FREQUENCY LOCALISING BASIS FUNCTIONS FOR WIDE-BAND IDENTIFICATION

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Abstract

A well known difficulty in system identification over a large bandwidth is the ill-conditioning of the normal matrix. This typically manifests itself as poor or erroneous estimates. Several methods have been proposed in the literature for addressing this issue. However, none appear to give an entirely satisfactory solution. Here we present a novel technique, utilising particular basis functions, aimed specifically at improving the numerical properties of the least squares normal matrix in transfer function estimation over a wide bandwidth. We show that, under some mild assumptions, the achieved condition number is actually independent of the frequency range. Several examples are presented showing the superior performance of the proposed method when applied to wide-band estimation problems.

1 Introduction

A linear single input single output (SISO) dynamic system can be characterised by the ratio of two polynomials. In system identification there is often a need, or desire, to estimate the coefficients of these polynomials from experimentally obtained data.

One can either perform the estimation in the time or frequency domain [9; 10]. Frequency domain identification is usually preferred over that of the time domain when the excitation signal is periodic. A number of advantages associated with frequency domain identification have been outlined in the literature [8; 14]. These include, the ease of noise reduction (only frequencies where excitation is provided are used in the estimation procedure), data reduction (through the use of a non-parametric model of the system in the frequency domain), the ease of combining data from different experiments and model validation (periodic excitation provides a good frequency response model at the excitation frequencies). Also, modelling of continuous time systems is much easier in the frequency domain.

Here we will use the continuous case for ease of presentation. However, it is well known [11] that discrete systems approach this continuous time format as the sampling rate increases.

We will model the system by a transfer function, $G(s)$, as follows:

$$G(s) = \frac{B(s)}{A(s)} \tag{1}$$

where the polynomials are defined by

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \tag{2}$$
$$A(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \tag{3}$$

The estimation problem of interest can be stated as the problem of minimising a cost function of the following generic form:

$$J = \frac{1}{T} \int_0^T e(t)^2 dt \tag{4}$$

where $e(t)$ denotes a suitably defined prediction error. For example, we could utilise an output error function where

$$e(t) = y(t) - \hat{y}(t) \tag{5}$$

and $\hat{y}(t)$ denotes the output of a model having transfer function

$$\hat{G} = \frac{\hat{B}(s)}{\hat{A}(s)} \tag{6}$$

where the super-hat indicates that the polynomials are evaluated at some estimate of the parameters i.e.

$$\hat{\theta} = [\hat{a}_0,\ \ldots,\ \hat{a}_{n-1}, \hat{b}_0,\ \ldots,\ \hat{b}_m] \tag{7}$$

Using Parseval's Theorem, Equation (4) can also be expressed (asymptotically) in the frequency domain as

$$J = \int_0^\infty |e(j\omega)|^2 \, d\omega \tag{8}$$

When the input comprises a finite set of sinusoidal components (say $\omega_1, \ldots, \omega_N$) then Equation (8) becomes

$$J = \sum_{k=1}^{N} |e(j\omega_k)|^2 \tag{9}$$

In the output error case, Equation (9) can also be written

$$J = \sum_{k=1}^{N} \left| \hat{H}(j\omega_k) - \hat{G}(j\omega_k) \right|^2 \phi_{uu}(\omega_k) \tag{10}$$

where $\hat{H}(j\omega_k)$ denotes the empirical transfer function estimate at ω_k and $\phi_{uu}(\omega_k)$ denotes the input power at frequency ω_k.

