DEVELOPMENT OF A SKEW μ UPPER BOUND

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Abstract

Exploitation of the NP hard, mixed μ problem structure provides a polynomial time algorithm that approximates μ with usually reasonable answers. When the problem is extended to the skew μ problem an extension of the existing method to the skew μ formulation is required. The focus of this paper is to extend the μ upper bound derivation to the skew μ upper bound and show its direct computation by way of a generalized eigenvalue solver and an LMI algorithm.

1 Introduction

μ analysis provides a general framework for robust analysis in the face of system/model uncertainties. However, in practice skewed μ problems commonly occur, perhaps even more so than μ problems. The most common example of a skew μ problem is the formulation for robust performance (see [10] among others). Ranges for uncertainties are chosen for a reason. Typically one does not want these uncertainties re-scaled in an analysis of system performance. The robust performance question that needs to be answered is "For uncertainties with these ranges, what level of performance can be expected from the system?". This is a skew μ problem. The physical uncertainties have a fixed range, and the performance block is allowed to vary to determine the point where the system could potentially become unstable.

Another common example of skew μ is stability analysis over a given frequency range. By reformulation of the problem using frequency as a perturbation variable, the gridded μ problem becomes a skew μ problem where no frequencies are missed in the search [9].

Both these skew μ problems can be approached as iterative μ problems in order to achieve a value of skew μ, however, the process is much slower than the results presented here. Where the speed of the computational algorithm provided here is on the order of the μ upper bound algorithm, an iterative μ procedure could easily be an order of magnitude slower.

The goal of this paper is to extend the theoretical concepts of a upper bound for skew μ and provide an outline of procedures for the calculation of an upper bound for skew μ.

1.1 Finding Skew μ

Systems in robust analysis are often modeled using the LFT (Linear Fractional Transform) as seen in figure 1. In the LFT, unknown (or uncompensated) dynamics and parameter variations are modeled by Δ, where Δ includes block structure requirements. The stability measure for an LFT is known as the SSV (Structured Singular Value) or μ, and is defined in definition (1). Values of $\mu(M) \geq 1$ provide the possibility of becoming unstable, while values of $\mu(M) < 1$ are stable. Because

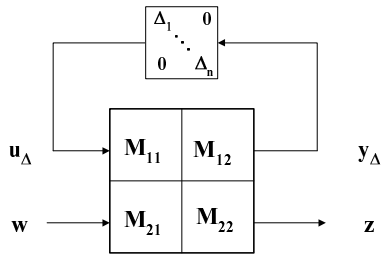

Figure 1: Basic LFT System Diagram

finding μ is extremely computationally intensive, bounds on μ are normally calculated, rather than calculating μ directly. Skew μ inherits most of the properties of μ, including the property related to computational difficulties. This necessitates the development of a skew μ upper bound similar to the μ upper bound.

2 Notation and Preliminaries

The notation used to provide mathematical descriptions is fairly standard and is essentially taken from Fan et. al. [2], in addition, the support basis for skew μ development comes from Young et. al. [11].

For any square complex matrix M, denote the complex conjugate transpose by M^H. The largest singular value and the structured singular value are denoted by $\overline{\sigma}(M)$ and $\mu_{\mathcal{K}}(M)$ respectively. The spectral radius is denoted $\rho(M)$ and the real spectral radius $\rho_R(M) = max\{|\lambda| : \lambda \; is \; a \; real \; eigenvalue \; of \; M\}$, with $\rho_R(M) = 0$ if M has no real eigenvalues. For any complex vec-

tor x, then x^H denotes the complex conjugate transpose, $\|x\|_2$ the Euclidean norm, and $\|x\|_\infty$ the infinity norm.

The definition of μ and skew μ are dependent upon the underlying block structure of the uncertainties, which is defined as follows. Given a matrix $M \in \mathbb{C}^{n\times n}$ and three non-negative integers m_r, m_c, and m_C with $m \triangleq m_r + m_c + m_C \leq n$, the block structure $\mathcal{K}(m_r, m_c, m_C)$ is an m-tuple of positive integers

$$\mathcal{K} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}, k_{m_r+m_c+1}, \ldots, k_m) \quad (1)$$

where the requirement is $\sum_{i=1}^{m} k_i = n$ in order that these dimensions are compatible with M. This determines the set of allowable perturbations, which are defined as

$$\begin{aligned} X_{\mathcal{K}} = \big\{\Delta = block\ diag(&\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \\ &\delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C) : \\ &\delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta_i^C \in \mathbb{C}^{k_{m_r+m_c+i}\times k_{m_r+m_c+i}}\big\}. \end{aligned} \quad (2)$$

Note that $X_{\mathcal{K}} \in \mathbb{C}^{n\times n}$ and that this block structure is sufficiently general to allow for repeated real scalars, repeated complex scalars, and full complex blocks. The purely complex case corresponds to $m_r = 0$, and the purely real case to $m_c = m_C = 0$.

