REDUCTION OF POSITIVE REAL NON INTEGER ORDER MODELS : INITIAL CONDITIONS DETERMINATION

Nicolas Guijarro, Geneviève Dauphin-Tanguy

L.A.I.L., Ecole Centrale de Lille, Cité Scientifique, BP48
59651 Villeneuve d'Ascq Cedex France
Phone : +33(0)320335414 Fax : +33(0)320335418
{nicolas.guijarro,gdt}@ec-lille.fr

Keywords: positive realness, non integer order model, bond graphs, approximation, realization.

Abstract

In this paper, a method to determine initial conditions for a class of positive real non integer order models is given. The exposed procedure makes use of a positive real approximation-reduction-realization process. An application, to a second generation CRONE control model, illustrates the proposed methodology.

1 Introduction

Following a certain approximation-reduction-realization process [13] of a non integer order model with given initial conditions, we need to know how to determine the initial conditions on the dynamic elements which compose the reduced-order model. To be more precise we must outline the context of this work.

Given a bond graph model [18, 20] or another type of model transformed into a bond graph type one, there exists powerful tools : to analyse [21], to synthesize and design control laws [15], for stabilization [16], for decoupling [3] and for disturbance rejection [2] (to name a few of them).

Unfortunately most of these tools are limited to finite dimensional models. But frequently, we have to deal with infinite dimensional models built when studying distributed parameter systems or delayed systems for example. When this kind of system is inserted into a bond graph it can be defined as a functional node [17]. This node prevents us from the use of classical bond graph tools.

To overcome this difficulty one could develop specific tools for bond graphs with functional nodes or, as it will be set out here, approximate the functional nodes by "classical" nodes (a combination of simple passive elements such as *R* (resistance) ,*C* (capacitance), *I* (inductance), *TF* (transformer) and *GY* (gyrator)), which amounts to build an equivalent electrical circuit, or bond graph, composed of *R*, *C*, *I* cells, *TF* and *GY*. The models considered in this paper are linear, time invariant and passive. In this case the passivity property can be characterized thanks to the following definition [7] :

Definition 1 *An L.T.I. finite or infinite dimensional model with transfer matrix function $H(s)$ is said to be passive (or $H(s)$ is a positive real function) if*

$$\overline{H(\bar{s})} = H(s),$$

$$H(s) \; is \; analytic \; in \; \Re(s) > 0,$$

$$H(s) + H^*(s) \geq 0 \; in \; \Re(s) > 0.$$

The input u and output y signals are chosen here conjugate power variables (voltage/current or force/velocity) so that $u.y$ denotes the power supplied to the system [1] [22].

1.1 Proposed Methodology

The general procedure, applied to the model studied here, can be summarized as follows [12] :

1. approximation of the infinite dimensional model by a large scale finite dimensional one (say of dimension N),
2. reduction of the model's dimension to obtain a model of dimension $n \ll N$,
3. realization of the latter model by a *R-C-I-TF-GY* network (a bond graph).

Steps *1)* and *2)* of the procedure must keep the key property of the initial model, *i.e.* its passivity.

In the third section some techniques are reviewed to approximate infinite dimensional models by a finite dimensional one. Next, the fourth section recalls briefly a technique used in reduced-order modelling. Then the fifth section gives a way to realize the low-dimension model by using lumped passive linear elements. Finally in the sequel an example and an application illustrate the whole procedure and concluding remarks are stated.

1.2 Infinite Dimensional Model Approximation

In a few words the approximation (or reduction) problem is the following one :

[1] We suppose in the following that the system is a 1-port, that means that the power is supplied to the system by one single access.

- *Given a model, find another one which is fast to compute, reduced-order and accurate.*

The first property, the rapidity of the computation, depends on the numerical methods involved. Typically it can be measured by the complexity. The second one (reduced-order) does not need to be developed. In fact it is easy to quantify the order decrease. Last but not least, the accuracy is the most subjective feature to define. Depending on the way one wants to quantify the error between the original model and the reduced-order one, one needs to choose a different norm (e.g. $\mathcal{H}_\infty$ for the supremum). Furthermore, the model will be called accurate if it retains some properties like stability or passivity. For example, balanced truncation retains the largest singular values and gives an evaluation of the error through the sum of the remaining ones. In the reduction process used in this procedure, moment matching is the key feature.

