DISCRETE-TIME ANTI-WINDUP: PART 1 - STABILITY AND PERFORMANCE

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Abstract

The anti-windup problem is formulated in discrete-time using a configuration which effectively decouples the nominal linear and nonlinear parts of a closed loop system with constrained plant inputs. Conditions are derived which ensure an upper bound on the induced l_2 norm of a certain mapping which is central to the anti-windup problem. Results are given for the full-order case, where a solution always exists, and for static and low-order cases, where a solution does not necessarily exist, but which is often more appealing from a practical point of view.

1 Introduction

Anti-windup compensation was a term which, some time ago, was associated with alleviating integral windup in control systems. This meaning has evolved to represent the now-common approach to dealing with systems containing control constraints: (i) firstly, design the nominal controller without directly taking into account control constraints; (ii) secondly, design a conditioning network to limit performance deterioration in the event of a control constraint being encountered.

There is a significant amount of practical merit in this two-stage approach, as compared to a one-stage synthesis. The first advantage is, assuming we are dealing with systems which are linear other than the input saturation, one can use all the linear control design tools for the nominal controller design. The second advantage is that anti-windup compensation can be introduced to an already designed controller, which perhaps functions perfectly unless saturation is encountered, which makes the approach more flexible than the one-stage alternative. The third advantage, and one which is sometimes overlooked, is that in terms of the optimisation procedure, the two-stage approach can often be less time consuming and less computationally demanding.

Although papers on discrete-time anti-windup have appeared, sporadically, in the literature (such as [11, 8]), many of these papers are perhaps not as rigorous as one would like. For example, most do not ensure stability of the overall *nonlinear* system and few tackle performance. Even fewer tackle performance in the spirit of the "true goal" of anti-windup compensation (see later). Notable exceptions exist of course; for example [1] tackled this problem some time ago, although their results were confined to single-loop systems. Recent work has also been contributed in [4], where the "true goal" of anti-windup compensation is considered explicitly. Our paper gives a precise formulation of the discrete-time anti-windup problem, which is roughly the discrete-time counterpart of [10] (see also [9] and [3]). This formulation is, in our opinion, central to the "true goal" of anti-windup compensation - that the saturated response should deviate as little as possible from the nominal linear response without saturation. This idea first appeared, at around the same time, in [7], [9] and [12] (which we follow here), but in the continuous time context. Using this problem definition, we then solve the problem for several types of anti-windup compensators, for the class of asymptotically stable linear plants [1]. The first type of compensator which we consider is that of full-order (order equal to that of the plant), where we show that a compensator always exists which solves the problem and we also give conditions, in terms of LMI's, which can be used to synthesise an optimal compensator. The second type of compensator is that of zeroth order, that is a static compensator: although one is not guaranteed to exist, in practical situations it is often preferable due to its low complexity. We then extend this static synthesis to include low-order compensators, which can often work for systems where static compensators would not, but are also of less complexity than full-order compensators.

The paper is organised as follows. The next section introduces notation and various preliminary concepts which we shall need in the remaining parts of the paper. Section 3 formulates the discrete-time anti-windup problem. Section 4 solves the full-order version of the problem, while Section 5 treats the static and low-order versions. Some concluding remarks are made in Section 6.

2 Preliminaries

2.1 Mathematical notation

The following notation is used throughout the paper. The l_2 norm of the time sequence of vectors $x(k)$ is defined as

$$\|x\|_2 := \sqrt{\sum_{k=0}^{\infty} \|x(k)\|^2}$$

where $\|\cdot\|$ is the Euclidean (induced) norm. Any sequence $x(k)$ with finite l_2-norm is said to belong to the space l_2. For the (nonlinear) operator $H : l_2 \mapsto l_2$ the induced l_2 norm is defined as $\|H\|_{i,2} := \sup_{0 \neq x \in l_2} \frac{\|H(x)\|_2}{\|x\|_2}$. If H is linear, the in-

[1] This class of plants is the only class for which the problem can be solved globally (in a strong sense).

duced l_2 norm reduces to the $\mathscr{H}_\infty$ norm $\|H\|_{i,2} = \|H\|_\infty = \sup_{z\in\bar{\mathscr{D}}} \|H(z)\|$, where $H(z)$ is the z-domain transfer function[2] of H and $\bar{\mathscr{D}}$ is the closure of the unit disk in the complex plane. Note that this definition assumes that H is *finite gain* l_2 stable. The saturation function is defined as

$$\text{sat}(u) := [\text{sat}_1(u_1), \ldots, \text{sat}_m(u_m))]$$

where $\text{sat}_i(u_i) := \text{sign}(u_i) \times \min\{|u_i|,\ \bar{u}_i\}$, where $\bar{u}_i > 0$ is the i'th saturation limit. The following identity holds

