ON MECHANICAL MIXED POTENTIAL, CONTENT AND CO-CONTENT

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Keywords: Modeling, mechanical networks, Brayton-Moser equations, energy and co-energy.

Abstract

In this paper a novel co-energy modelling framework is presented for a relevant class of linear and nonlinear mechanical systems. The approach uses the classical Brayton-Moser equations which are deduced from a (port-)Hamiltonian description. The approach allows classical results from electrical circuit synthesis and analysis to be carried over exactly to the mechanical domain. It also enables one to apply (nonlinear) control techniques like Power Shaping as recently proposed in [7]. Illustrative examples are provided to facilitate the theoretical developments.

1 Introduction

Commonly appreciated tools in the dynamical modelling of electrical and mechanical systems are the Lagrangian and (port-)Hamiltonian framework, see e.g. [6, 9] and the references therein. Based on the energy and interconnection properties, one is able to derive the equations of motion in a systemic way. Starting from the energy-balance of the system these formalisms have proven to be very useful for gaining insight in the behavior of the system, stability analysis and the design of (nonlinear) stabilizing controllers, see e.g. [6, 8]. However, there are some disadvantages of using the Lagrangian or (port-)Hamiltonian descriptions. One of these disadvantages is that the choices of the state variables are not always the most intuitively meaningful ones. For example, as pointed out in [4], in the electrical domain, the (port-)Hamiltonian uses the inductor fluxes and capacitor charges as the state variables. Apart from the fact that Ohm's law is formulated in terms of the currents and voltages (or flows and efforts, respectively), the fluxes and charges are in practice not the most common states or outputs to measure when designing a feedback controller. This also holds for mechanical systems where the dissipation is usually described in terms of the Rayleigh dissipation function depending on the velocities in the system instead of the momenta. Besides that, in a practical situation one rather measures velocity instead of momenta.

Recently, in [3] we have proposed an alternative modelling framework for a limited class of nonlinear electro-mechanical systems that differs from the conventional modelling setting. The formalism is based on an explicit definition of Brayton-Moser's mixed potential and the co-energy of the system. In this way the equations of motion are directly expressed in terms of the ('easily' measurable) flows and efforts of the system. However, the approach in [3] treats the conservative forces stemming from some potential energy function as external ports. In case of gravity this can be considered natural, but if the system contains translational or rotational springs we may treat this differently. Also, in [3] we have a 'canonical' description of the Brayton-Moser equations, whereas a more general form that includes a larger class of systems will be treated here. In this paper we present a complete mechanical analog of Brayton and Moser's *mixed potential* function in terms of the conservative forces and generalized velocities acting in the system. This leads to the notion of mechanical *content* and mechanical *co-content*, which represent the characteristics of the velocity- and force controlled mechanical dissipators, respectively.

Notation: By $T_x(x,\cdot)$ we denote the partial derivative of $T(x,\cdot)$ with respect to x, i.e, $T_x(x,\cdot) = \partial T(x,\cdot)/\partial x$. Consequently, $T_{xx}(x,\cdot) = \partial^2 T(x,\cdot)/\partial x^2$, etc.. Furthermore, by $\hat{y}(x)$ we define the vector of the constitutive relations for the, say k, (x–controlled) elements of a certain type (resistance, inductance, etc.), $\hat{y}(x) = \mathrm{col}(\hat{y}_1(x), \ldots, \hat{y}_k(x))$.

2 Topologically Complete Mechanical Systems

In this section we aim at a precise analog of the classical Brayton and Moser framework [1] to describe the equations of motion for a practical class of mechanical systems. During our developments we restrict ourselves to mechanical systems that can be described by a class of the celebrated port-Hamiltonian equations [9], and we will rewrite them in a similar form as the Brayton-Moser equations. In the construction we do not want to elaborate on the existence of a mixed potential function of

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mechanical type in general. For example, in [2] it is argued that there may exist state-modulated systems that can not be generated by one mixed potential. Instead, we will focus on a topological construction of such function and for now we simply assume that such function exists.