As the cost function, Equation (10), is nonlinear in the parameters, to apply linear least squares to estimate the parameters of the polynomials, it was suggested [7] that the problem be linearised by multiplying through by the denominator $\hat{A}(j\omega_k)$. Equation (10) then becomes

$$J = \sum_{k=1}^{N} \left| \hat{A}(j\omega_k)\hat{H}(j\omega_k) - \hat{B}(j\omega_k) \right|^2 \frac{\phi_{uu}(\omega_k)}{\left|\hat{A}(j\omega_k)\right|^2} \tag{11}$$

If we replace $\hat{A}(j\omega_k)$, in the denominator, by an estimate $E(j\omega_k)$, then we obtain the following quadratic criterion

$$\acute{J} = \sum_{k=1}^{N} \left| \hat{A}(j\omega_k)\hat{H}(j\omega_k) - \hat{B}(j\omega_k) \right|^2 \frac{\phi_{uu}(\omega_k)}{|E(j\omega_k)|^2} \tag{12}$$

In early literature, the polynomial $E(j\omega_k)$ was taken as 1. However, this technique exhibits a number of undesirable properties. Firstly, high frequency errors are inherently given more weighting which correspondingly yields poor results at low frequency. Secondly, the normal matrix formed using this method is susceptible to extreme ill-conditioning. This ill-conditioning arises due to the large dynamic range of the entries in the normal matrix as well it is highly correlated since every frequency will influence every coefficient that is being estimated. If, for instance, the frequency range spans 3 decades and the system is of $5th$ order then the dynamic range will be, at a minimum, 10^{15} : 1.

An iterative method was proposed [15] to overcome the poor low frequency fit by applying a weighting of $1/|\hat{A}'(j\omega_k)|^2$ which is based on the estimate from the previous iteration. However, this did not address the ill-conditioning of the normal matrix.

Frequency scaling [3; 14] is a means of improving the numerical stability of the normal matrix. One common approach is to scale the frequency axis by the arithmetic mean of the minimum and maximum frequencies of interest, i.e.

$$\omega_{scale} = \frac{\omega_{min} + \omega_{max}}{2} \tag{13}$$

The use of orthogonal polynomials [1; 3] have been used to further improve the numerical properties of the normal matrix. Specifically, the above authors discuss the use of Tchebychev polynomials in the re-parameterisation of the model, Equation (1), such that

$$B(j\omega) = \sum_{k=0}^{m} b_k q^k(j\omega) \tag{14}$$

$$A(j\omega) = 1 + \sum_{k=1}^{n} a_k q^{k-1}(j\omega) \tag{15}$$

where q^k are kth order modified Tchebychev polynomials.

Interest has also focussed on re-parameterising the model using orthonormal basis functions [16; 17; 2; 4]. For example, the following basis function was proposed [2]

$$F_k(s) \triangleq \frac{\sqrt{2 * Re\{p_k\}}}{s + p_k}\varphi_{n-1}(s) \,, \tag{16}$$

$$\varphi_n(s) \triangleq \prod_{l=1}^{k} \frac{s - \bar{p}_l}{s + p_l} \,, \varphi_0(s) \triangleq 1 \,. \tag{17}$$

The model, given in Equation (1), is then re-parameterised as

$$\frac{B(s)}{E(s)} = b_0 + \sum_{k=1}^{n} b_k F_k(s) \tag{18}$$

$$\frac{A(s)}{E(s)} = 1 + \sum_{k=1}^{n} a_k F_k(s) \tag{19}$$

where $E(s) = \prod_{l=1}^{n}(s + p_l)$

The basis functions formed in Equations (16) and (17) when $p_k = p \in \mathbb{R}$ correspond to the well known Laguerre functions [16]. These orthonormal basis functions allow for perfect conditioning of the normal matrix for a white input spectrum. However, it has also been shown [13] that these basis functions exhibit some degree of robustness with respect to spectral colouring of the input.

In general when solving the least squares problem, the normal matrix should not be explicitly formed. There exists a number of techniques that offer methods to solve the least squares problem without explicitly forming and inverting the normal matrix [5]. These include Cholesky Factorisation, Househoulder Transformation and QR Factorisation. The above methods work towards constructing a matrix factorisation of the least squares normal equation from orthogonal components involving numerically robust equations. These methods amount to improving the conditioning by taking the square root of the regression matrix. Clearly this improves the numerical properties relative to brute force approaches. Nonetheless, if the condition number of the regression matrix is very poor, then these methods will not resolve the problem.