The in-between model is the result of a finite dimension approximation of the initial model. Moreover, some properties, like stability and passivity, must remain. The most important one is passivity because our models (bond graphs) need energetic coherence. As a matter of fact, passivity has a closure characteristic. Unlike stability, in the interconnection of passive networks, the passivity property holds. According to the sort of infinite dimensional model the plan to follow will differ. A detailed discussion could be found in [13].

1.3 Large Scale Model Reduction

In this section *Krylov* subspaces techniques used in reduced-order modelling are recalled. The previous section methods provide us with an approximate but large scale finite dimensional model. In state-space formalism we can describe it as follows [8] :

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{c}^T\mathbf{x}(t) \end{cases} . \tag{1}$$

In (1) $\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{N\times N},\ \mathbf{b} \in \mathbb{R}^N$, and $\mathbf{c} \in \mathbb{R}^N$ are given matrices. The matrices $\mathbf{E}$ and $\mathbf{A}$ are allowed to be singular but we assume that the pencil $(s\mathbf{E} - \mathbf{A})$ is regular. Applying the *Laplace* transform on (1), one obtains :

$$H(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}. \tag{2}$$

Similarly for the reduced-order model (n) :

$$\begin{cases} \mathbf{E}_n\dot{\mathbf{z}}(t) = \mathbf{A}_n\mathbf{z}(t) + \mathbf{b}_n\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) = \mathbf{c}_n^T\mathbf{z}(t) \end{cases} , \tag{3}$$

and :

$$H_n(s) = \mathbf{c}_n^T(s\mathbf{E}_n - \mathbf{A}_n)^{-1}\mathbf{b}_n. \tag{4}$$

The model (3) must, in a certain sense, approximate the model (1). The classical reduction methods, like balanced truncation or *Hankel*-norm optimal approximations, do not take advantage of the structure of the matrices. In our case, the matrices are large scale, sparse and/or structured. A reduction method, which makes use of *Krylov* subspaces techniques, exploits the characteristics of the matrices. The reduced-order transfer function must match $2n$ moments of certain series expansions of the initial model transfer function. A table, which summarizes the different matchings and how to obtain the approximant, can be found in [10].

1.4 Realization

Knowing a state-space representation of the reduced-order model a network or a bond graph can be realized.

In the 1-port case the determination of the electrical network or bond graph component's values can be achieved thanks to the *Brune* process [5]. In the *m*-port case various processes described in [4] could be used.

Of course, the network synthesis can be done thanks to state-space procedures like the one outlined in [1].

2 Initial conditions problem : non integer order models case

Our objective is to design a method to determine the initial conditions of the integer order model approximating the non integer order one. We need to know how to take into account the initial values of the output and its derivatives.

Here we recall the results of *F. Mainardi* and *R. Gorenflo* [9]. That is important to understand why we use the *Caputo* definition of the non integer order derivation operator. We deal first with ordinary differential equations to make our reasoning more comprehensible.

The relaxation or first order differential equation is given by :

$$D^1 y(t) = y^{'}(t) = -y(t) + u(t),\ t \geq 0,$$

where $u(t)$ is a continuous function, admits the solution :

$$y(t) = c_0 e^{-t} + \int_0^t u(t-\tau)e^{-\tau}d\tau,$$

where $c_0 \doteq y(0^+)$ is the initial condition.

Likewise the second order equation is given by :

$$D^2 y(t) = y^{''}(t) = -y(t) + u(t),$$

and admits the solution :

$$y(t) = c_0\cos(t) + c_1\sin(t) + \int_0^t u(t-\tau)\sin(\tau)d\tau,$$

where $c_0 \doteq y(0^+)$ and $c_1 \doteq y'(0^+)$ are the initial conditions.

If we want to generalize these results to a non integer order α, the following type of equation has to be considered :

$$D^\alpha_* y(t) = D^\alpha \left(y(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(0^+) \right) = -y(t) + u(t). \tag{5}$$

In (5) m is the integer such that $m-1 < \alpha \leq m$ and the initial conditions are defined through $y^{(k)}(0^+) \doteq c_k$ with $k = 0, ..., m-1$. The solution, in the integer case ($\alpha = m$), could be summed up in the following result :

$$y(t) = \underbrace{\sum_{k=0}^{m-1} c_k y_k(t)}_{\text{response due to the initial conditions}} + \underbrace{\int_0^t u(t-\tau) y_\delta(\tau) d\tau}_{\text{response due to the output}},$$

$$y_k(t) = J^k u_0(t),\ u_k^{(h)}(0^+) = \delta_{kh},\ h, k = 0, 1, ..., m-1,$$

$$y_\delta(t) = -y_0^{'}(t).$$

If $m = 1$ (relaxation) we get $y_0(t) = e^{-t} = y_\delta(t)$ and if $m = 2$ (oscillation), $y_0(t) = \cos(t)$ and $y_1(t) = J y_0(t) = \sin(t) = \cos(t - \pi/2) = y_\delta(t)$.