2.1 Brayton and Moser's Equations

The Brayton-Moser framework was originally developed to model the dynamics of (possibly nonlinear) topologically complete[1] electrical RLC circuits and is based on the introduction of a single scalar function. In general, the Brayton-Moser equations [1] are described by

$$C(v)\frac{dv}{dt} = P_v(v,i), \quad -L(i)\frac{di}{dt} = P_i(v,i), \tag{1}$$

where v represents the vector of independent capacitor voltages and i represents the vector of independent inductor currents. Furthermore, the matrices $C(v)$ and $L(i)$ denote the incremental capacitor and inductor characteristics, respectively, and $P(v,i)$ is a scalar function called the *mixed potential* function. This function captures the physical structure, like interconnection and dissipation, of the circuit and can also be used as a Lyapunov-type function to determine stability under certain conditions on the circuit elements.

To this end, we do not elaborate any further on the mixed potential function of electrical type. Instead, we will show, in a piece by piece manner, how to construct this function starting from a port-Hamiltonian description for mechanical systems.

2.2 Port-Hamiltonian Mechanics

It is well-known that a rather general class of (possibly nonlinear) mechanical systems defined on a n-dimensional differentiable state-space manifold $\mathcal{M}$, with local coordinates (x,p), admit a port-Hamiltonian description with Dirac structure $\mathcal{D}$, see [9]. The class of port-Hamiltonian systems we consider herein are in local coordinates given by

$$\begin{aligned} \dot{x} &= \gamma^\top H_p(x,p) \\ \dot{p} &= -\gamma H_x(x,p), \end{aligned} \tag{2}$$

where $x = \mathrm{col}(x_1,\ldots,x_\varrho) \in \mathbb{R}^\varrho$ denote the generalized displacement coordinates and $p = \mathrm{col}(p_1,\ldots,p_\varsigma) \in \mathbb{R}^\varsigma$ denote the corresponding generalized momenta (thus, $n = \varrho + \varsigma$). Furthermore, $\gamma \in \mathcal{D}$ is a $\varsigma \times \varrho$ matrix which may depend on the coordinates[2], and the scalar function $H(x,p)$ denotes the Hamiltonian which in this case is defined as the sum of the kinetic energy $T(x,p)$ and potential energy $V(x)$, i.e., $H(x,p) = T(x,p) + V(x)$. In the sequel, we assume that the mechanical systems under consideration allow a Hamiltonian of the form $H(x,p) = \frac{1}{2}\sum_{j,k}\{M^{-1}\}_{jk}(x)p_jp_k + V(x)$, where $\{M^{-1}\}_{jk}(x)$ refers to the (j,k)-th element of $M^{-1}(x)$, with $M(x)$ a positive definite symmetric $\varsigma \times \varsigma$ matrix called the inertia or generalized mass matrix. In this coordinate setting the definition of the potential energy has the general form

$$V(x) = \int_0^x \hat{f}(x')dx', \tag{3}$$

where $f = \hat{f}(x)$, with $f = \mathrm{col}(f_1,\ldots,f_\varrho) \in \mathbb{R}^\varrho$, are the conservative forces generated by the system. In order to be able to find a Brayton-Moser type description of (2) we need to define the total co-energy $H^*(f,v) = T^*(f,v) + V^*(f)$. If the constitutive relations are bijective this is easily accomplished by performing a Legendre transformation on the energy variables $x \mapsto f$, i.e.,

$$V^*(f) = \int_0^f \hat{x}(f')df'. \tag{4}$$

The Legendre transformation on the momenta $p \mapsto v$, where $v = \mathrm{col}(v_1,\ldots,v_\varsigma) \in \mathbb{R}^\varsigma$ denote the generalized velocities, yields the kinetic co-energy

$$T^*(x,v) = \int_0^v \hat{p}(x,v')dv'. \tag{5}$$

The resulting co-energy is then given by

$$\begin{aligned} H^*(f,v) &= T^*(x,v)\big|_{x=\hat{x}(f)} + V^*(f) \\ &= \frac{1}{2}\sum_{j,k}\{M\}_{j,k}(f)v_jv_k + V^*(f). \end{aligned} \tag{6}$$

It is interesting to remark that $H^*(f,v)$ is often called the co-Hamiltonian. Again we point out that $V^*(f)$ is only well-defined if there exists some bijective constitutive relation $f = \hat{f}(x)$. In case the system is subject to, for example, gravitational fields, such transformation does not exist. We come back to this in Section 4.

Remark 1 Notice that the port-Hamiltonian system (2) can also be considered as a topologically complete system, i.e., there should exist an independent set of forces (resp. displacements) and velocities (resp. momenta) such that each branch (e.g., masses and springs) is determined by at least one element from the set of forces (resp. displacements) and velocities (resp. momenta). An example of a topologically non-complete mechanical system is treated in the following section.