$$\text{Dz}(u) = u - \text{sat}(u)$$

where $\text{Dz}(u)$ is the deadzone function. A decentralised nonlinear element $\mathscr{N}(.) = \text{diag}(n_1(.), \ldots, n_m(.))$ is said to belong to the Sector$[0,I]$ if all $n_i(.)$ belong to the Sector$[0,1]$, that is:

$$n_i^2(u_i) \le u_i n_i(u_i) \le u_i^2,\ u_i \in \mathbb{R} \tag{1}$$

Note that both the saturation and deadzone operators belong to Sector$[0,I]$. In fact, the results derived later in the paper actually hold for all Lipschitz nonlinear elements in the Sector$[0,I]$, rather than just the narrow class we consider here. For a decentralised Sector$[0,I]$ nonlinearity, it follows that there exists a diagonal matrix W such that

$$\mathscr{N}(u)'W(u - \mathscr{N}(u)) \ge 0,\ u \in \mathbb{R}^m \tag{2}$$

We define the set $\mathscr{U}$ as

$$\mathscr{U} := [-\bar{u}_1,\ \bar{u}_1] \times, \ldots, \times [-\bar{u}_m,\ \bar{u}_m]$$

It is evident that $\text{sat}(u) = u,\ \text{Dz}(u) = 0 \quad \forall u \in \mathscr{U}$ and that this is the set in which the saturation element behaves linearly. The distance from a vector x to a set $\mathscr{X}$ is defined as $dist(x,\mathscr{X}) := \inf_{w\in\mathscr{X}} \|x - w\|$. $\mathscr{R}^{i\times j}$ represents the space of all real rational transfer function matrices of dimension $i \times j$.

2.2 Anti-windup configuration

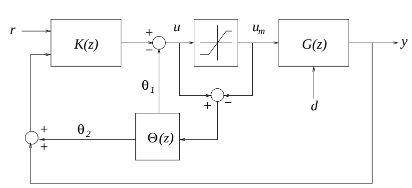

Figure 1: A general anti-windup configuration

We consider the design of compensators, $\Theta(z)$, for the framework of Figure 1, which we call a "general" anti-windup configuration. In such a configuration, the signals generated by the anti-windup compensator are fed into the controller output and the controller input. We also consider the system in Figure 2 where the anti-windup compensation is introduced into the system using the free parameter $M(z)$. As noted in [12], most linear conditioning schemes can be interpreted in the framework of Figure 2, and, indeed, this interpretation is central to our paper.

[2]For linear systems, we often do not distinguish explicitly between the operator and its transfer function.

The stabilisable and detectable plant, $G(z) = [G_1(z)\ G_2(z)]$, has the following state-space description

$$G(z) \sim \begin{cases} x_p(k+1) = A_p x_p(k) + B_p u_m(k) + B_{pd} d(k) \\ \ \ y(k) \quad\ = C_p x_p(k) + D_p u_m(k) + D_{pd} d(k) \end{cases} \tag{3}$$

where $x_p(k) \in \mathbb{R}^{n_p}$ is the plant state, $u_m(k) \in \mathbb{R}^m$ is the actual control input to the plant, $d(k) \in \mathbb{R}^{n_d}$ is some disturbance, $y(k) \in \mathbb{R}^q$ is the output which is fed back to the controller and $G_1(z) \sim (A_p, B_{pd}, C_p, D_{pd})$ and $G_2(z) \sim (A_p, B_p, C_p, D_p)$ represent the disturbance feedforward and feedback transfer functions of $G(z)$. As the work here is seeking global results, we are necessarily forced to assume that $G(z)$ is asymptotically stable; that is $|\lambda_{max}(A_p)| < 1$. This is necessary in the approach we take, as will be clear later.

We assume the following stabilisable, detectable, linear controller $K(z) = [K_1(z)\ K_2(z)]$ has been designed to control the plant $G(z)$,

$$K(z) \sim \begin{cases} x_c(k+1) = A_c x_c(k) + B_c y(k) + B_{cr} r(k) \\ \ \ y_c(k) \quad = C_c x_c(k) + D_c y(k) + D_{cr} r(k) \end{cases} \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $y_c \in \mathbb{R}^m$ is the controller output - the desired control signal, $r \in \mathbb{R}^{n_r}$ is the reference command and $K_1(z) \sim (A_c, B_{cr}, C_c, D_{cr})$, $K_2(z) \sim (A_c, B_c, C_c, D_c)$. The plant input u_m is given by $u_m = sat(u)$. We make the following formal assumption on the closed-loop systems