For clarity, in this paper we use frequency response data to estimate the two frequency dependant polynomials which characterise a dynamic linear SISO system. However, due to the Parseval relationship between time and frequency domain, the same general conclusions will hold for time domain methods. We will also utilise a form of estimation with quadratic cost as in Equation (12). However, readers who are familiar with this field will appreciate that similar qualitative conclusions hold for various iterative algorithms where $E(j\omega_k)$ is based on past estimates.

A key point in our approach is that our method restricts the dynamic range over which each coefficient is estimated by the use of frequency localising basis functions (FLBF's) which span the desired frequency region. This also reduces the amount of correlation in the normal matrix. By numerical examples we show how this method is superior to others outlined above for a system with a large dynamic range.

The layout of the remainder of the paper is as follows: In Section 2 we outline the problem formulation. Section 3 describes the frequency localising basis functions (FLBF's) and how they are utilised in the least squares problem. We then present a result on the condition number for the normal matrix which utilises the FLBF's in Section 4. Simulation results are presented in Section 5. An example involving real data is discussed in Section 6. Conclusions are drawn in Section 7.

2 Frequency Domain Estimation

We consider the least squares form of the estimation problem as in Equation (12). This cost function can be written in several equivalent fashions, e.g.

$$\acute{J} = \sum_{k=1}^{N} \left| \frac{\hat{A}(j\omega_k)}{E(j\omega_k)} Y(j\omega_k) - \frac{\hat{B}(j\omega_k)}{E(j\omega_k)} U(j\omega_k) \right|^2 \tag{20}$$

where $\{Y(j\omega_k)\}$, $\{U(j\omega_k)\}$ denotes the transform of the output and input data respectively. We then expand the quantities $\frac{\hat{A}(s)}{E(s)}$ and $\frac{\hat{B}(s)}{E(s)}$ in terms of appropriate basis functions as

$$\frac{\hat{A}(s)}{E(s)} = 1 + \sum_{k=1}^{n} \alpha_k F_k(s) \tag{21}$$

$$\frac{\hat{B}(s)}{E(s)} = \sum_{k=1}^{n} \beta_k F_k(s) \tag{22}$$

Substituting Equations (21) and (22) into Equation (20) leads to a quadratic function in the unknown parameters $\theta = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. The least squares solution is given by

$$\hat{\theta} = [X^* X]^{-1} [X^* Y] \tag{23}$$

where * denotes complex conjugate transpose,

$$\hat{\theta} = \left[\hat{\alpha}_1,\ \ldots\ , \hat{\alpha}_n,\ \hat{\beta}_1,\ \ldots,\ \hat{\beta}_n \right]^T \tag{24}$$

$$X = \begin{bmatrix} -Y_{\omega_1}^{f_1} & \cdots & -Y_{\omega_1}^{f_n} & U_{\omega_1}^{f_1} & \cdots & U_{\omega_1}^{f_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -Y_{\omega_N}^{f_1} & \cdots & -Y_{\omega_N}^{f_n} & U_{\omega_N}^{f_1} & \cdots & U_{\omega_N}^{f_n} \end{bmatrix} \tag{25}$$

$$Y = [Y(j\omega_1),\ \ldots\ ,\ Y(j\omega_N)]^T \tag{26}$$

$$Y_{\omega_k}^{f_l} = F_l(j\omega_k) Y(j\omega_k) \ \text{ and } \ U_{\omega_k}^{f_l} = F_l(j\omega_k) U(j\omega_k).$$

The normal matrix can be identified in Equation (23) as $[X^* X]$. If the normal matrix is 'near singular' then the solution for the least squares method becomes ill-conditioned. The main goal of the current paper is to study the impact of using different choices for the basis functions $\{F_l(s)\}$ on the conditioning of the least squares problem. (Of course, the parameter vector θ will be different for each choice of basis functions).