In the remaining of the document, we assume that the mechanical system is defined on $\mathbb{R}^n$ and hence the approach can be considered to be global. We are now ready to define a Brayton-Moser description by introducing a mixed potential function of mechanical type.

[1] A circuit is called 'topologically complete' if it can be described by an independent set of inductor currents and capacitor voltages such that Kirchhoff's laws are satisfied. For a detailed treatment, the reader is referred to [10].

[2] In many mechanical systems γ is the identity matrix (symplectic), hence $\varsigma = \varrho$.

2.3 Mechanical Mixed-Potential, Content and Co-Content

Our purpose is to write the equations obtained in the previous subsection in a Brayton-Moser form for the framework of mechanical systems. To do this, we have to search for the suitable function P which allows us to write (2) in a form

$$\begin{aligned} \tfrac{d}{dt} H_f^*(f,v) &= P_f(f,v) \\ -\tfrac{d}{dt} H_v^*(f,v) &= P_v(f,v). \end{aligned} \tag{7}$$

Completely analog to [1], we have the following proposition:

Proposition 1 *Assume that $\hat{x}(f)$ and $\hat{f}(x)$ are bijective functions smoothly defined on $\mathbb{R}^{\varrho}$, and let $P : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined as*

$$P(f,v) = v^\top \gamma f, \tag{8}$$

then the equations of motion for a complete mechanical system described by (2) can be rewritten in terms of the co-energy variables (f,v) in the form (7)–(8).

Proof: Let $\mathcal{F}$ denote a linear n-dimensional space spanned by the forces (f,f'), where $\dim\{f'\} = \varsigma$, acting in the system and let $\mathcal{V}$ denote a linear n-dimensional space spanned by the velocities (v,v'), $\dim\{v'\} = \varrho$, generated by the system. Then from Tellegen's Theorem[3] we know that $\mathcal{F}$ and $\mathcal{V}$ are orthogonal subspaces spanning the total configuration space $\mathbb{R}^{2n}$. Furthermore, let Γ be a one-dimensional curve in $\mathbb{R}^{2n}$ with projections on $\mathcal{F}$ and $\mathcal{V}$ denoted by the $2n$ forces and velocities, respectively. Since $(v,v') \in \mathcal{V}$ then also $(dv,dv') \in \mathcal{V}$. Hence, by letting $f' = \frac{d}{dt} H_v^*(f,v)$, $v' = \frac{d}{dt} H_f^*(f,v)$ and using Tellegen's Theorem we have that

$$\int_\Gamma \sum_{i=1}^{\varrho} v_i' df_i + \int_\Gamma \sum_{j=1}^{\varsigma} v_j df_j' = 0, \tag{9}$$

or by integrating the second line integral by parts,

$$\int_\Gamma \sum_{i=1}^{\varrho} v_i' df_i - \int_\Gamma \sum_{j=1}^{\varsigma} f_j' dv_j + \sum_{k=1}^{\varsigma} v_k f_k' \Big|_\Gamma = 0. \tag{10}$$

Hence, by using the Legendre transformation $v = H_p(x,p)$ and $f = H_x(x,p)$ we obtain from (2) that $v' = \gamma^\top v$ and $f' = -\gamma f$, see (2), $P(f,v)$ is defined by

$$P(f,v) = \sum_{k,j} v_k \gamma_{kj} f_k \Big|_\Gamma, \ \gamma_{kj} \in \gamma. \tag{11}$$

Notice that $P(f,v)$ is a function depending only on the end points of Γ and therefore only depends on the variables f and v. This concludes the proof. ■

[3] Dirac structures [9] provide a natural generalization of this theorem, characterizing in an elegant geometrical language the key notion of power preserving interconnections of the system elements. Due to the assumed form of the port-Hamiltonian equations (2), which in the electrical domain just constitute the Kirchhoff laws, we can still apply Tellegen's Theorem in its classical definition.

Remark 2 Our aim of the proof was to emphasize the geometrical structure and origin of the mixed potential function. The proof is constructed along the same lines as in [1].