Assumption 1 *1. The poles of $(I - K_2G_2)^{-1}(z)$ are in the open unit disk.*

2. $\lim_{z\to\infty}(I - K_2(z)G_2(z))^{-1}$ exists □

The first statement ensures the nominal closed-loop system is stable; the second ensures it is mathematically well-posed and, in state-space terms, is equivalent to the existence of the matrices

$$\Delta := (I - D_p D_c)^{-1}, \quad \tilde{\Delta} := (I - D_c D_p)^{-1}$$

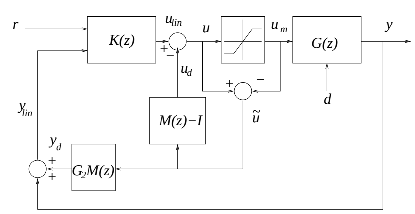

Figure 2: Conditioning with $M(z)$

A novel way of representing most anti-windup configurations was introduced in [12], where one interprets the conditioning of controllers in terms of a single transfer function $M(z)$. The discrete-time equivalent of this system is shown in Figure 2. In [12], it was shown that such a scheme could exhibit an attractive decoupled structure. This also holds for discrete-time systems and, with all signals labelled identically, Figure 2 can be re-drawn as Figure 3. Notice that this configuration reveals a useful decoupling into nominal linear system, nonlinear loop and disturbance filter. This was analysed, in continuous time, in terms of existing schemes in [13] and extended to static and low-order compensators in [10].

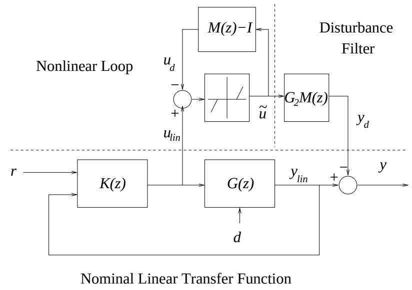

Figure 3: Equivalent representation of conditioning with $M(z)$

2.3 Problem definition

Bearing in mind our argument in the introduction that the true goal of anti-windup compensation is to ensure that the response of the saturated system deviates as little as possible from that of the nominal linear system, it stands to reason in terms of the framework of Figure 3 that the performance of our anti-windup compensator can be judged on how small the size of y_d is in response to u_{lin}. As $y = y_{lin} - y_d$, the size of y_d is a direct measure of the saturated system's deviation from the nominal linear performance in response to u_{lin}. In [10], the continuous time version of the mapping $\mathscr{T} : u_{lin} \mapsto y_d$ was picked as a measure of the anti-windup compensator's performance. Similar to [10], we shall choose to minimise $\|\mathscr{T}\|_{i,2}$ in our anti-windup synthesis. We now formally define the problem we seek to solve in the remainder of the paper.

Definition: The anti-windup compensator $\Theta(z)$ is said to solve the anti-windup problem if the closed loop system in Figure 3 is internally stable and well-posed and if

1. $\text{dist}(u_{lin}, \mathscr{U}) = 0, \quad \forall t \geq 0$ then $y_d = 0, \quad \forall t \geq 0$ (assuming zero initial conditions for $\Theta(z)$).
2. $\text{dist}(u_{lin}, \mathscr{U}) \in l_2$, then $y_d \in l_2$.

The anti-windup compensator $\Theta(z)$ is said to *solve strongly the anti-windup problem* if, in addition, the following condition is satisfied.

3. The operator $\mathscr{T} : u_{lin} \mapsto y_d$ is well-defined and finite gain l_2 stable. □

Remark 1: In this paper we actually only deal with the strong version of the anti-windup problem. The weaker version is included to enable the reader to make comparisons with other techniques in the literature, such as that in [10]. □□

3 Full-order anti-windup synthesis

Before considering the actual anti-windup synthesis, we first make the following observation:

Observation 1 *There always exists a full-order compensator which solves strongly the anti-windup problem*

Proof: To see this note that setting $M = I$ results in the nonlinear "loop" reducing to a pure deadzone, that is, there is no stability issue. So the operator $\mathscr{T} = Dz(G_2(.))$ for which $\|\mathscr{T}\|_{i,2} = \|G_2\|_\infty$ since $\|Dz(\cdot)\|_{i,2} = 1$. By stability and linearity of $G_2(z)$, condition 1 of the problem is solved. □□

By choosing $M = I$, we have effectively recovered the globally stable internal model control scheme (IMC) (see [14]). It is well known [2] that although IMC always ensures stability, it can lead to very poor performance. This therefore motivates the search for compensators which improve on the performance of the IMC scheme, but also retain its stability properties.