Additionally note that all the results which follow are easily generalized to the case where the full complex blocks need not be square, and the blocks may come in any order. These restrictions in equation (2) are made purely for notational convenience.

In order to more easily generalize the case of the μ upper bound to the skew μ case, the formal definition of μ is given as:

Definition 1 *The structured singular value $\mu(M)$ of a matrix $M \in \mathbb{C}^{n\times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is defined as:*

$$\mu_{\mathcal{K}}(M) = \frac{1}{\min\limits_{\Delta\in X_{\mathcal{K}}} \{\overline{\sigma}(\Delta) |\ det(I - \Delta M) = 0\, for\ structured\ \Delta\}} \quad (3)$$

with $\mu_{\mathcal{K}}(M) = 0$ if no $\Delta \in X_{\mathcal{K}}$ solves $det(I - \Delta M) = 0$.

This definition and associated block structure format is well known. Using the general structure of the definition of μ, a definition for skew μ can now be created.

2.1 Scaling Set Notation

In order to develop the skew μ upper bound, some sets of block diagonal scaling matrices will need to be used. These matrices are defined here for easy reference, and are used throughout the remainder of this paper. These scaling sets are also dependent on the underlying block structure ($\mathcal{K}(m_r, m_c, m_C)$), and similar to those used in [12].

$$\begin{aligned} \mathcal{D}_{\mathcal{K}} = \big\{&block\ diag(D_1, \ldots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}} \ldots, d_{m_C} I_{k_m}) \\ &: \det(D_i) \neq 0, D_i \in \mathbb{C}^{k_i\times k_i}, d_i \neq 0, d_i \in \mathbb{C}\big\}. \end{aligned} \quad (4)$$

$$\begin{aligned} \hat{\mathcal{D}}_{\mathcal{K}} = \big\{&block\ diag(D_1, \ldots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}} \ldots, d_{m_C} I_{k_m}) \\ &: 0 < D_i = D_i^H \in \mathbb{C}^{k_i\times k_i}, 0 < d_i \in \mathbb{R}\big\}. \end{aligned} \quad (5)$$

$$\begin{aligned} \mathcal{G}_{\mathcal{K}} = \Big\{&block\ diag(G_1, \ldots, G_{m_r}, 0_{k_{m_{r+1}}}, \ldots, 0_{k_m}) : \\ &G_i = G_i^H \in \mathbb{C}^{k_i\times k_i}\Big\}. \end{aligned} \quad (6)$$

$$\hat{\mathcal{G}}_{\mathcal{K}} = \Big\{block\ diag(g_1, \ldots, g_{n_r}, 0_{k_{n_{r+1}}}, \ldots, 0_{k_n}) : g_i \in \mathbb{R}\Big\}, \quad (7)$$

where $n_r = \sum_{i=1}^{m_r} k_i$.

3 Formal Definition of Skew μ

For a system M, with a set of perturbations, the definition of skew μ is the smallest SSV of a subset of perturbations that destabilizes the system M with the remainder of the perturbations being of fixed range. Formally developing this:

Given a set of allowable perturbations

$$\begin{aligned} Y_{\mathcal{K}} = \big\{\Delta_v = block\ diag(&\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \\ &\delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C) : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \\ &\Delta_i^C \in \mathbb{C}^{k_{m_r+m_c+i}\times k_{m_r+m_c+i}}\big\}, \end{aligned} \quad (8)$$

and a secondary set of perturbations with structure $\widehat{\mathcal{K}}$ defined as

$$\begin{aligned} Z_{\widehat{\mathcal{K}}} = \{\Delta_f = block\ diag(&\delta_1^r I_{\hat{k}_1}, \ldots, \delta_{m_r}^r I_{\hat{k}_{m_r}}, \delta_1^c I_{\hat{k}_{m_r+1}}, \ldots, \\ &\delta_{m_c}^c I_{\hat{k}_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C) : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \\ &\Delta_i^C \in \mathbb{C}^{\hat{k}_{m_r+m_c+i}\times \hat{k}_{m_r+m_c+i}}\}, \end{aligned} \quad (9)$$

where $Z_{\widehat{\mathcal{K}}}$ are restricted to the unit ball

$$\mathbf{B}Z_{\widehat{\mathcal{K}}} = \{\Delta_f \in Z_{\widehat{\mathcal{K}}} : \overline{\sigma}(\Delta_f) \leq 1\}. \quad (10)$$