Next, we like to include the effect of a set of external and dissipative forces and velocities on the system. An ideal (translational or rotational) mechanical dissipator is defined as an object which exhibits no mass or spring effects, but only dissipative forces or velocities. If the dissipation is nonlinear we may distinguish between velocity-controlled, force-controlled and one-to-one (both velocity and force controlled) dissipators. For the velocity-controlled dissipators we consider the usual description in terms of the Rayleigh dissipation function, defined as

$$D(v) = \int_0^v \hat{\delta}(v')dv',$$

where $\hat{\delta}(v)$ represents the vector of functions describing the characteristics of the mechanical dissipation depending on the velocities. Note that the content is a known function of the dissipator velocity so that $D_v(v)$ represents the dissipator force. It is interesting to notice that this framework also admits a Rayleigh dissipation co-function in terms of the forces, i.e.,

$$K(f) = \int_0^f \hat{\kappa}(f')df',$$

where $\hat{\kappa}(f)$ represents the vector of functions describing the characteristics of the mechanical dissipation depending on the forces. Since the co-content is a known function of the dissipator force, $K_f(f)$ represents the dissipator velocity.

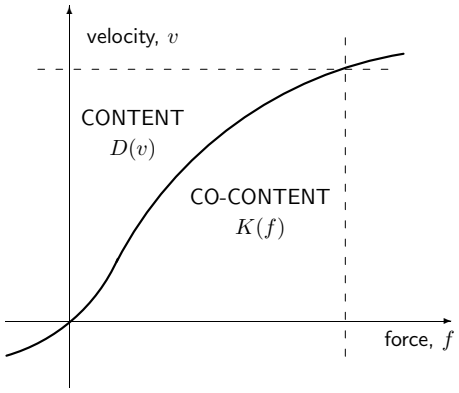

Figure 1: Mechanical content and co-content

In order to be fully consistent with the electrical domain, where current-controlled and voltage-controlled resistors are called *content* and *co-content*, respectively, we may refer to the velocity-controlled dissipators $D(v)$ as the *mechanical content* and the force-controlled dissipators $K(f)$ as the *mechanical co-content*, see also Figure 1. Originally, this terminology has been introduced by W. Millar and C. Cherry in the early-fifties (see [1] and the references therein). For a one-to-one dissipator the sum of the content and co-content yields the total absorbed

power, i.e.,

$$P_{\text{diss}}(v, f) = D(v) + K(f).$$

The external forces (control inputs), $\tau = \text{col}(\tau_1, \ldots, \tau_m) \in \mathbb{R}^m$, can be included through the total supplied power defined by $P_{\text{in}}(v, \tau) = v^\top B\tau$, where $B \in \mathbb{R}^{m\times n}$ reflects the degree of (under-)actuation. Hence, in a similar fashion as the proof of Proposition 1, the total mixed potential function of mechanical type is constructed as

$$P(f, v) = D(v) - K(f) + v^\top \gamma f - v^\top B\tau. \qquad (12)$$

Concerning the mechanical content and co-content, we observe that, in case $H^*(f, v)$ is quadratic, the power-balance is given by

$$\dot{H}^*(f, v) = P_{\text{in}}(v, \tau) - \underbrace{\left\{f^\top K_f(f) + v^\top D_v(v)\right\}}_{\text{dissipated power}}, \qquad (13)$$

which implies that (7) defines a passive port with power-port variables (v, τ). Equations (7) define the complete mechanical analog of (1), except for the fact that the mechanical co-content does not include an analog of the electrical current sources. This would suggest something in the direction of velocity sources. However, if necessary the velocity sources can be included as supplied co-content in a similar fashion as the external forces in the supplied content. Notice that if the kinetic co-energy does not depend on the conservative forces f, then (7) reduces to the canonical form of the Brayton-Moser equations of mechanical type. Let us conclude this section by a simple example of a mechanical system.

Example 1 Consider the mechanical mass-spring-damper system depicted in Figure 2.