As in [12] for continuous-time, $M(z)$ can be chosen as a coprime factor of $G_2(z)$. So if $G_2(z) = N(z)M^{-1}(z)$, we can search for a coprime factor of $G(z)$ such that the anti-windup closed-loop has the best performance in terms of the gain of $\mathscr{T}$. This approach is also related to that of [7] and, to a lesser extent, that of [6]. To achieve full-order stabilisation we would like to choose coprime factors, which share the same state space and are of order equal to that of $G_2(z)$. Employing Figure 3, such coprime factorisations can be characterised by

$$\begin{bmatrix} M(z) - I \\ N(z) \end{bmatrix} \sim \left[\begin{array}{c|c} A_p + B_pF & B_p \\ \hline F & 0 \\ C_p + D_pF & D_p \end{array}\right] \quad (5)$$

where $\tilde{u}(k) = Dz(u_{lin}(k) - u_d(k))$. Note that these equations are parameterised by the free parameter F and therefore we attempt to choose F such that $\|\mathscr{T}\|_{i,2}$ is minimised.

Theorem 1 *There exists a dynamic compensator $\Theta(z)$ of order n_p which solves strongly the anti-windup problem if there exist matrices $Q > 0, W = \text{diag}(\mu_1, \ldots \mu_m) > 0, L \in \mathbb{R}^{(m+q)\times m}$ and a scalar $\mu > 0$ such that the following linear matrix inequality is satisfied*

$$\begin{bmatrix} -Q & -L' & 0 & QC_p + L'D_p' & QA_p + L'B_p' \\ \star & -2U & I & UD_p' & UB_p' \\ \star & \star & -\mu I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0 \quad (6)$$

Furthermore, if this inequality is satisfied, a suitable F for (5) achieving $\|\mathscr{T}\|_{i,2} < \gamma = \sqrt{\mu}$, is given by $F = LQ^{-1}$.

Proof: Let us choose a Lyapunov function candidate as $V(k) = x(k)'Px(k) > 0$. We define the Lyapunov difference as $\Delta V(k) := V(k+1) - V(k)$. Next we consider the function $\Delta\tilde{V}(k)$ which is defined as

$$\begin{aligned} \Delta\tilde{V}(k) \quad &:= \quad \Delta V(k) + 2\tilde{u}(k)'W[u_{lin}(k) - u_d(k) - \tilde{u}(k)] \\ &\quad + \|y_d(k)\|^2 - \gamma^2\|u_{lin}(k)\|^2 \end{aligned} \quad (7)$$

This function is a combination of the Lyapunov difference (first term), the sector bounds from (2) associated with the deadzone nonlinearity (second term), and terms which ensure we have a certain level of l_2 performance. If we can ensure that equation (7) is negative definite, we have

1. **Asymptotic stability** When $u_{lin}(k) = 0$ and

$$\Delta\tilde{V}(k) = \Delta V(k) + \underbrace{2\tilde{u}(k)'W[u_{lin}(k) - u_d(k) - \tilde{u}(k)]}_{\geq 0}$$

$$+\underbrace{\|y_d(k)\|^2}_{\geq 0} < 0$$

for $[x(k)\ \tilde{u}(k)\ u_{lin}(k)] \neq 0$ asymptotic stability is implied from $\Delta V(k) < 0$.

2. l_2 **gain** $< \gamma$.

Summing $\Delta\tilde{V}(k)$ from 0 to ∞ gives:

$$\sum_{k=0}^{\infty} \Delta V(k) + 2\sum_{k=0}^{\infty} \tilde{u}(k)'W[u_{lin}(k) - u_d(k) - \tilde{u}(k)]$$

$$+\|y_d\|_2^2 - \gamma^2\|u_{lin}\|_2^2 < 0$$

As $\sum_{k=0}^{\infty} \Delta V(k) = V(\infty) - V(0) > 0$ we get

$$\underbrace{V(\infty)}_{>0} - V(0) + \underbrace{2\sum_{k=0}^{\infty} \tilde{u}(k)'W[-u_d(k) - \tilde{u}(k)]}_{\geq 0}$$

$$+\|y_d\|_2^2 - \gamma^2\|u_{lin}\|_2^2 < 0$$

which implies $\|y_d\|_2 < \gamma\|u_{lin}\|_2 + V(0)$ and $\|\mathscr{T}\|_{i,2} < \gamma$.

