PARAMETERIZED H_∞ CONTROLLER DESIGN FOR ADAPTATIVE TRADE-OFF BY FINITE DIMENSIONAL LMI OPTIMIZATION

M. Dinh△ ‡, G. Scorletti△, V. Fromion‡, E. Magarotto△

△ LAP, ISMRA, 6 bd du Maréchal Juin, 14050 Caen cedex, France. {scorletti@greyc.ismra.fr}
‡ LASB, INRA–Montpellier, 2 place Viala, 34060 Montpellier, France.

Keywords: robust control, LPV synthesis, set of trade-offs, parameter dependent Lyapunov function.

Abstract

We consider the design of a control law $K(s,\theta)$ such that a set of trade-offs parameterized by a scalar θ are satisfied for a given LTI system. The trade-off is formulated as an optimization problem involving a weighted H_∞ norm. Matrices of the state space representation of $K(s,\theta)$ have to be obtained as explicit functions of θ. This problem, close to the LPV control design, can be naturally formulated as a convex but infinite dimensional optimization problem. The contribution of this paper is to propose design methods based on finite dimensional LMI optimization problem, avoiding the "naive" gridding approach.

1 Introduction

During the last twenty years, dramatic advances were accomplished in the design of Linear Time-Invariant (LTI) control laws for LTI systems using the frequency domain approach. In the so-called H_∞ control [20, 21], the specifications are translated as constraints, defined by weighting functions, on the magnitude of the closed-loop transfer function. Controller design boils down to optimization on the H_∞ norm of the closed loop transfer function involving the plant augmented with the weighting functions [11]. Nevertheless, the existing methods focus on designing one particular LTI control law for one particular trade-off of the design specifications. In some applications, it is important to adjust the control law parameters in order to satisfy different trade-offs. The road adaptative active suspension is an example [7] in which control gains are tuned, in real time, to adapt the trade-off between limited suspension deflection and comfort depending on a parameter (the road conditions). In the sequel, the considered trade-offs are parameterized by a single parameter, denoted θ.

In this paper, we propose several methods for designing controllers whose state space representation explicitly depends on this parameter θ. To this purpose, we extend and adapt design methods proposed for LPV control. A more detailed presentation of the results can be found in [5].

Outline of the paper In section 2, the problem is formulated. Problem difficulties and related approaches are then discussed. Two possible solutions are proposed in sections 3 et 4.

Notations I_n denotes the $n \times n$ identity matrix. I and 0 denote respectively the identity and the zero matrices of appropriate size. $P > 0$ denotes that P is positive definite. The Redheffer star product [21] is denoted by $\star$. The "state space" matrices of an LFT [21] $M(\Delta)$ are denoted A_M, B_M, C_M and D_M, that is: $M(\Delta) = \Delta \star \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right]$. In a matrix, $\star$ denotes the transpose of the symmetric block.

2 Considered problem

2.1 Problem formulation

We consider the augmented plant:

$$P(s,\theta)\begin{cases} \dot{x}(t) = & Ax(t) + \quad B_w w(t) + \quad B_u u(t) \\ z(t) = & C_z(\theta)x(t) + D_{zw}(\theta)w(t) + D_{zu}(\theta)u(t) \\ y(t) = & C_y x(t) + \quad D_{yw} w(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input and $\theta \in [0,1]$ is the parameter. The *data* matrix $[C_z(\theta)\ D_{zw}(\theta)\ D_{zu}(\theta)]$ is an explicit function of θ. Such dependence can be obtained when output weighting functions depend on θ as follows:

$$W(s,\theta) = \frac{1}{s} I \star \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W(\theta) & D_W(\theta) \end{array}\right]. \quad (2)$$

PROBLEM Given $P(s,\theta)$ as defined in (1), compute γ_{opt} such that

$$\gamma_{opt} = \min_{K(s,\theta)} \max_{\theta \in [0,1]} \|P(s,\theta) \star K(s,\theta)\|_\infty \quad (3)$$

with $K(s,\theta) = \frac{1}{s} I \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array}\right]$ that ensures for each $\theta \in [0,1]$ the internal stability of $P(s,\theta) \star K(s,\theta)$.

2.2 Illustrative example

Let us consider a plant $G(s) = \frac{1}{s+1}$ controlled by a one degree of freedom controller. The purpose is to design a control law ensuring that the closed loop system output is able to track step reference signals with a specified transient time response. We want to design a θ dependent controller which achieves different trade-offs between transient time response and control input energy. Such a problem is addressed by mixed sensitivity H_∞

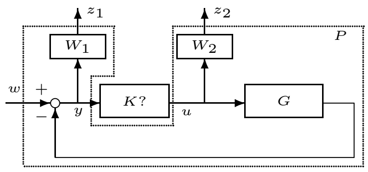

Figure 1: Mixed sensitivity problem

design [18] (see figure 1). The usual H_∞ problem is for a given trade-off, that is, for a given $\theta_i \in [0,1]$:

$$\gamma_{\theta_i} = \min_{K_{\theta_i}(s)} \left\| \begin{matrix} W_1(s,\theta_i)\dfrac{1}{1+G(s)K_{\theta_i}(s)} \\ W_2(s,\theta_i)\dfrac{K_{\theta_i}(s)}{1+G(s)K_{\theta_i}(s)} \end{matrix} \right\|_\infty . \quad (4)$$

If γ_{θ_i} is close to 1 (see *e.g.* [18]), condition (4) ensures that:

1. tracking specification is given by $W_1(s,\theta_i) = k\dfrac{s+\beta_{\theta_i}}{s+\epsilon}$ where ϵ is set to a small value (1.7×10^{-3}) for ensuring a small error tracking;
2. a 6 dB modulus margin is guaranteed by setting k of W_1 to 0.5;
3. control input energy is constrained by the choice of W_2. W_2 is such that the inverse of W_2 is a low pass filter.