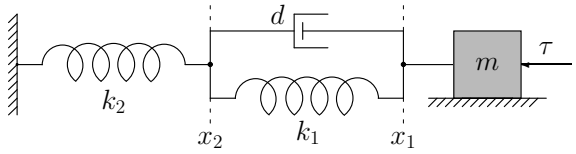

Figure 2: Mechanical system with co-content

Assume that the springs are linear with spring constants k_1 and k_2. Furthermore, let m be the mass of the block, d the resistance of the damper and let $\hat{\delta}(v_1)$ denote the constitutive relation for the friction of the mass with the surface. The externally supplied force is denoted by τ and the total co-energy is given by

$$H^*(f_1, f_2, v_1) = \frac{m}{2}v_1^2 + \frac{1}{2k_1}f_1^2 + \frac{1}{2k_2}f_2^2.$$

Then, the mixed potential for this system is defined by

$$P(f, v) = -\frac{1}{2d}(f_2 - f_1)^2 - \tau v_1 + \int_0^{v_1} \hat{\delta}(v_1')dv_1' + v_1 f_2. \qquad (14)$$

Hence, the equations of motion are determined by substitution of (14) together with the total co-energy $H^*(f_1, f_2, v_1)$ into (7), i.e.,

$$\begin{aligned}
\frac{1}{k_1}\dot{f}_1 &= \frac{1}{d}(f_2 - f_1) \\
\frac{1}{k_2}\dot{f}_2 &= v_1 - \frac{1}{d}(f_2 - f_1) \\
m\dot{v}_1 &= \tau - f_2 - \hat{\delta}(v_1).
\end{aligned}$$

3 Topologically Non-Complete Mechanical Systems

Although topological completeness can, in some sense as discussed in [1, 10] for electrical circuits, be considered as typical, in general it may occur that a system is not topologically complete. To see this we consider a topologically complete mechanical mass-spring-damper system with $m_2 \neq 0$ depicted in Figure 3. Assume that the spring is linear, i.e., $f_1 = k_1 x_1$, the masses are constant, d_1 is velocity-controlled with constitutive relation given by $\hat{\delta}_1(v_2 - v_1)$ and d_2 is linear with $\hat{\delta}_2(v_2) = d_1 v_2$. There is no friction between the masses and the surfaces.

If $m_2 > 0$ the system is obviously complete since the velocity v_2 is well-defined by the corresponding displacement of m_2. The mixed potential function is easily computed to be

$$P(f_1, v_1, v_2) = f_1(v_2 - v_1) + \int_0^{v_2 - v_1} \hat{\delta}_1(s)ds + \tfrac{1}{2}d_2 v_2^2 - v_1\tau.$$

Suppose now that $m_2 = 0$, then v_2 is no longer an independent variable and must be eliminated be letting $P_{v_2}(f_1, v_1, v_2) = 0$, see [1], or by setting

$$\tau = \hat{\delta}_1(v_2 - v_1) + d_2 v_2 + f_1,$$

which leads to $\tau - f_2 - d_2 v_1 = \hat{\delta}_1(v_2 - v_1) + d_2(v_2 - v_1)$, or equivalently

$$v_2 - v_1 = \hat{h}(\tau - f_1 - d_2 v_1).$$

Hence, after substituting the latter into the mixed potential $P(f_1, v_1, v_2)$ function yields a new mixed potential $\tilde{P}(f_1, v_1)$ in terms of the independent variables f_1 and v_1, i.e.,

$$\begin{aligned}
\tilde{P}(f_1, v_1) = f_1\hat{h}(\tau - f_1 - d_2 v_1) + \int_0^{\hat{h}(\tau - f_1 - d_2 v_1)} \hat{\delta}_1(s)ds + \\
\tfrac{1}{2}d_2[\hat{h}(\tau - f_1 - d_2 v_1) + v_1]^2 - v_1\tau.
\end{aligned}$$

If we assume that d_1 is also linear, i.e., $\hat{\delta}_1[\hat{h}(\tau - f_1 - d_2 v_1)] = d_1\hat{h}(\tau - f_1 - d_2 v_1)$ and thus $v_2 = \frac{1}{d_1 + d_2}(\tau + d_1 v_1 - f_1)$, then the equations of motion become

$$\begin{aligned}
\frac{1}{k_1}\dot{f}_1 &= \frac{\tau - f_1}{d_1 + d_2} - \frac{d_2}{d_1 + d_2}v_1 \\
m_1\dot{v}_1 &= \frac{d_1}{d_1 + d_2}(\tau + d_1 v_1 - f_1) - d_1 v_1.
\end{aligned}$$

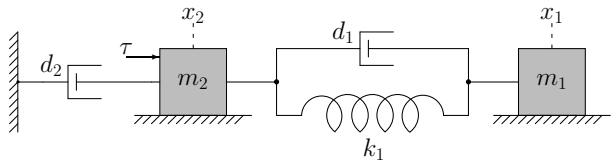

Figure 3: Topologically complete mechanical system system for $m_2 \neq 0$; topologically non-complete for $m_2 = 0$.