Substituting for $x(k), u_d(k)$ and $y_d(k)$ in (7), we have that $\Delta\tilde{V}(k) < 0$ if

$$\begin{bmatrix} x(k) \\ \tilde{u}(k) \\ u_{lin}(k) \end{bmatrix}' \underbrace{\begin{bmatrix} V_{F11} & V_{F12} & 0 \\ \star & V_{F22} & W \\ \star & \star & -\gamma^2 I \end{bmatrix}}_{V_F} \begin{bmatrix} x(k) \\ \tilde{u}(k) \\ u_{lin}(k) \end{bmatrix} < 0 \qquad (8)$$

where

$$\begin{aligned} V_{F11} &= (A_p + B_pF)'P(A_p + B_pF) - P \\ &\quad + (C_p + D_pF)'(C_p + D_pF) &(9) \\ V_{F12} &= (A_p + B_pF)'PB_p - F'W + (C_p + D_pF)'D_p &(10) \\ V_{F22} &= -2W + D_p'D_p + B_p'PB_p &(11) \end{aligned}$$

The remainder of the proof follows by standard Schur complement and congruence transformation arguments to show that $V_F < 0$ is equivalent to (6). Note that as there is no 'direct feedthrough' term in the nonlinear loop, well-posedness is ensured. To see that condition 1 of the anti-windup problem definition is satisfied, note that $\Theta(z)$ is linear and Schur stable. □□

Remark 2: It is easy to see that in terms of the configuration in Figure 1, we have $\Theta_1(z) = M(z) - I$ and $\Theta_2(z) = G_2(z)M(z)$. □□

4 Static anti-windup synthesis

Full-order anti-windup synthesis may often lead to the responses which deteriorate the least during saturation. However, such compensators require a significant amount of extra on-line computation to be performed, which may not always be possible in some applications. This is good motivation for static compensators, which we consider in this section.

4.1 Representing $M(z)$

The closed loop analysis is examined in terms of conditioning via $M(z)$. Hence, the first step in the static synthesis is to derive an expression for $M(z)$ in terms of the now-static compensator, $\Theta = [\Theta_1'\ \Theta_2']' \in \mathbb{R}^{(m+q)\times m}$. To do this, we proceed in much the same manner as in [10]. Comparing Figure 1 with Figure 2, we see that we have the following expressions

$$\begin{aligned} u &= K_1 r + K_2 y + (K_2\Theta_2 - \Theta_1)\tilde{u} &(12) \\ u &= K_1 r + K_2 y - [(I - K_2G_2)M - I]\tilde{u} &(13) \end{aligned}$$

Obviously, for the two schemes to be equivalent, we must have

$$M = (I - K_2G_2)^{-1}(-K_2\Theta_2 + \Theta_1 + I).$$

Note that $M(z)$ is well defined by virtue of Assumption 1. From this we can form the state-space realisation

$$\begin{bmatrix} M(z) - I \\ G_2(z)M(z) \end{bmatrix} \sim \begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + (\bar{B}_0 + \bar{B}\Theta)\tilde{u}(k) \\ u_d = \bar{C}_1\bar{x}(k) + (\bar{D}_{01} + \bar{D}_1\Theta)\tilde{u}(k) \\ y_d = \bar{C}_2\bar{x}(k) + (\bar{D}_{02} + \bar{D}_2\Theta)\tilde{u}(k) \end{cases} \qquad (14)$$

where Θ is a static matrix. A "minimal realisation" of the remaining matrices is given in the appendix.

4.2 Solution of problem

The following theorem, which is the discrete-time equivalent of Theorem 1 in [10], is the main result of the section:

Theorem 2 *There exists a static compensator $\Theta = [\Theta_1'\Theta_2']' \in \mathbb{R}^{(m+q)\times m}$ which solves strongly the anti-windup problem if there exist matrices $Q > 0, U = \mathrm{diag}(\mu_1, \ldots, \mu_m) > 0, L \in \mathbb{R}^{(m+q)\times m}$ and a positive real scalar $\mu > 0$ such that the following LMI is satisfied*

$$\begin{bmatrix} -Q & -Q\bar{C}_1' & 0 & Q\bar{C}_2' & Q\bar{A}' \\ \star & -X & I & UD_{02}' + L'\bar{D}_2' & UB_0 + L'\bar{B}' \\ \star & \star & -\mu I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0 \qquad (15)$$

where $X = 2U + D_{01}U + \bar{D}_1L + UD_{01}' + L'\bar{D}_1'$. Furthermore, if this inequality is satisfied a suitable Θ achieving $\|\mathscr{T}\|_{i,2} < \gamma = \sqrt{\mu}$ is given by $\Theta = LU^{-1}$.