In our problem, a trade-off is defined by β_{θ_i} and the bandwidth of W_2. We consider two extreme trade-offs:

1. for $\theta_i = 1$, fast response with high control energy: $\beta_{\theta_i} = 3.45$, $W_2(s) = 100\dfrac{s+5}{s+1580}$;
2. for $\theta_i = 0$, slow response with low control energy: $\beta_{\theta_i} = 0.86$, $W_2(s) = 1800\dfrac{s+0.28}{s+1580}$.

Between both trade-offs, β_{θ_i} is divided by 4. W_1 and W_2, for intermediate values of θ, are obtained by interpolating the previous two pairs of W_1 and W_2.

2.3 Solution formulation

For a given θ, two solutions were proposed to solve PROBLEM using convex optimization involving Linear Matrix Inequality (LMI) constraints [8, 15]. Using the elimination lemma [8], the existence and the computation of a solution are performed in two shots. In order to avoid these two shots, the approach of [15] is considered. From [15], we have:

Theorem 2.1 PROBLEM *has the following solution:*
γ_{opt} *is the minimum value of* γ *such that there exist*

- *symmetric matrices of functions* $\mathcal{X}(\theta), \mathcal{Y}(\theta) \in \mathbb{R}^{n\times n}$*;*
- *matrices of functions* $\mathcal{A}(\theta) \in \mathbb{R}^{n\times n}$, $\mathcal{B}(\theta) \in \mathbb{R}^{n\times n_y}$, $\mathcal{C}(\theta) \in \mathbb{R}^{n_u\times n}$ *and* $\mathcal{D}(\theta) \in \mathbb{R}^{n_u\times n_y}$

satisfying for each $\theta \in [0,1]$*:*

$$\begin{bmatrix} \mathcal{X}(\theta) & I \\ I & \mathcal{Y}(\theta) \end{bmatrix} > 0 \quad (5)$$

$$\left[\begin{array}{c|c|c|c} \begin{matrix} A\mathcal{X}(\theta)+\mathcal{X}(\theta)A^T+ \\ B_u\mathcal{C}(\theta)+(B_u\mathcal{C}(\theta))^T \end{matrix} & \star & \star & \star \\ \hline \begin{matrix} \mathcal{A}(\theta)+ \\ (A+B_u\mathcal{D}(\theta)C_y)^T \end{matrix} & \begin{matrix} A^T\mathcal{Y}(\theta)+\mathcal{Y}(\theta)A+ \\ \mathcal{B}(\theta)C_y+(\mathcal{B}(\theta)C_y)^T \end{matrix} & \star & \star \\ \hline (B_w+B_u\mathcal{D}(\theta)D_{yw})^T & (\mathcal{Y}(\theta)B_w+\mathcal{B}(\theta)D_{yw})^T & -\gamma I & \star \\ \hline \begin{matrix} C_z(\theta)\mathcal{X}(\theta)+ \\ D_{zu}(\theta)\mathcal{C}(\theta) \end{matrix} & \begin{matrix} C_z(\theta)+ \\ D_{zu}(\theta)\mathcal{D}(\theta)C_y \end{matrix} & \begin{matrix} D_{zw}(\theta)+ \\ D_{zu}(\theta)\mathcal{D}(\theta)D_{yw} \end{matrix} & -\gamma I \end{array}\right] < 0 \quad (6)$$

Then, the state space matrices of $K(s,\theta)$ *are obtained by:*

$$\begin{aligned} \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array}\right] &= \begin{bmatrix} L(\theta) & -M(\theta) & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix} \times \cdots \\ \cdots \times &\left(\begin{bmatrix} I_n & 0 \\ 0 & B_u \\ 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} \mathcal{A}(\theta) & \mathcal{B}(\theta) \\ \mathcal{C}(\theta) & \mathcal{D}(\theta) \end{bmatrix} \begin{bmatrix} \mathcal{X}(\theta)^{-1} & 0 \\ -C_y & I_{n_y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A & 0 \\ 0 & 0 \end{bmatrix} \right) \end{aligned} \quad (7)$$

where $[\,L(\theta) \;\; -M(\theta)\,] = \left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathcal{X}(\theta) \, [\,I_n \;\; \mathcal{Y}(\theta)\,] \begin{bmatrix} I_n & 0 & 0 \\ 0 & -I_n & I_n \end{bmatrix} \right) \star I_n$.

The optimization problem presented in Theorem 2.1 boils down to a convex optimization problem which is a desirable feature. The main difficulty is that it is infinite dimensional:

1. as functions of θ, decision variables ($\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$, $\mathcal{A}(\theta)$, $\mathcal{B}(\theta)$, $\mathcal{C}(\theta)$ and $\mathcal{D}(\theta)$) belongs to an infinite dimensional space;
2. as parameterized by θ, there is an infinite number of constraints.

In the next section, we discuss the possible ways to bypass these difficulties.