Application of the *Laplace* transform to the equation (5) leads us to :

$$\begin{aligned} Y(s) &= \sum_{k=0}^{m-1} c_k \frac{s^{\alpha-k-1}}{1+s^\alpha} + \frac{1}{1+s^\alpha} U(s), \\ Y(s) &= \sum_{k=0}^{m-1} c_k H_k(s) + H(s) U(s). \end{aligned}$$

It has to be noted that the *Laplace* transform of $D^\alpha_* y(t)$ is given by :

$$\begin{aligned} \mathcal{L}(D^\alpha_* y(t)) &= s^\alpha Y(s) - \sum_{k=0}^{m-1} y^{(k)}(0^+) s^{\alpha-k-1}, \\ &= s^{\alpha-m} \left(s^m Y(s) - \sum_{k=0}^{m-1} y^{(k)}(0^+) s^{m-k-1} \right), \\ &= s^{\alpha-m} \mathcal{L}(D^m y(t)). \end{aligned} \tag{6}$$

Thus we obtain the following time domain expressions :

$$y(t) = c_0 y_0(t) + \int_0^t u(t-\tau) y_\delta(\tau) d\tau, \tag{7}$$

where :

$$\begin{cases} y_0(t) = \int_0^\infty e^{-rt} K_{\alpha,0}(r) dr, \\ y_\delta(t) = -\int_0^\infty e^{-rt} K_{\alpha,-1}(r) dr, \end{cases} \tag{8}$$

with $y_0(0^+) = 1$, $y_\delta(0^+) = \infty$, $0 < \alpha < 1$ and :

$$K_{\alpha,k}(r) \doteq \frac{(-1)^k}{\pi} \frac{r^{\alpha-1-k} \sin(\alpha\pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1}.$$

Remark 1 : We limit in the sequel our study to the case where α is between zero and one because we are only interested in positive real transfer functions. The transfer function $H_0(s)$ is positive real because it results of the positive real interconnection of two positive real functions $H_0^1(s) \doteq 1/s$ and $H_0^2(s) \doteq s^{1-\alpha}$. We get :

$$H_0(s) = \frac{H_0^1(s)}{1 + H_0^1(s) H_0^2(s)}.$$

Expression (5) can be compared to the one otained in [19, chapter 1, section 1.4]. In this reference an approximate definition is given by :

$$\mathcal{L}(D^\alpha f(t)) = s^\alpha F(s) - \sum_{i=0}^{\infty} s^{\alpha-i-1} f^{(i)}(0) + \sum_{i=0}^{\infty} s^{-i-1} f^{(n+i)}(0). \tag{9}$$

(9) is a generalization to the non integer case of the well known expression :

$$\mathcal{L}(D^n f(t)) = s^n F(s) - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0),$$

where n is an integer. This expression makes use of an infinite number of dependant initial conditions. In the expression (6), derived thanks to the use of the *Caputo* definition of the non integer derivation operator, only a finite independant number of initial conditions is needed.

An analog remark could be made about the response to an arbitrary input to a second generation CRONE control with initial conditions different from zero. The equation corresponding to such a control in the case of an unitary relaxation constant [19, chapter 1, section 1.5]) is given by :

$$D^\alpha y(t) = -y(t) + u(t),$$

in the time domain, and by :

$$Y(s) = \sum_{k=0}^{\infty} y^{(k)}(0) \frac{s^{\alpha-k-1}}{1+s^\alpha} -$$

$$- \sum_{k=0}^{\infty} y^{(\alpha+k)}(0) \frac{s^{-k-1}}{1+s^\alpha} + \frac{1}{1+s^\alpha} U(s),$$

in the *Laplace* domain.

The next section gives several realization solutions in terms of passive elements (R, I and C) of the transfer functions $H_0(s)$ and $H(s)$.