Proof: Let us choose a Lyapunov function candidate as $V(k) = x(k)'Px(k) > 0$. As with the proof of Theorem 1, if we can find a function $\Delta\tilde{V}(k)$ such that for $[x(k)\ \tilde{u}(k)\ u_{lin}(k)] \neq 0$

$$\begin{aligned} \Delta\tilde{V}(k) &= \Delta V(k) + 2\tilde{u}(k)'W[u_{lin}(k) - u_d(k) - \tilde{u}(k)] \\ &\quad + \|y_d(k)\|^2 - \gamma^2\|u_{lin}(k)\|^2 < 0 &(16) \end{aligned}$$

we know that we have *asymptotic stability* (when $u_{lin}(k) = 0$) and l_2 gain less than γ. Substituting for $x(k), u_d(k)$ and $y_d(k)$, it follows that

$$\Delta\tilde{V}(k) = \begin{bmatrix} \bar{x} \\ \tilde{u} \\ u_{lin} \end{bmatrix}' \underbrace{\begin{bmatrix} V_{R11} & V_{R12} & 0 \\ \star & V_{R22} & W \\ \star & \star & -\gamma^2 I \end{bmatrix}}_{V_R} \begin{bmatrix} \bar{x} \\ \tilde{u} \\ u_{lin} \end{bmatrix} \qquad (17)$$

where

$$\begin{aligned} V_{R11} &= \bar{A}'P\bar{A} - P + \bar{C}_2'\bar{C}_2 \\ V_{R12} &= \bar{A}'P(B_0 + \bar{B}\Theta) - \bar{C}_1'W + \bar{C}_2'(D_{02} + \bar{D}_2\Theta) \\ V_{R22} &= -2W - W(D_{01} + \bar{D}_1\Theta) - (D_{01} + \bar{D}_1\Theta)'W + \\ &\quad (D_{02} + \bar{D}_2\Theta)'(D_{02} + \bar{D}_2\Theta) + (B_0 + \bar{B}\Theta)'P(B_0 + \bar{B}\Theta) \end{aligned}$$

and $\Delta\tilde{V}(k)$ is negative definite for $V_R < 0$. The LMI in the theorem follows from $V_R < 0$ using standard Schur complement and congruence transformation procedures.

Thus far we have proved internal stability of the closed loop and condition (3) of the anti-windup problem (from which condition (2)) follows. To see that condition (1) is satisfied, note that Θ is linear and static. It thus remains to prove well-posedness. To aid us in this, the following lemma (based on a result from [3]) is required.

Lemma 1 *Assume that $-2V - \tilde{D}V - V\tilde{D}' < 0$ for some diagonal positive definite matrix $V > 0$. Also assume that the map $\Pi(w(t)) : \mathbb{R}^l \mapsto \mathbb{R}^{l\times l}$ is unique for all $w(t)$. Then $I + \tilde{D}\Pi(w(t))$ is nonsingular for all matrices $\Pi \in \mathscr{P}$, where*

$$\mathscr{P} := \left\{\Pi(w(t)) : \Pi = diag(\pi_1(w_1(t)), \ldots, \pi_l(w_l(t))\right\}$$

and $\pi_i(w_i(t)) \in [0,1]\ \forall i$. □

Note that in order for the anti-windup system to be well-posed, we must have existence and uniqueness of the equations in the nonlinear loop. For this, we must be able to determine $u_d(k)$ uniquely from the expression

$$u_d(k) = \bar{C}_1\bar{x}(k) + (D_{01} + \bar{D}_1\Theta)\tilde{u}(k) \tag{18}$$

where $\tilde{u}(k) = Dz(u(k))$. Note that we equivalently write $\tilde{u}(k) = H(u(k))u(k)$ for some discrete-time varying diagonal gain $H(u(k)) = \text{diag}\left(h_1(u(k)), \ldots, h_m(u(k))\right)$, where $h_i(u(k)) \in [0,1]\ \forall\ u(k)$ (by virtue of $Dz(.) \in \text{Sector}[0,I]$). Furthermore, as $Dz(.)$ is a well defined operator we know that $H(u(k))$ is unique for each $u(k)$. Hence existence and uniqueness of (18) are equivalent to studying existence and uniqueness of solutions to

$$u_d(k) = \bar{C}_1\bar{x}(k) + \tilde{D}H(u(k))u_{lin}(k) - \tilde{D}H(u(k))u_d(k) \tag{19}$$

where we have defined $\tilde{D} := (D_{01} + \bar{D}_1\Theta)$. A solution (or solutions) exists iff $I + \tilde{D}H(u(k))$ is invertible for all $u(k)$. Note that, in terms of Lemma 1, $H(u(t)) \in \mathscr{P}$, and hence $I + \tilde{D}H(u(t))$ will be invertible for $u(t)$, as the map $H(.) : \mathbb{R}^m \mapsto \mathbb{R}^{m\times m}$ is unique, if $-2V - \tilde{D}V - V\tilde{D}' < 0$ for some positive definite diagonal matrix V: but this will be the case if the LMI (15) is satisfied for $\Theta = LU^{-1}$ (from looking at the 2,2 term). To prove uniqueness of solutions is somewhat harder and, due to space restrictions, is omitted. □□

5 Low-order compensator synthesis

Low order compensators are potentially useful as they combine some of the advantages of both full-order and static schemes while some of the disadvantages of static anti-windup schemes can be prevented: for certain plant-controller combinations, static anti-windup is not feasible and the lack of frequency shaping can lead to robustness problems.