2.4 Possible approaches to manage infinite dimension

Gridding A first approach is to grid θ: for a finite number of values θ_i, compute the solution of Theorem 2.1, that is, compute the solution of the following standard H_∞ problem: $\gamma_{\theta_i} = \min_{K_{\theta_i}(s)} \|P(s,\theta_i) \star K_{\theta_i}(s)\|_\infty$. The θ dependent controller is then obtained by interpolating the designed ones. The major theoretical drawback is the lack of guarantee for the values of θ which ar not contained in the grid. Moreover, interpolation of controllers is a difficult problem.

LPV A more interesting approach is to directly design the θ dependent controller from Theorem 2.1. This is strongly related to Linear Parameter-Varying (LPV) control, in which the purpose is to design a controller depending on θ that ensures stability and $\mathcal{L}_2$ gain performance [19] for a θ dependent augmented plant. Note that, in [7], the design of a road adaptative active suspension is recast as the design of an LPV control law. Unfortunately, in this paper, the particular features of the considered problem are not exploited.

As in our problem, the main difficulty in LPV control is that obtained conditions are naturally in the form of an infinite dimensional problem, that is an infinite number of decision variables and an infinite number of constraints. The proposed solutions in LPV context are to first restrict the set of "data"[1] and decision variables, functions of θ, to a finite dimensional set. This step leads to a finite number of decision variables but introduces conservatism. Then, in order to turn the infinite number of constraints into a finite one, two classes of approaches were proposed. When the θ dependant matrix "data" defines a polytope, the first one allows to check the conditions only at the vertices of the polytope [3, 2, 9]. The second one is based on Linear Fractional Transformation (LFT) properties and the use of the $\mathcal{S}$-procedure [13, 1, 6, 12, 17, 16].

Some important works on LPV are classified in the non exhaustive Table 1. Other approaches are discussed in [5].

[1] State space matrices of the to-be-controlled plant.

Data	$\mathcal{X}, \mathcal{Y}$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	Polytopic	LFT
affine	constant	affine	[3, 2]	
rational	constant	rational		[13, 1, 12, 17, 16]
affine	affine	affine	[9]	[6]

Table 1: Considered sets of functions (with respect to θ)

Each solution corresponds to a particular trade-off between conservatism and complexity. Our contribution is to extend and adapt existing approaches of LPV control to our problem with an interesting trade-off between conservatism and complexity. In sections 3 and 4, in order to avoid the infinite number of decision variables, we use the set of rational functions. In order to bypass the infinite number of constraints, a polytopic approach is explored in section 3. This approach can lead to strong restrictions on the possible sets of functions. We then develop the LFT approach in section 4 which allows to avoid these restrictions.

3 Particular approach

In this section, we investigate the use of rational functions with the polytopic approach.

3.1 A specific result

In the sequel, we assume that the data matrix is a degree one rational function *i.e.*: with $d > -1$ and $\theta \in [0, 1]$
$[C_{z0}\ D_{zw0}\ D_{zu0}] + \frac{\theta}{1+d\theta}[C_{z1}\ D_{zw1}\ D_{zu1}]$.

Theorem 3.1 *An upper bound γ_{pub} of γ_{opt} can be computed by minimizing γ such that there exist*

- *symmetric matrices $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{Y}_0$, $\mathcal{Y}_1$;*
- *matrices $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{D}_0$, $\mathcal{D}_1$*

satisfying:

1. *the following Linear Matrix Equalities (LMEs):*
$$C_{z1}\mathcal{X}_1 + D_{zu1}\mathcal{C}_1 = 0 \qquad D_{zu1}\mathcal{D}_1 = 0; \tag{8}$$
2. *the following LMIs for $\alpha \in \{0, 1\}$:*
$$\begin{bmatrix} \mathcal{X}_0 + \frac{\alpha}{1-d}\mathcal{X}_1 & I \\ I & \mathcal{Y}_0 + \frac{\alpha}{1-d}\mathcal{Y}_1 \end{bmatrix} > 0 \tag{9}$$
$$T_0 + \frac{\alpha}{1-d}T_1 < 0 \tag{10}$$

where

$$T_0 = \left[\begin{array}{c|c|c|c} \begin{array}{c} A\mathcal{X}_0 + \mathcal{X}_0 A^T + \\ B_u\mathcal{C}_0 + (B_u\mathcal{C}_0)^T \end{array} & \star & \star & \star \\ \hline \mathcal{A}_0 + (A + B_u\mathcal{D}_0 C_y)^T & \begin{array}{c} A^T\mathcal{Y}_0 + \mathcal{Y}_0 A + \\ \mathcal{B}_0 C_y + (\mathcal{B}_0 C_y)^T \end{array} & \star & \star \\ \hline (B_w + B_u\mathcal{D}_0 D_{yw})^T & (\mathcal{Y}_0 B_w + \mathcal{B}_0 D_{yw})^T & -\gamma I & \star \\ \hline C_{z0}\mathcal{X}_0 + D_{zu0}\mathcal{C}_0 & C_{z0} + D_{zu0}\mathcal{D}_0 C_y & D_{zw0} + D_{zu0}\mathcal{D}_0 D_{yw} & -\gamma I \end{array}\right]$$