In all cases, we utilise the Householder transformation to improve the numerical properties of the estimates to ensure that each method is considered on an equal footing. The least squares solution can then be expressed as:

$$\hat{\theta} = R^{-1} \eta_1 \tag{27}$$

where R is an upper triangular matrix such that

$$\Psi X = \begin{bmatrix} R \\ 0 \end{bmatrix} \ \text{ and } \ \Psi Y = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

Here Ψ is an orthonormal matrix constructed by a product of Householder transformations [6].

3 Frequency Localising Basis Functions

The traditional choices of basis functions described in the literature have the property that they are exactly orthogonal for a moving average (MA) type model ($\alpha_i = 0$ for $i = 1, \ldots, n$) and a white noise input, or more simply, they form an orthonormal set in L_2. By way of contrast, we propose a set of basis functions, which are not exactly orthogonal for any standard input signal but which are "nearly orthogonal" for a wide range of inputs. Thus we trade an exact property for an approximate property with the aim of achieving robustness. This line of reasoning mirrors the usual trade-off that exists between performance (under some nominal conditions) versus robustness (under other conditions).

The basis functions that we propose have the following form:

$$F_k(s) = \prod_{l=1}^{k} \frac{s^{k-1} p_k}{s + p_l}; \qquad k = 1, \ldots, n \tag{28}$$

Note that these filters have approximately 0dB gain in the range (p_{k-1}, p_k) for basis function F_k. Outside of this range the gain decreases by, at least 20dB, per decade. The poles are usually chosen such that the test signal frequencies correspond to the center of the filter passbands. Thus we parameterise the system via

$$\frac{A(s)}{E(s)} = 1 + \sum_{l=1}^{n} a_{l-1}^{\perp} F_l(s) \tag{29}$$

$$\frac{B(s)}{E(s)} = \sum_{l=1}^{n} b_{l-1}^{\perp} F_l(s) \tag{30}$$

$$\text{where} \qquad E(s) = \prod_{l=1}^{n} (s + p_l) \tag{31}$$

$$= s^n + e_{n-1} s^{n-1} + \cdots + e_0 \tag{32}$$

It is clearly seen that the form of parameterisation is of the same nature as that of other methods discussed in Section 1.

The key point regarding these filters is that they are non-zero essentially only for a small set of frequencies that lie in their pass-band. Hence the least squares normal matrix will take on a near block diagonal form.

The model parameterisation can be easily expressed in terms of the original polynomial by using the following transformation.

Let

$$M^l(s) = F_l(s)E(s) \tag{33}$$
$$= m^l_{n-1}s^{n-1} + \ldots + m^l_{l-1}s^{l-1}; \quad l = 1, \ldots, n$$

where l represents the lth basis function, then

$$\mathbf{a} = M\mathbf{a}^{\perp} + \mathbf{e} \tag{34}$$

where $\mathbf{a}$ and $\mathbf{e}$ are the parameter vectors of $A(s)$ and $E(s)$ respectively, $\mathbf{a}^{\perp}$ is the vector of parameters in Equation (29) and

$$M = \begin{bmatrix} m^1_0 & 0 & \ldots & 0 \\ m^1_1 & m^2_1 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ m^1_{n-1} & m^2_{n-1} & \ldots & m^n_{n-1} \end{bmatrix} \tag{35}$$

Remark: The basis function representation given in Equations (29) and (30) is specifically for a strictly proper model, $G(s)$. If G(s) is bi-proper then the re-parameterisation for $A(s)$ remains the same, however re-parameterisation for $B(s)$ becomes:

$$\frac{B(s)}{E(s)} = \sum_{l=1}^{n} b^{\perp}_{l-1}F_l(s) + \frac{b^{\perp}_n s}{s + p_n} \tag{36}$$

where p_n is the high frequency breakpoint.