The approach we take mirrors the continuous time case given in [10], where we split the anti-windup compensator into two parts: a dynamic part which is chosen by the designer; and a static part which is synthesised in an optimal fashion. Note that this type of approach is definitely sub-optimal, but it has yielded good results in various case studies.

5.1 Representing $M(z)$

The approach we take to synthesising a low-order compensator is based on the static approach. Let $\Theta(z)$ be described by

$$\Theta(z) = \begin{bmatrix} \Theta_1(z) \\ \Theta_2(z) \end{bmatrix} = \begin{bmatrix} F_1(z)\tilde{\Theta}_1 \\ F_2(z)\tilde{\Theta}_2 \end{bmatrix} \in \mathscr{R}^{(m+q)\times m} \tag{20}$$

where $F_1(z) \in \mathscr{R}^{m\times m}$ and $F_2(z) \in \mathscr{R}^{q\times q}$ are transfer function matrices and $\tilde{\Theta}_1 \in \mathbb{R}^{m\times m}$ and $\tilde{\Theta}_2 \in \mathbb{R}^{q\times m}$ are constant matrices. Since $F_1(z)$ and $F_2(z)$ are chosen by the designer, only $\tilde{\Theta}_1, \tilde{\Theta}_2$ are synthesised in an optimal way, similar to the pure static synthesis described earlier. Obviously the resulting compensator will be sub-optimal in terms of its $\mathscr{L}_2$ gain but simulation results have shown that using relatively simple choices for $F_1(z), F_2(z)$, such as first order low pass filters (which suppress high frequency signals in y_d), good responses can be obtained. As before we obtain that $M = (I - K_2G_2)^{-1}(-K_2\Theta_2 + \Theta_1 + I)$. Similar to before, this implies that

$$\begin{bmatrix} M(z) - I \\ G_2(z)M(z) \end{bmatrix} \sim \begin{cases} \bar{x}(k) = \bar{A}\bar{x}(k) + (B_0 + \bar{B}\tilde{\Theta})\tilde{u}(k) \\ u_d(k) = \bar{C}_1\bar{x}(k) + (D_{01} + \bar{D}_1\tilde{\Theta})\tilde{u}(k) \\ y_d(k) = \bar{C}_2\bar{x}(k) + (D_{02} + \bar{D}_2\tilde{\Theta})\tilde{u}(k) \end{cases} \tag{21}$$

where the matrices are described in the appendix and $\tilde{\Theta}$ is given as $\tilde{\Theta} := [\tilde{\Theta}_1' \quad \tilde{\Theta}_2']'$.

5.2 A sub-optimal synthesis routine

Theorem 3 *Given $F_1(z), F_2(z)$, where $\deg(F_1(z)) = n_1$ and $\deg(F_2(z)) = n_2$, and $\tilde{\Theta} = [\tilde{\Theta}_1' \quad \tilde{\Theta}_2']' \in \mathbb{R}^{(m+q)\times m}$, then there exists an $n_1 + n_2$'th order compensator of the form of (20) which solves strongly the anti-windup problem if there exist matrices $Q > 0$, $U = diag(\mu_1, \ldots, \mu_m) > 0$, $L \in \mathbb{R}^{(m+q)\times m}$ and a positive real scalar γ such that the LMI (15) is satisfied (with the state-space realisation of equation (21)). From (15), a suitable $\tilde{\Theta}$ achieving $\|\mathscr{T}\|_{i,2} < \gamma$ is given by $\tilde{\Theta} = LU^{-1}$.* □

Proof: The proof is similar, *mutatis mutandis*, to that of Theorem 2.

Remark 3: We have only given a guide on one way to choose a low-order compensator. Another way of doing so is to solve the full-order Theorem for a full-order compensator, then, using Hankel Model reduction, reduce the order of this compensator, making sure it works well in simulation. Then solve the LMI (15) using the given reduced-order compensator, the directionality matrix, U and search for a new Lyapunov matrix, Q and a performance bound γ. Note that with this procedure, the state space matrices change from those given in the appendix as $F(z)$ is no longer partitioned into $F_1(z), F_2(z)$. An alternative approach to this was used in [5]. □□

6 Conclusion

This paper has formulated a discrete time anti-windup problem and has presented several types of solution: full-order, static and low-order. These solutions can be found by solving a set of linear matrix inequalities, which for plants of small-moderate size is reasonably easy using modern computers. Moreover, in our opinion, the problem has been posed in a manner that is central to the "true" anti-windup problem: the deviation of the saturated system's response from that of the nominal linear system has been minimised explicitly.