$$T_1 = \left[\begin{array}{c|c|c|c} \begin{array}{c} A\mathcal{X}_1 + \mathcal{X}_1 A^T + \\ B_u\mathcal{C}_1 + (B_u\mathcal{C}_1)^T \end{array} & \star & \star & \star \\ \hline \mathcal{A}_1 + (B_u\mathcal{D}_1 C_y)^T & \begin{array}{c} A^T\mathcal{Y}_1 + \mathcal{Y}_1 A + \\ \mathcal{B}_1 C_y + (\mathcal{B}_1 C_y)^T \end{array} & \star & \star \\ \hline (B_u\mathcal{D}_1 D_{yw})^T & (\mathcal{Y}_1 B_w + \mathcal{B}_1 D_{yw})^T & -\gamma I & \star \\ \hline \begin{array}{c} C_{z1}\mathcal{X}_0 + C_{z0}\mathcal{X}_1 + \\ D_{zu1}\mathcal{C}_0 + D_{zu0}\mathcal{C}_1 \end{array} & \begin{array}{c} C_{z1} + (D_{zu1}\mathcal{D}_0 + \dots \\ D_{zu0}\mathcal{D}_1)C_y \end{array} & \begin{array}{c} D_{zw1} + (D_{zu1}\mathcal{D}_0 + \dots \\ D_{zu0}\mathcal{D}_1)D_{yw} \end{array} & -\gamma I \end{array}\right]$$

The θ dependent controller is then obtained from equation (7).

The infinite number of decision variables is avoided by choosing the set of rational functions of degree one for all the decision variables. Under the LMEs (8), this leads to a convex problem but with an infinite number of constraints. The infinite number of constraints can be reduced with a polytopic property (see [10]). (The obtained result has strong connections with the result of [9].) Unfortunately, the use of the polytopic property restrict the allowed sets of functions to the set of rational functions of degree one.

Remark In Theorem 3.1, further conservatism (with respect to the choice of the set of rational functions of degree one) is introduced to bypass the infinite number of constraints (see LMEs (8)).

Computation Theorem 3.1 has introduced two kind of constraints:

- *LME ones:* the set of decision variables which satisfy conditions (8) can be linearly parameterized using the well-known Gauss-Seidel iteration. It leads to a new set of decision variables;
- *LMI ones:* minimizing γ such that there exist decision variables in this last set satisfying conditions (9) and (10) is a linear cost minimization problem over LMI constraints [4].

3.2 Numerical example

Let us consider the example introduced in section 2.2 and assume that the weighting functions are given by $W_1(s,\theta) = \frac{1}{s}I \star \left[\begin{array}{c|c} -0.0017 & 1.32 \\ \hline 0.33-\theta & 0.5 \end{array}\right]$ and $W_2(s,\theta) = \frac{1}{s}I \star \left[\begin{array}{c|c} -1580 & 400 \\ \hline -7170 + 6770\ \theta & 1800 - 1700\ \theta \end{array}\right]$ which correspond to a simple linear interpolation.

Results analysis $K(s,\theta)$ is computed by solving optimization problem of Theorem 3.1. We obtain a value of 1.92 for γ_{pub}. $|S| = \frac{1}{|1+GK|}$ and $|KS|$ are represented figure 2 with $K \in \{K(s,\theta_i),\ \theta_i \in \{0, 0.5, 1\}\}$. Note, for instance, that be-

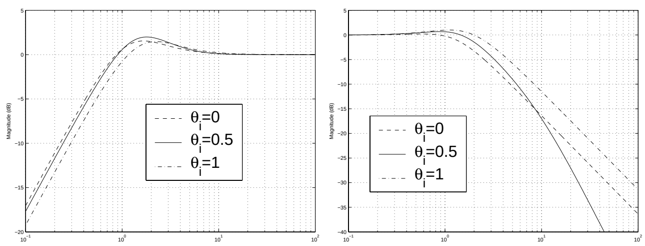

Figure 2: $|S|$ and $|KS|$, $K \in \{K(s,\theta_i), \theta_i \in \{0, 0.5, 1\}\}$

tween extreme values of θ, the bandwidth of S is multiplied by 1.3. When the controllers are obtained by solving the standard H_∞ problem: $\gamma_{\theta_i} = \min_{K_{\theta_i}(s)} \|P(s,\theta_i) \star K_{\theta_i}(s)\|_\infty$ ("*pointwise controllers*"), for $\theta = 0$ and $\theta = 1$, the bandwidth of S is multiplied by 4. The poor performance of the obtained $K(s,\theta)$ is confirmed by inspecting the output transient response to a step reference signal (see figure 3).

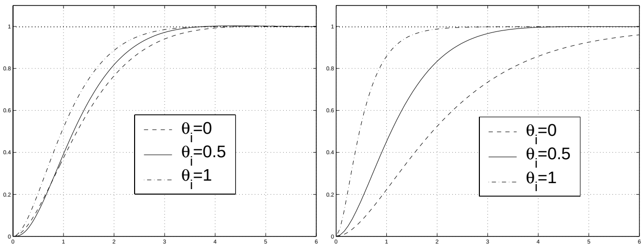

Figure 3: Plant output with $K(s, \theta_i)$ (left) and $K_{\theta_i}(s)$ (right), $\theta_i \in \{0, 0.5, 1\}$

Discussion Let us explain the poor results obtained with $K(s, \theta)$. In Theorem 3.1, we minimize γ which is independent of θ and necessarily greater than the maximum γ_{θ_i} (approximatively 1.38) obtained with standard H_∞ problems. For $\theta_i = 1$, γ_{θ_i} is about 1, that is, 35% of discrepancy, which explains the poor results. In addition, the conservatism of Theorem 3.1 increases this effect since γ_{pub} is 1.92. If we adapt the approaches of [3, 2, 9, 6] to our problem, affine functions are also considered for the interpolation: the same problem would arise.