4 Condition Number

To give further credibility to the FLBF's described above, we next explore the condition number of the normal matrix for a special case. In particular, we assume:

A.1 That an MA type structure is used (i.e. $a^{\perp}_k = 0; \quad k = 1, \ldots, n$)

A.2 The excitation signal consists of sine waves of unit strength at frequencies $(\omega_1, \omega_2, \ldots\omega_n)$ at the geometric mean of the pass-band for each basis function

A.3 The breakpoints for the basis function are logarithmically spaced such that

$$p_{l+1} = \gamma p_l \quad \text{where } \gamma > 1 \tag{37}$$

Now the model, re-parameterised by the frequency localising basis functions, can be expressed for a single frequency as

$$Y(j\omega) = \begin{bmatrix} F_1(\omega) & \ldots & F_m(\omega) \end{bmatrix} \theta U(j\omega)$$

hence the least squares normal matrix becomes

$$\mathbf{X} = [F^*U^*FU]$$
$$= \sum_{k=1}^{N} \begin{bmatrix} F^*_1(\omega_k) \\ \vdots \\ F^*_n(\omega_k) \end{bmatrix} \begin{bmatrix} F_1(\omega_k) & \ldots & F_n(\omega_k) \end{bmatrix} |U(j\omega_k)|^2 \tag{38}$$

where $|U(j\omega_k)|^2 = 1$ from (A.2).

Now we denote by $\bar{h}_{ij}$ the gain applied to the component at frequency ω_i by filter F_j. Then the normal matrix can be expressed as

$$\mathbf{X} = \begin{bmatrix} \sum_{i=1}^{N} \bar{h}^2_{i1} & \sum_{i=1}^{N} \bar{h}_{i1}\bar{h}_{i2} & \ldots & \sum_{i=1}^{N} \bar{h}_{i1}\bar{h}_{iN} \\ \vdots & \sum_{i=1}^{N} \bar{h}^2_{i2} & \ldots & \sum_{i=1}^{N} \bar{h}_{i2}\bar{h}_{iN} \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{N} \bar{h}_{iN}\bar{h}_{i1} & \ldots & \ldots & \sum_{i=1}^{N} \bar{h}^2_{iN} \end{bmatrix} \tag{39}$$

We consider here a worst case i.e. when the filters roll off at 20dB / decade. (Note that the filters in Equation (28) actually roll off at a greater rate on the down frequency side). Under these conditions $\bar{h}_{ij}$ depends only on the absolute value of $i-j$. Hence let

$$h_k \triangleq \bar{h}_{i,i+k} \quad \text{for all } k. \tag{40}$$

We then have

Theorem 1
Provided assumptions (A.1) to (A.3) hold and the filter spacing is chosen such that

$$h_0 \geq 2\sum_{k=1}^{\infty} |h_k| \quad ^1 \tag{41}$$

Then the condition number of the least squares normal matrix, $\mathbf{X}$, formed using frequency localising basis functions is bounded independent of the dynamic frequency range of the system.

Proof: The normal matrix $\mathbf{X}$ can be expressed as:

$$\mathbf{X} = \mathbb{S}\,\mathbb{S} \tag{42}$$

where

$$\mathbb{S} = \begin{bmatrix} h_0 & h_1 & \ldots & h_{N-1} \\ h_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ h_{N-1} & & \ldots & h_0 \end{bmatrix} \tag{43}$$

which is symmetric. Hence the eigenvalues of $\mathbf{X}$ are the square of the eigenvalues of $\mathbb{S}$.

Now from Gershgorin's Theorem [5], the eigenvalues of $\mathbb{S}$ lie in the union of circles with centres being the diagonal elements and radii the sum of the modulus of the off diagonal terms.

It then follows that

$$\lambda_{max}(\mathbb{S}) \leq h_0 + 2\sum_{k=1}^{\infty} |h_k| \tag{44}$$

$$\lambda_{min}(\mathbb{S}) \leq h_0 - 2\sum_{k=1}^{\infty} |h_k| \tag{45}$$

[1] For an infinite number of filters this condition is satisfied when $\gamma = 4$.