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A State-space matrices

For the static case, the state-space matrices are given as

$$\bar{x} := \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad \bar{A} := \begin{bmatrix} A_p + B_p\tilde{\Delta}D_cC_p & B_p\tilde{\Delta}C_c \\ B_c\Delta C_p & A_c + B_c\Delta D_pC_c \end{bmatrix}$$

$$B_0 := \begin{bmatrix} B_p\tilde{\Delta} \\ B_c\Delta D_p \end{bmatrix}, \bar{B} := \begin{bmatrix} B_p\tilde{\Delta} & -B_p\tilde{\Delta}D_c \\ B_c\Delta D_p & -B_c\Delta \end{bmatrix}, \Theta := \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$$

$$\bar{C}_1 := [\tilde{\Delta}D_cC_p \ \tilde{\Delta}C_c],\ D_{01} := \tilde{\Delta}D_cD_p,\ \bar{D}_1 := [I + \tilde{\Delta}D_cD_p \ \ -\tilde{\Delta}D_c]$$

$$\bar{C}_2 := [\Delta C_p \ \Delta D_pC_c],\ D_{02} := \Delta D_p,\ \bar{D}_2 := [\Delta D_p \ \ -\Delta D_pD_c]$$

For the low-order case, if we assign the state-space realisations

$$\Theta_3(z) = \Theta_1(z) + I \sim \begin{cases} x_1(k+1) = A_1x_1(k) + B_1\tilde{\Theta}_1\tilde{u}(k) \\ y_1(k) = C_1x_1(k) + (D_1\tilde{\Theta}_1 + I)\tilde{u}(k) \end{cases}$$

$$\Theta_2(z) \sim \begin{cases} x_2(k+1) = A_2x_2(k) + B_2\tilde{\Theta}_2\tilde{u}(k) \\ y_2(k) = C_2x_2(k) + D_2\tilde{\Theta}_2\tilde{u}(k) \end{cases}$$

where $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$. Some tedious algebra then yields the state-space matrices for a minimal realisation of $[M' - I \quad (GM)']'$ as

$$\bar{x} = \begin{bmatrix} x \\ x_c \\ x_2 \\ x_1 \end{bmatrix} \in \mathbb{R}^{\bar{n}}$$

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & -B_{12}C_2 & B_{11}C_1 \\ A_{21} & A_{22} & -B_{22}C_2 & B_{21}C_1 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix} \in \mathbb{R}^{\bar{n}\times\bar{n}}$$

$$\bar{B} = \begin{bmatrix} B_{11}D_1 & -B_{12}D_2 \\ B_{21}D_1 & -B_{22}D_2 \\ 0 & B_2 \\ B_1 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}\times(m+q)}$$

$$B_0 = \begin{bmatrix} B_{11} \\ B_{21} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}\times m}$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 \\ \tilde{\Theta}_2 \end{bmatrix} \in \mathbb{R}^{(m+q)\times m}$$

$$\begin{aligned} \bar{C}_1 &= [C_{11} \quad C_{12} \quad -D_{12}C_2 \quad D_{11}C_1] \in \mathbb{R}^{q\times\bar{n}} \\ D_{01} &= D_{11} - I \in \mathbb{R}^{q\times m} \\ \bar{D}_1 &= [D_{11}D_1 \quad -D_{12}D_2] \in \mathbb{R}^{q\times(m+q)} \\ \bar{C}_2 &= [C_{21} \quad C_{22} \quad -D_{22}C_2 \quad D_{21}C_1] \in \mathbb{R}^{m\times\bar{n}} \\ D_{02} &= D_{21} \in \mathbb{R}^{m\times m} \\ \bar{D}_2 &= [D_{21}D_1 \quad -D_{22}D_2] \in \mathbb{R}^{m\times(m+q)} \end{aligned}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := \begin{bmatrix} A_p + B_p\tilde{\Delta}D_cC_p & B_p\tilde{\Delta}C_C \\ B_c\Delta C_p & A_c + B_c\Delta D_pC_c \end{bmatrix}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} := \begin{bmatrix} B_p\tilde{\Delta} & B_p\tilde{\Delta}D_c \\ B_c\Delta D_p & B_c\Delta \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} := \begin{bmatrix} \tilde{\Delta}D_cC_p & \tilde{\Delta}C_c \\ \Delta C_p & \Delta D_pC_c \end{bmatrix}$$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} := \begin{bmatrix} I + \tilde{\Delta}D_cD_p & \tilde{\Delta}D_c \\ \Delta D_p & \Delta D_pD_c \end{bmatrix}$$

and $\bar{n} := n_p + n_c + n_2 + n_1$