This analysis is the motivation for (i) improving the weighting functions interpolation in order that γ_{θ_i} remains constant for any value θ_i and for (ii) decreasing the conservatism of proposed conditions. In the previous example, the weighting functions interpolation improvement requires more general rational functions than the functions considered by theorem 3.1. One of the interests of the general approach presented in the next section is to consider general rational functions.

4 General approach

4.1 Preliminary remarks

Finite set of rational functions and the LFT realization In this general approach, we consider the finite dimensional set of rational functions $\Phi_{N,\{d_i\},m,p}$ defined as:

$$\left\{ H_0 + \frac{\sum_{i=1}^{N} \theta^i H_i}{1 + \sum_{i=1}^{N} \theta^i d_i} \,\middle|\, H_i \in \mathbb{R}^{m\times p}, \forall \theta \in [0,1], 1 + \sum_{i=1}^{N} \theta^i d_i \neq 0 \right\}$$

where N, m and p are three positive integers. This choice is interesting since it is a large set of functions. It encompasses all the sets presented Table 1.

For a given N, when the scalars d_i are fixed, $\Phi_{N,\{d_i\},m,p}$ is convex, more precisely affine in H_i. Note that, when the sets $\{d_i\}$ are different, the sets of functions $\Phi_{N,\{d_i\},m,p}$ are also different even for given N, m and p. The set $\Phi_{N,\{d_i\},m,p}$ with appropriate m and p is now denoted $\Phi_{N,\{d_i\}}$.

An element of the finite dimensional set of rational functions $\Phi_{N,\{d_i\}}$ admits an LFT realization. More precisely, one has:

Lemma 4.1 *Any rational matrix $H(\theta)$ of the set $\Phi_{N,\{d_i\},m,p}$ can be expressed as:*

$$\theta I \star \left[\begin{array}{cccccc|c} 0 & \cdots & \cdots & 0 & -d_N I_m & H_N \\ I_m & \ddots & & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & I_m & -d_1 I_m & H_1 \\ \hline 0 & \cdots & \cdots & 0 & I_m & H_0 \end{array}\right] = \theta I \star \left[\begin{array}{c|c} A_H & I\ 0 \\ \hline C_H & 0\ I \end{array}\right] \times \begin{bmatrix} H_N \\ \vdots \\ H_1 \\ H_0 \end{bmatrix} \quad (11)$$

From equation (11), a decision variable $H(\theta) \in \Phi_{N,\{d_i\}}$ can be factorized as the product of an LFT and of $\mathcal{R}_H = [H_N \cdots H_1\ H_0]^T$. When $H(\theta)$ is symmetric of size m, $\mathcal{R}_H$ has a special structure: it is the concatenation of symmetric matrices. For notational convenience, the following set is defined as: $\Lambda_{N,m} = \left\{ \mathcal{R}_H \middle| H_i = H_i^T \in \mathbb{R}^{m\times m} \right\}$. Note that restricting the decision variable $H(\theta)$ to belong to $\Phi_{N,\{d_i\}}$ is equivalent in choosing $\mathcal{R}_H$ as a decision variable.

The infinite number of constraints and the LFT realization We now illustrate the main interest of rational functions or the LFT approach when we have to test an infinite number of matrix inequalities. Let us consider *e.g.* condition (5) of Theorem 2.1. We assume that $\mathcal{X}(\theta)$ and $\mathcal{Y}(\theta)$ are rational functions and we want to test if condition (5) is satisfied, *i.e.* if, for each $\theta \in [0,\ 1]$, $M(\theta) + M(\theta)^T > 0$ with

$$M(\theta) = \begin{bmatrix} \frac{1}{2}\mathcal{X}(\theta) & I \\ 0 & \frac{1}{2}\mathcal{Y}(\theta) \end{bmatrix}. \quad (12)$$

Following Lemma 4.1, $M(\theta)$ can be expressed as an LFT and condition (5) can be interpreted as a passivity condition of the static system $M(\theta)$ [17]. From [14][2], we have the following lemma.

Lemma 4.2 *The following inequality is satisfied:*

$$M(\theta) + M(\theta)^T > 0 \quad (13)$$

for each $\theta \in [0, 1]$ if and only if there exist $\mathcal{S} = \mathcal{S}^T > 0$ and $\mathcal{G} = -\mathcal{G}^T$ such that

$$\begin{bmatrix} (\mathcal{S}+\mathcal{G})A_M^T + A_M(\mathcal{S}-\mathcal{G}) - 2\mathcal{S} & (\mathcal{S}+\mathcal{G})C_M^T - B_M \\ C_M(\mathcal{S}-\mathcal{G}) - B_M^T & -D_M - D_M^T \end{bmatrix} < 0 \quad (14)$$

Note that the infinite number of constraints defined by (13) is replaced by a finite dimensional LMI problem (14). Moreover, since the LFT realization of $M(\theta)$ is obtained from the LFT realizations of $\mathcal{X}(\theta)$ and $\mathcal{Y}(\theta)$, it only depends on the decision variables $\mathcal{R}_\mathcal{X}$ and $\mathcal{R}_\mathcal{Y}$. The interesting feature is that only B_M and D_M depend on $\mathcal{R}_\mathcal{X}$ and $\mathcal{R}_\mathcal{Y}$, in an affine way. As a consequence, condition (14) of Lemma 4.2 is a matrix inequality, affine in $\mathcal{S}$, $\mathcal{G}$,$\mathcal{R}_\mathcal{X}$ and $\mathcal{R}_\mathcal{Y}$.