Although this is a very simple example, which could be modelled with much simpler techniques, the method becomes of interest if rather complex systems have to be studied. A similar but detailed systematic solution to such problems for electrical circuits is proposed in [10]. The general procedure applied to mechanical systems is as follows.

Procedure:

1. If possible, make the system topologically complete by adding additional dynamic elements (e.g., masses or springs). Derive the mixed potential function $P(f, v, f_{\text{add}}, v_{\text{add}})$ for the extended topologically complete system, where $f_{\text{add}} = \text{col}(f_{\varrho+1}, \ldots, f_\mu)$ and $v_{\text{add}} = \text{col}(v_{\varsigma+1}, \ldots, v_\nu)$ with $\mu - \varrho$ and $\nu - \varsigma$ the number of additional elements, respectively.

2. Find the relations $f_{\text{add}} = \hat{f}_{\text{add}}(f, v)$ and $v_{\text{add}} = \hat{v}_{\text{add}}(f, v)$, substitute the latter into step 1 and solve
$$P_{f_{\text{add}}}(f, v, f_{\text{add}}, v_{\text{add}}) = P_{v_{\text{add}}}(f, v, f_{\text{add}}, v_{\text{add}}) = 0.$$

3. Obtain the mixed potential function $\tilde{P}(f, v)$ by inserting $f_{\text{add}} = \hat{f}_{\text{add}}(f, v)$ and $v_{\text{add}} = \hat{v}_{\text{add}}(f, v)$ into $P(f, v, f_{\text{add}}, v_{\text{add}})$.

A necessary condition for $f_{\text{add}} = \hat{f}_{\text{add}}(f, v)$ and $v_{\text{add}} = \hat{v}_{\text{add}}(f, v)$ to exist is that the content and co-content need to be continuous functions. Furthermore, the Hessians of $P(f, v, f_{\text{add}}, v_{\text{add}})$ with respect to f_{add} and v_{add} need to be regular.

4 Non-Bijective Potential Fields

So far we have defined a precise mechanical analog of the Brayton-Moser equations. We have seen that this is only possible if the Legendre transformations $p \mapsto v$ and $x \mapsto f$ are well-defined. Unfortunately, in general this is not always the case. For example, if a systems operates under the influence of gravity the mapping $x \mapsto f$ simply does not exist. For that reason we extend the previous developments to systems having non-bijective potential fields introducing a parameterized version of the Brayton-Moser equations as follows.

Let $\lambda \leq \varrho$ denote the number of non-bijective functions depending on the partitioned generalized displacements $x = \text{col}(x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_\varrho)$ and let the corresponding total potential energy be given by

$$V(x) = \sum_{j=1}^{\lambda} V_j(x). \tag{15}$$

The remaining $\varrho - \lambda$ bijective relations should be expressible in terms of the related conservative forces $f = \text{col}(f_{\lambda+1}, \ldots, f_\varrho)$ with the corresponding potential co-energy

$$V^*(f) = \sum_{k=1+\lambda}^{\varrho} V_k^*(f). \tag{16}$$

Consequently, we have that the first equation of (7) becomes $\dot{x}_1 = v_1, \ldots, \dot{x}_\lambda = v_\lambda$ and $\frac{d}{dt} H^*_{f_{\lambda+1}} = P_{f_{\lambda+1}}, \ldots, \frac{d}{dt} H^*_{f_\varrho} = P_{f_\varrho}$. Hence, by defining $\tilde{x} = \text{col}(x_1, \ldots, x_\lambda)$ and $\tilde{v} = \text{col}(v_1, \ldots, v_\lambda)$, the complete description becomes

$$\begin{aligned} \tfrac{d}{dt}\tilde{x} &= \tilde{v} \\ \tfrac{d}{dt} H^*_f(\tilde{x}, f, v) &= P_f(\tilde{x}, f, v) \\ -\tfrac{d}{dt} H^*_v(\tilde{x}, f, v) &= P_v(\tilde{x}, f, v). \end{aligned} \tag{17}$$