3 Initial conditions problem : the second generation *CRONE* control case

The equation corresponding to the second generation *CRONE* control for a unitary relaxation constant is given by [19] :

$$D^{\alpha}_{*}y(t) = D^{\alpha}\left(y(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}y^{(k)}(0^+)\right) = -y(t) + u(t), \tag{10}$$

In (10), m is the integer such that $m-1 < \alpha \leq m$ and the initial conditions are given by $y^{(k)}(0^+) \doteq c_k$ with $k = 0, ..., m-1$.

Applying the *Laplace* transform to (10) we get :

$$\begin{aligned} Y(s) &= \sum_{k=0}^{m-1} c_k \frac{s^{\alpha-k-1}}{1+s^{\alpha}} + \frac{1}{1+s^{\alpha}}U(s), \qquad (11)\\ Y(s) &= \sum_{k=0}^{m-1} c_k H_k(s) + H(s)U(s). \end{aligned}$$

If $0 < \alpha \leq 1$ then (11) becomes :

$$Y(s) = c_0 H_0(s) + H(s)U(s),$$

$$Y(s) = c_0 \frac{s^{\alpha-1}}{1+s^{\alpha}} + \frac{1}{1+s^{\alpha}}U(s).$$

Thanks to the positive realness property of the two transfer functions $H_0(s)$ and $H(s)$ we could obtain a realization by means of a passive bond graph model after an approximation-reduction process.

Remark 2 : The effect of the initial conditions is obtained through the realization of $H_0(s)$ but it has a cost in terms of additional dynamical elements. This number of additional elements is given by the rank of the chosen approximate model.

Several approximation-reduction scheme could be applied to $H(s)$. A method developed for a fractional power pole model [11] could be used with a shift of the *Laplace* variable. After that, we obtain an approximate model for $1/s^{\alpha}$. Then a unitary feedback loop gives us an approximate model of $1/(1+s^{\alpha}) = H(s)$.

An interesting alternative method could be found in [14]. It consists also in the construction of an approximate model of s^{α} for α between minus one and one.

Concerning the transfer function $H_0(s)$ we just need to add a feedback loop comprising an integrator to an approximate model of $s^{\alpha-1}$. The resulting approximate model will be positive real by construction.

Example 1 : For $\alpha = 1/2$ and $c_0 = 1$, we get :

$$\begin{aligned} H_0(s) &\doteq \frac{s^{-1/2}}{1+s^{1/2}} = \frac{\frac{1}{s^{1/2}}}{1+s\frac{1}{s^{1/2}}},\\ H(s) &\doteq \frac{1}{1+s^{1/2}}. \end{aligned}$$

If the approximate model of $s^{1/2}$ is derived from the approximation-reduction of a fractional power pole model [11], we get :

$$\begin{aligned} H^6_0(s) &= \frac{1}{s+\frac{1}{H^6_{kl}((s-1)p_T)}},\\ H^6(s) &= \frac{1}{1+\frac{1}{H^6_{kl}((s-1)p_T)}},\\ H^6_{kl}\left((s-1)p_T\right) &= \sum_{i=1}^{6}\frac{r_i}{p_T}.\frac{1}{s-(1+p_i/p_T)}. \end{aligned}$$

where the parameters r_i and p_i are those of table 1.

The transfer function of the reduced-order model is given by :

$$H^{kl}_6(s) = \sum_{i=1}^{6} r_i \frac{1}{s-p_i},$$

Poles (p_i)	**Residues (r_i)**
-63.95	0.250
-18.02	0.173
-5.076	0.132
-1.430	0.106
-0.4033	0.0972
-0.1132	0.241

Table 1: Pole and residues values of the reduced-order model of $1/s^{1/2}$

Example 2 : The lower frequencies precision $H^6_{kl}\left((s-1)p_T\right)$ depends on the closeness to zero of the least large pole, i.e. the value of $1 + p_0/p_T$. For example if the order is equal to six the *Krylov-Lanczos* method gives a value of 0.0932 and the singularity function method [6] a value of 0.256. To reach the same value with the latter method we need to increase the order to nine.

To have an idea of the performance achieved, if the reduced-order model for $s^{-1/2}$ is given by $H^6_{kl}\left((s-1)p_T\right)$ then the relative gain error is less than 0.5% on a pulsation range from 1 to 135 rad.s^{-1}. A significant improvement could be considered with a careful choice of the interpolation points anticipating the variable shift.

4 Conclusion

In this paper we have exposed an approximation-reduction-realization process and a method to take into account the initial conditions applied to the original model in case of a non integer order model.

It has to be noted that the initial conditions on the reduced-order model are obtained through a passive realization. In consequence the more we want the model to be precise, the more dynamic elements we need to construct the approximate model of the non integer order operator.