4.2 Proposed solution

If the data matrix is assumed to be a rational function of θ, that is in the LFT form:

$$\begin{bmatrix} C_z(\theta)\ D_{zw}(\theta)\ D_{zu}(\theta) \end{bmatrix} = \theta I \star \left[\begin{array}{c|c} \tilde{A} & B_{C_z}\ B_{D_{zw}}\ B_{D_{zu}} \\ \hline \tilde{C} & D_{C_z}\ D_{D_{zw}}\ D_{D_{zu}} \end{array}\right]$$

[2] In fact, it is strongly related to the μ upper bounds which is known to be exact when one uncertain parameter is considered.

then, one has:

Theorem 4.1 *An upper bound γ_g of γ_{opt} can be computed by minimizing γ such that there exist*

- *symmetric positive definite matrices $\mathcal{S}_0$, $\mathcal{S}$, skew-symmetric matrices $\mathcal{G}_0$, $\mathcal{G}$;*
- *matrices $\mathcal{R}_\mathcal{X} \in \Lambda_{N_\mathcal{X},n}$, $\mathcal{R}_\mathcal{Y} \in \Lambda_{N_\mathcal{Y},n}$ and $\mathcal{R}_\mathcal{V} \in \mathbb{R}^{(N_\mathcal{V}+1)(n+n_u)\times(n+n_y)}$*

satisfying:

$$\begin{bmatrix} (\mathcal{S}_0+\mathcal{G}_0)A_{\Omega_0}^T + A_{\Omega_0}(\mathcal{S}_0-\mathcal{G}_0) - 2\mathcal{S}_0 & \begin{matrix}(\mathcal{S}_0+\mathcal{G}_0)C_{\Omega_0}^T - \\ B_{\Omega_0}\mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})\end{matrix} \\ C_{\Omega_0}(\mathcal{S}_0-\mathcal{G}_0) - \mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})^T B_{\Omega_0}^T & \begin{matrix}-D_{\Omega_0}\mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})\dots \\ -\mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})^T D_{\Omega_0}^T\end{matrix} \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} (\mathcal{S}+\mathcal{G})A_{\Omega}^T + A_{\Omega}(\mathcal{S}-\mathcal{G}) - 2\mathcal{S} & \begin{matrix}(\mathcal{S}+\mathcal{G})C_{\Omega}^T + \\ B_{\Omega}\mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})\end{matrix} \\ C_{\Omega}(\mathcal{S}-\mathcal{G}) + \mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})^T B_{\Omega}^T & \begin{matrix}D_{\Omega}\mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})\dots \\ +\mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})^T D_{\Omega}^T\end{matrix} \end{bmatrix} < 0 \quad (16)$$

where

$$\Omega_0(\theta) = \theta I \star \left[\begin{array}{cc|cccc} A_\mathcal{X} & 0 & \frac{1}{2}I & 0 & 0 & 0 \\ 0 & A_\mathcal{Y} & 0 & 0 & \frac{1}{2}I & 0 \\ \hline C_\mathcal{X} & 0 & 0 & \frac{1}{2}I & 0 & 0 \\ 0 & C_\mathcal{Y} & 0 & 0 & 0 & \frac{1}{2}I \end{array}\right]$$

$$\mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y}) = \begin{bmatrix} \mathcal{R}_\mathcal{X} & \begin{bmatrix}0\\2I\end{bmatrix} \\ 0 & \mathcal{R}_\mathcal{Y} \end{bmatrix}$$

$$\Omega(\theta) = \theta I \star \left[\begin{array}{c|cccccccc} \tilde{A} & 0 & B_{D_{zu}} & B_{C_z} & 0 & 0 & B_{D_{zw}} & 0 & 0 \\ \hline 0 & 0 & B_u & A & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & A^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_w^T & B_w^T & 0 & -\frac{1}{2}I & 0 \\ \tilde{C} & 0 & D_{D_{zu}} & D_{C_z} & 0 & 0 & D_{D_{zw}} & 0 & -\frac{1}{2}I \end{array}\right] \times$$

$$\theta I \star \left[\begin{array}{ccc|ccccccc} A_\mathcal{V} & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_\mathcal{X} & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\mathcal{Y} & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline C_\mathcal{V} & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & C_\mathcal{X} & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & C_\mathcal{Y} & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right]$$

$$\mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y}) = \left[\begin{array}{c|cccc} \mathcal{R}_\mathcal{V} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}_\mathcal{X} & \begin{bmatrix}0\\I\end{bmatrix} & 0 & 0 \\ 0 & \mathcal{R}_\mathcal{Y} & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 & \gamma I \end{array}\right] \times \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & C_y & D_{yw} & 0 \\ \hline & & I & \end{array}\right]$$

The controller is then obtained with equation (7).

Computation As $\mathcal{W}(\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})$ and $\mathcal{Z}(\mathcal{R}_\mathcal{V},\mathcal{R}_\mathcal{X},\mathcal{R}_\mathcal{Y})$ are affine in $\mathcal{R}_\mathcal{V}$, $\mathcal{R}_\mathcal{X}$ and $\mathcal{R}_\mathcal{Y}$, testing conditions (15) and (16) is a linear cost minimization problem over LMI constraints [4].