The total co-energy $H^*(\tilde{x}, f, v)$ is defined as $H^*(\tilde{x}, f, v) = T^*(\tilde{x}, v) + V^*(f)$ and the mixed potential function $P : \mathbb{R}^{\varrho+\varsigma} \to \mathbb{R}$ takes the form

$$\begin{aligned} P(\tilde{x}, f, v) = D(\tilde{x}, v) \; - \; K(\tilde{x}, f) + v^\top \gamma f + \tilde{v}^\top V_{\tilde{x}}(\tilde{x}) \\ - \int_0^v H^*_{\tilde{x}}(\tilde{x}, f, v') dv' - v^\top B\tau, \end{aligned}$$

where γ is now a $\varsigma \times (\varrho - \lambda)$ matrix not necessarily constant, i.e., γ may depend on $\tilde{x}$, f and v. Notice that $\dot{\tilde{x}} = \tilde{v}$ represents the constraints on the additional velocity. The term

$$\int_0^v H^*_{\tilde{x}}(\tilde{x}, f, v') dv'$$

corresponds to the power generated by the 'geometrical' forces $H^*_x(x, v)$, which is part of the forces called the Coriolis and centrifugal forces represented by $H^*_{v\tilde{x}}\dot{\tilde{x}} - H^*_{\tilde{x}}$, see [9] for more details. One easily recognizes in (17) the underlying partial Lagrangian structure, i.e., in case $\lambda = \varrho$, by noting that $\tilde{x} = x$, $\dot{x} = v$ and by writing $P_v(x, v)$ explicitly, (17) reduces to

$$-\tfrac{d}{dt} H^*_v(x, v) + H^*_x(x, v) = -D_v(x, v) + V_x(x) + B\tau,$$

where we assume that $H^*(x, v) = T^*(x, v)$ and $K(f) = 0$. Hence, by defining $L(x, v) = T^*(x, v) - V(x)$, the latter equation can be written as

$$-\tfrac{d}{dt} L_v(x, v) + L_x(x, v) = -D_v(x, v) + B\tau.$$

This is just the well-known Euler-Lagrange equation with Lagrangian $L(x, v)$.

We conclude this section with the remark that if the kinetic co-energy depends on the generalized displacements (or possibly forces as well), then (17) establishes a set of generalized parameterized Brayton-Moser equations. In this case it is not possible to find a canonical set of equations due to the fact that the Coriolis and centrifugal forces are workless [3].

5 Concluding Remarks

In this paper we have presented a precise mechanical analog of the Brayton and Moser framework. The method uses the kinetic and potential co-energy and a mixed potential deduced from the topological constraints of the system. A necessary condition for the framework to be valid is that the kinetic and potential energy are homogeneous quadratic functions. Although this seems quite a strong restriction there is still a broad class of systems that can be approached by the method. On the other hand, if the potential energy is non-quadratic we may try to find a description of the system locally or we can use a parameterized version of the Brayton-Moser equations proposed in the previous section. As shown in an example, it may occur that one has to deal with a mechanical system having force controlled dampers or friction. In contrast to the (port-)Hamiltonian equations, this type of elements can be naturally incorporated in the framework.

An advantage of using the mixed potential function is that we can apply Brayton and Moser's stability criteria to investigate the stability of the equilibrium points of a mechanical system or use it to find tuning rules for feedback controllers, like the passivity-based control algorithms [5]. Also, in the proposed framework we are able to apply the recently developed novel nonlinear control technique, called *Power Shaping*, as proposed in [7], to the class of mechanical systems considered here. This control method aims at shaping the mixed potential function in order to stabilize the system towards a desired (non-zero) equilibrium. However, it is not precisely clear to what extend the method can be used in general. What we do know is that for a given system described by a function, say $f(z,\cdot) = P_z(z,\cdot)$, the existence of a mixed potential function $P(z,\cdot)$ hangs upon the fact that the Jacobian of $f(z,\cdot)$ should be symmetric. As stated in e.g. [2] there may exist (internally modulated) Dirac structures that can not be generated by one mixed potential function due to a particular kind of nonlinearity. An example of such system is a rigid body spinning around its center of mass described by Euler's equations, see e.g., [2, 9]. For that reason, future research should be devoted to find conditions for the existence of a mixed potential function of mechanical type.

Acknowledgment

The authors would like to thank Romeo Ortega and Jesús Clemente-Gallardo for their insightful remarks and discussions regarding the material presented in this paper.

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