Due to assumption (A.3),

$$\lambda_{max}(\mathbb{S}) < \infty \tag{46}$$

Also, Equation (41) implies

$$\lambda_{min}(\mathbb{S}) > 0. \tag{47}$$

This concludes the proof. □

5 Simulations

To illustrate the performance of the FLBF's we choose a highly resonant system of large order that spans several decades of frequency. The system is described by

$$G_k(s) = \sum_{k=1}^{9} \frac{b_k \omega_k^2}{(s^2 + 2\zeta_k \omega_k s + \omega_k^2)} \tag{48}$$

where $\{\omega_k\}$ spans 9 decades. A bode magnitude plot of the true system is shown in Figures 1 and 2 as a dashed line. The frequency response for the system was evaluated at 36 logarithmically spaced points that span the entire frequency range of the system. No noise was added to the frequency response data of the system for this comparison. The model was then estimated using the following methods:

i. Nonlinear Least Squares (NLS)

ii. Nonlinear Least Squares using Frequency Scaling as in Equation (13) (NLSFS)

iii. Tchebyshev Polynomials (TP)

iv. Laguerre Basis Functions with the poles chosen by a discretised search between the minimum and maximum frequency, so as to give the best fit, according to the criteria in Table 1 (LBF)

v. FLBF's with break points chosen logarithmically spaced between minimum and maximum frequency. (FLBF)

It is also noted that Kautz basis functions where evaluated, but given the low number of excitation signals the authors were unable to obtain a satisfactory fit.

Estimation was carried out based on measured frequency response data using criterion (12) taking $\phi_{uu}(j\omega_k) = 1 \quad \forall k$.

Table 1 compares the condition number of the normal matrix for the above choices. Note that the condition number for all the choices save for the FLBF's is very poor indeed. The table also compares the mean square errors and maximum error between the estimated model and the true model at the frequencies of excitation. It is observed that the FLBF method provides the superior model fit.

Figures 1 and 2 compare the magnitudes of the estimated frequency response with that of the true system. The true system is plotted as a dashed line, the estimates as a solid line.

Method	Cond. #	L_2 Cost	L_∞ Cost
NLS	5.8950e+175	2.7292e+003	105.2499
NLSFS	1.0642e+044	848.2818	68.5336
TP	6.0597e+022	898.3536	68.2660
LBF	8.0111e+017	69.5152	37.9211
FLBF	4059	9.5976e-023	5.3152e-011

Table 1: Comparisons of Different Methods

Remark: It can be seen from Figures 2 that the FLBF's are the only procedure to give an acceptable fit to the data over the full frequency range. The authors acknowledge that it may well be possible to choose the various "free parameters" in the other methods to obtain a good fit. Some steps were taken to achieve this but clearly we cannot claim to have exhausted all possibilities. However, it does seem that the FLBF's are particularly easy to "tune" and give excellent results (at least for this example).

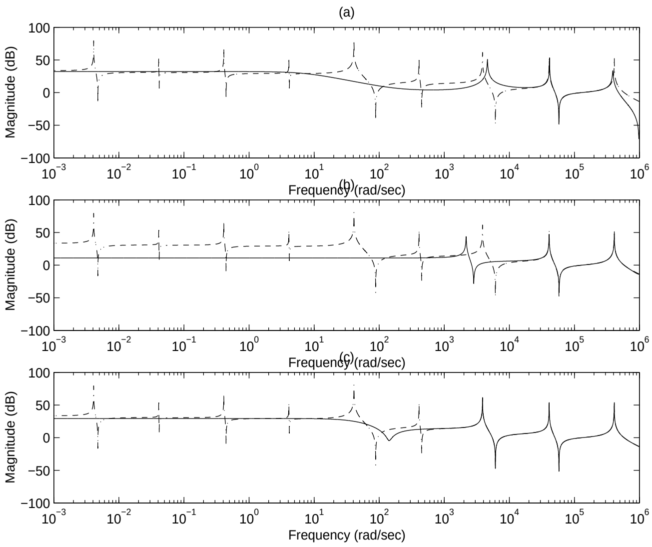

Figure 1: Magnitude plots of estimates using (a) Nonlinear least squares, (b) Nonlinear least squares with frequency scaling, (c) Tchebyshev polynomials. The true system appears in all plots as a dashed line.