Conservatism of the conditions In contrast with section 3, the obtain conditions are only conservative with respect to the finite parametrisation of the decision variables. Actually, Theorem 4.1 gives necessary and sufficient conditions for the following problem:

Given

- $P(s,\theta)$ defined in (1), where $[\,C_z(\theta)\;\; D_{zw}(\theta)\;\; D_{zu}(\theta)\,]$ is a rational function of θ;
- $(N_\mathcal{X}, \{d\}_\mathcal{X})$, $(N_\mathcal{Y}, \{d\}_\mathcal{Y})$, $(N_\mathcal{V}, \{d\}_\mathcal{V})$ where $\mathcal{V}(\theta) = \begin{bmatrix}\mathcal{A}(\theta) & \mathcal{B}(\theta) \\ \mathcal{C}(\theta) & \mathcal{D}(\theta)\end{bmatrix}$

compute $\gamma_g = min\ \gamma$ such that there exist

- symmetric matrices $\mathcal{X}(\theta) \in \Phi_{N_\mathcal{X},\{d\}_\mathcal{X}}$, $\mathcal{Y}(\theta) \in \Phi_{N_\mathcal{Y},\{d\}_\mathcal{Y}}$;
- matrice $\mathcal{V}(\theta) \in \Phi_{N_\mathcal{V},\{d\}_\mathcal{V}}$

satisfying (5) and (6).

Since the conservatism only arises from the restriction of decision variables to finite subset of functions, a set of rational functions of sufficient dimension N allows to decrease the conservatism of Theorem 4.1. Moreover, the size of the LMIs (15) and (16) grows linearly with N while the number of decision variables grows quadratically with it. Reducing the conservatism of Theorem 4.1 does not cause an explosion in the dimension of the optimization problem.

4.3 Numerical example

Problem definition Let us consider the example presented in section 2.2. In contrast with the previous approach, rational interpolation allows to obtain a nearly constant value of γ_{θ_i} for each θ_i. This is obtain under the use of a third pair W_1 and W_2 for $\theta_i = 0.6$: $W_1(s) = 0.5\frac{s+1.73}{s+0.0017}$, $W_2(s) = 500\frac{s+1}{s+1580}$. The least square approximation gives: $C_{W_1}(\theta) = 0.33 + \frac{0.33\ \theta}{1-0.67\ \theta}$ and $[C_{W_2}(\theta)\ D_{W_2}(\theta)] = [-7170\ 1800] + \frac{\theta}{1+1.17\ \theta}[14670\ {-3680}]$.

With these weighting functions, by solving H_∞ standard problem for given values $\theta_i \in [0,1]$, the obtained value of γ_{θ_i} is close to 1: between 1.02 for $\theta = 0.3$ and 1.045 (lower bound of γ_{opt}) for $\theta = 0.9$, with a variation of 2.5%. $P(s,\theta)$ can be obtained where the data matrix belongs to $\Phi_{2,\{0.5,-0.78\}}$. We use the same set of functions for the decision variables.

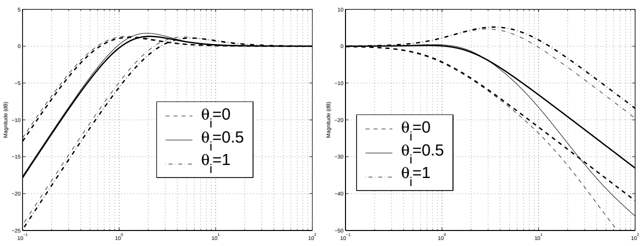

Figure 4: $|S|$ and $|KS|$, $K \in \{K(s,\theta_i), \theta_i \in \{0, 0.5, 1\}\}$

Results analysis The computation of Theorem 4.1 solution leads to $\gamma_g = 1.12$. The conservatism is low as the difference between this upper bound and the lower bound is only 7%.
Let us focus on the obtained closed-loop transfert functions. $|S| = \frac{1}{|1+GK|}$ and $|KS|$ are represented figure 4 with $K \in \{K(s,\theta_i),\ \theta_i \in \{0, 0.5, 1\}\}$. In contrast with the controller obtained section 3, the bandwidth of S is multiplied by 4 between extreme values of θ, as with the pointwise controllers. Our approach is thus weakly conservative.

Bode diagrams of $K(s,\theta_i)$ with $\theta_i \in \{0, 0.5, 1\}\}$ and the corresponding pointwise controllers $K_{\theta_i}(s)$ are presented in figure 5. The transient responses obtained with these controllers are

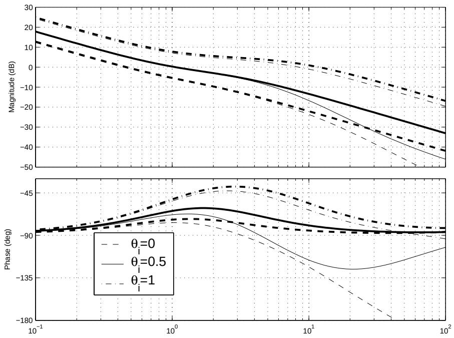

Figure 5: Bode diagrams of $K(s, \theta_i)$ and $K_{\theta_i}(s)$, $\theta_i \in \{0, 0.5, 1\}$

presented figure 6. Note that, in both figures, thick curves refer to $K_{\theta_i}(s)$ and thin curves to $K(s, \theta)$. Note that both sets of curves are close.