References

[1] B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis : A Modern Systems Theory Approach*. Pentice-Hall, Englewoord Cliffs, New Jersey, 1973.

[2] J. Arib, C. Sueur, and G. Dauphin-Tanguy. Disturbance Rejection by Measurement Feedback for Bond Graph Models. In *Proc. of the 3rd Mathmod Conference, Vienne (Autriche)*, volume 2, pages 709–714, 2000.

[3] J. Arib, C. Sueur, and G. Dauphin-Tanguy. Decoupling Problem by Measurement Feedback for Bond Graph Models. In *Proc. of ICBGM'01, Phoenix (Arizona), U.S.A.*, 2001.

[4] V. Belevitch. *Classical Network Theory*. Holden-Day, San Francisco, 1968.

[5] Otto Brune. Synthesis of a Finite Two-Terminal Network whose Driving-Point Impedance is a Prescribed Function of Frequency. *Journal Of Mathematics and Physics (MIT)*, 10:191–236, 1931.

[6] A. Charef, H. H. Sun, Y. Y. Tsao, and B. Onaral. Fractal System as Represented by Singularity Function. *IEEE Transactions on Automatic Control*, 37(9):1465–1470, September 1992.

[7] R.F. Curtain. Old and New Perspectives on the Positive-Real Lemma in Systems and Control Theory. *ZAMM*, 79:579–590, 1999.

[8] R.W. Freund. Reduced-Order Modeling Techniques Based on Krylov Subspaces and Their Use in Circuit Simulation. *J. Comput. Appl. Math.*, 123:395–421, November 2000.

[9] R. Gorenflo and F. Mainardi. *Fractals and Fractional Calculus in Continuum Mechanics*, chapter Fraction Calculus : Integral and Differential Equations of Fractional Order, pages 223–276. Springer Verlag, Wien and New York, Carpinteri, A. and Mainardi, F. edition, 1997.

[10] E. J. Grimme. *Krylov Projection Methods for Model Reduction*. PhD thesis, University of Illinois at Urbana Champaign, 1997.

[11] N. Guijarro, L. Lefèvre, and G. Dauphin-Tanguy. Infinite Dimensional Models : Approximation and Realization. In *Proceedings of the* 39^{th} *IEEE Conference on Decision and Control*, volume 4, pages 3295–3300, Sydney, Australia, 2000.

[12] N. Guijarro, L. Lefèvre, and G. Dauphin-Tanguy. Approximation, Reduction and Realization of Passive Delay Systems. In *Proceedings of the* 15^{th} *IFAC World Congress*, Barcelona, Spain, 2002.

[13] Nicolas Guijarro. *Contribution à l'étude des systèmes passifs de dimension infinie : application aux bond graphs*. PhD thesis, Université des Sciences et Technologies de Lille, 2002.

[14] H.E. Jones and B.A. Shenoi. Maximally Flat Lumped-Element Approximation to a Fractional Operator Immittance Function. *IEEE Transactions on Circuit Theory*, pages 125–128, 1970.

[15] S. Junco. A Bond Graph Approach To Control Systems Synthesis. In *Proc. of ICBGM'01, Phoenix (Arizona), U.S.A.*, 2001.

[16] S. Junco. Lyapunov Second Method and Feedback Stabilization Directly on Bond Graphs. In *Proc. of ICBGM'01, Phoenix (Arizona), U.S.A.*, 2001.

[17] Laurent Lefèvre. *De l'introduction d'éléments fonctionnels dans la théorie des bond graphs*. PhD thesis, Université des Sciences et Technologies de Lille, 1999.

[18] D. Margolis, D. Karnopp, and R. Rosenberg. *System Dynamics: A Unified Approach, second edition*. John Wiley & Sons, New York, 1990.

[19] A. Oustaloup. *La dérivation non entière : théorie, synthèse et applications*. Hermès, Paris, 1995.

[20] H. Paynter. *Analysis and Design of Engineering Systems*. MIT Press, Cambridge, MA, 1961.

[21] C. Sueur and A. Rahmani. *Les bond graphs*, chapter Analyse et commande des modèles bond graphs, pages 313–350. Hermès Science, Paris, Geneviève Dauphin-Tanguy edition, 2000.

[22] Jan C. Willems. Dissipative Dynamical Systems Part I : General Theory. *Arch. Rational Mech. Anal.*, 45:321–350, 1972.