6 Real Data from a Resonant Beam

Finally, we present estimation results for real frequency response data collected from a resonant beam [12]. The experimental data spans approximately 2 decades and is shown in Figure 3 by the dots. Also shown in Figure 3 is the estimated frequency response obtained by applying the FLBF method. Again it would seem that an excellent fit has been obtained.

Remark: Again the authors hasten to add that other researchers have been able to obtain excellent fits to the frequency domain data by other methods. We are only able to

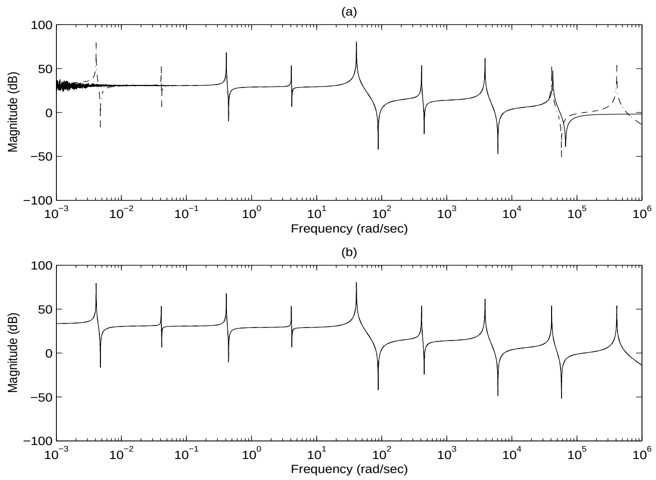

Figure 2: Magnitude plots of estimates using (a) Laguerre basis functions, (b) Frequency Localising Basis Functions. The true system appears in all plots as a dashed line.

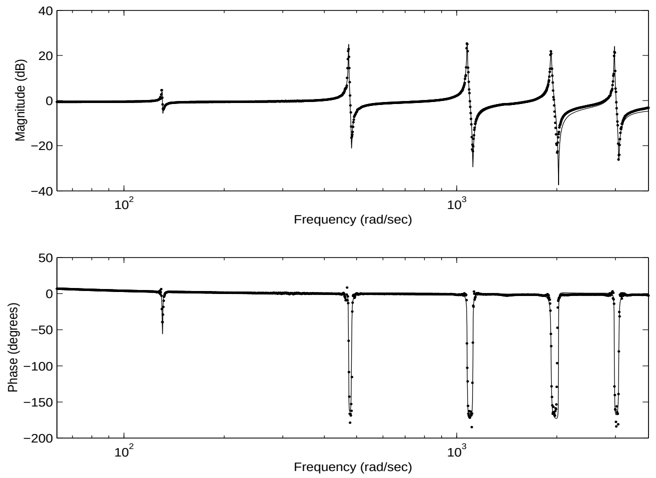

Figure 3: Resonant Beam Data. Dots - Experimental Data, Solid - Estimated Model using FLBF's

state that the FLBF method was again easy to tune (with break frequencies being logarithmically spaced) and appears to give an excellent result.

7 Conclusion

We have described in this paper **F**requency **L**ocalising **B**asis **F**unctions (FLBF's) aimed at improving the numerical conditioning of the normal matrix in least squares estimation. Specifically we have shown for a MA model type structure the condition number of the normal matrix is independent of the dynamic frequency range. We have also compared a number of different methods using a simulated example and real data from a resonant beam.

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