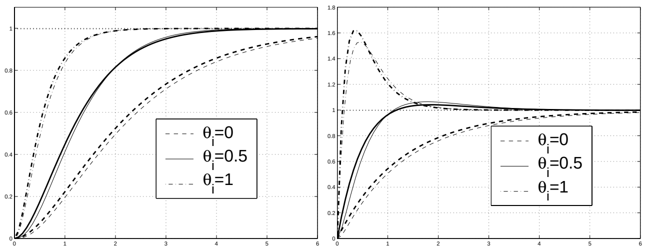

Figure 6: Output (left) and u (right) for $\theta_i \in \{0, 0.5, 1\}$

Quadratic parameter independent Lyapunov approach Our approach proposed in Theorem 4.1 is much more demanding than an approach with a θ independent Lyapunov function ($\mathcal{X}(\theta) = \mathcal{X}_0$ and $\mathcal{Y}(\theta) = \mathcal{Y}_0$). Thus, it is important to assess the benefits (if any) of our approach with respect to a θ independent Lyapunov function approach. Let us compute the solution of Theorem 4.1 when the Lyapunov function is independent of θ and $\mathcal{V}(\theta)$ is still a rational function of θ, as before. The most interesting is that, practically, the obtained $K(s, \theta)$ is independent of θ (see figure 7).

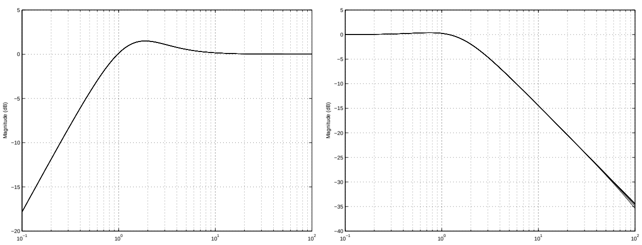

Figure 7: $|S|$ and $|KS|$, $K \in \{K(s, \theta_i)$ by step of 0.1 in $\theta_i\}$

References

[1] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled H_∞ controllers. *IEEE Trans. Aut. Control*, 40(5):853–864, May 1995.

[2] P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled H_∞ control of linear parameter-varying systems: a design example. *Automatica*, 31(9):1251–1261, September 1995.

[3] G. Becker, A. Packard, D. Philbrick, and G. Balas. Control of parametrically-dependent linear systems: a single quadratic lyapunov approach. In *Proc. American Control Conf.*, pages 2795–2799, San Fransisco, California, USA, June 1993.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Appllied Mathematics*. SIAM, Philadelphia, USA, June 1994.

[5] M. Dinh, G. Scorletti, V. Fromion, and E. Magarotto. Parameterized H_∞ controller design for adaptative trade-off by finite dimensional lmi optimization. Technical report, L.A.P. Caen, France, http://www.greyc.ismra.fr/EquipeAuto/Gerard_S/robust.html, 2003.

[6] E. Feron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems via parameter-dependent lyapunov functions. *IEEE Trans. Aut. Control*, 41(7):1041–1046, July 1996.

[7] I. Fialho and G. Balas. Road adaptive active suspension design using linear parameter-varying gain-scheduling. *IEEE Trans. on Control Systems Technology*, 10(1):43–54, January 2002.

[8] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *Int. J. Robust and Nonlinear Control*, 4(4):421–448, October 1994.

[9] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter-dependent lyapunov functions and real parametric uncertainty. *IEEE Trans. Aut. Control*, 41(3):436–442, March 1996.

[10] F. Garofalo, G. Celentano, and L. Glielmo. Stability robustness of interval matrices via Lyapunov quadratic forms. In *IEEE Trans. Aut. Control*, volume 38, pages 281–284, February 1993.

[11] K. Glover and J.C. Doyle. State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity. *Syst. Control Letters*, 11:167–172, 1988.

[12] A. Helmersson. μ synthesis and LFT gain scheduling with mixed uncertainties. In *Proc. European Control Conf.*, pages 153–158, Roma, Italy, July 1995.

[13] A. Packard. Gain scheduling via linear fractional transformations. *Syst. Control Letters*, 22:79–92, 1994.

[14] L. Rossignol, G. Scorletti, and V. Fromion. Filter design and related problems: a finite dimensional convex optimization approach. *Int. J. Robust and Nonlinear Control*, to appear, 2003.

[15] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective Output-Feedback control via LMI optimization. *IEEE Trans. Aut. Control*, 42(7):896–911, July 1997.

[16] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 73:361–375, 2001.

[17] G. Scorletti and L. El Ghaoui. Improved LMI conditions for gain scheduling and related problems. *Int. J. Robust and Nonlinear Control*, 8(10):845–877, August 1998.

[18] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control, Analysis and Design*. John Wiley & Sons, Chichester, England, 1996.

[19] F. Wu, X.H. Yang, A. Packard, and G. Becker. Induced l_2-norm control for LPV systems with bounded parameter variations rates. In *Proc. American Control Conf.*, pages 2379–2383, Seattle, Washington, USA, June 1995.

[20] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Aut. Control*, AC-26(2):301–320, April 1981.

[21] K. Zhou, J.C. Doyle, and K. Glover. *Robust and optimal control*. Prentice Hall, New Jersey, 1995.