{1}{3}(a_1-b_1)(t_N-t_1)} \left(\frac{1-w_1^2(t)}{1-w_1^2(T_0)}\right)^{\frac{1}{3}} \|\dot{w}(t)\|\, dt.$$

The integrand is a product of an integrable function $e^{-\frac{1}{3}(a_1-b_1)t}$, since $a_1 - b_1 = a_1 - \bar{b}_1\epsilon > 0$ and the function which tends to zero as $\epsilon \to 0$. Indeed, the smooth variables have small derivatives, $\|\dot{w}\| \to 0$ as $\epsilon \to 0$.

It follows $I_\epsilon \to 0$ as $\epsilon \to 0$, and the matrix H in Lemma 1 tends to zero too. By Lemma 1 the limit cycle is stable.

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