The composite (Δ) perturbations are defined as

$$W_{\mathcal{K},\widehat{\mathcal{K}}} = \{\Delta = block\ diag(\Delta_f, \Delta_v)\}, \quad (11)$$

or

$$\Delta = \begin{bmatrix} \Delta_f & 0 \\ 0 & \Delta_v \end{bmatrix}. \quad (12)$$

These specifications allow the skewed SSV definition:

Definition 2 *The skewed structured singular value $\mu_s(M)$ of a matrix $M \in \mathbb{C}^{n\times n}$ with respect to a block structure $\mathcal{K}(m_{r_f}, m_{c_f}, m_{C_f}, m_{r_v}, m_{c_v}, m_{C_v})$ is defined as:*

$$\mu_s(M) = \frac{1}{\min\limits_{\Delta\in W_{\mathcal{K},\widehat{\mathcal{K}}}} \{\overline{\sigma}(\Delta_v) |\ det(I - \Delta M) = 0\, for\ structured\ \Delta\}} \quad (13)$$

with $\mu_s(M) = 0$ if no $\Delta \in W_{\mathcal{K},\widehat{\mathcal{K}}}$ solves $det(I - \Delta M) = 0$.

Note that $W_{\mathcal{K},\widehat{\mathcal{K}}} \subset \mathbb{C}^{n\times n}$ and that this block structure is sufficiently general to allow for repeated real scalars, repeated complex scalars, and full complex blocks in both the fixed range and varying perturbations.

4 Developing a Skew Mu Upper Bound

As previously mentioned, the skew μ problem is computationally hard. Because of the difficulty in calculating skew μ, upper and lower bounds are used to provide information concerning the value of skew μ. The skew μ upper bound is defined the following section.

4.1 Defining the Skew μ Upper Bound

In this paper, the upper bound on skew μ will be referred to as ν, however, first it must be shown that ν is an upper bound on skew μ. This is done by initially defining a matrix of the block form

$$S = \left[\begin{array}{c|c} I_f & 0 \\ \hline 0 & \nu I_v \end{array}\right], \tag{14}$$

where the S matrix is partitioned such that the blocks I_f and νI_v are sized to correspond to the fixed range and varying uncertainties Δ_f and Δ_v respectively, as in equation (12). Then let the matrix M_s be defined as

$$M_s = S^{-1}M. \tag{15}$$

From the definition of skew μ, there must exist an M_s for some choice of ν such that $\mu(M_s) = 1$, if not, then $\mu_s(M) = 0$. From this, the following theorem can be derived.

Theorem 1 *The value of ν is an upper bound on $\mu_s(M)$.*

Proof: The desire is to have a ν that sets the $\det(I - M_s\Delta) = 0$, thus it can be stated that

$$\begin{gathered} (I - M_s\Delta)x = 0 \\ x = M_s\Delta x \\ \|x\| \leq \|M_s\Delta\|\|x\| \\ 1 \leq \|M_s\Delta\| \leq \|M_s\|\|\Delta\| \end{gathered}$$

therefore

$$\frac{1}{\overline{\sigma}(\Delta)} \leq \overline{\sigma}(M_s) = \overline{\sigma}(S^{-1}M). \tag{17}$$

Since $\mu(M_s) = 1$ for skew μ, then ν is chosen such that $\overline{\sigma}(S^{-1}M) = 1$, where $\overline{\sigma}(S^{-1}M)$ is an upper bound on μ and thus ν is an upper bound on $\mu_s(M)$. □

Remarks: This is predicated on the fact that $\mu(S^{-1}M) = 1$, which follows from the definition of skew μ.

4.2 Implications of Using $\sqrt{S^{-1}}M\sqrt{S^{-1}}$ Versus $S^{-1}M$

There are several ways to form the product of the skewing matrix S and the system matrix M. One way is to form the product $S^{-1}M$. A second method is to form the product $\sqrt{S^{-1}}M\sqrt{S^{-1}}$. The advantage of the second method can be seen by dissecting the products in block form. When this is done, two things are apparent: first, both systems have the same eigenvalues and second, if M is norm-balanced (via Osborne balancing [8]), then the second method causes less upset on the matrix balance than the first method. Hence, the second method is the recommended implementation in software using Osborne balancing.

4.3 Direct Computation for a Skew μ Upper Bound

The first method proposed for finding an upper bound on skew μ is via a direct computation for the upper bound in terms of a generalized eigenvalue problem. To facilitate this method requires some re-arrangement of M and Δ. WLOG, elementary transformations can be used to arrange M and Δ such that the interacting components of Δ_f and M are arranged to allow Δ_f to occupy the upper left quadrant of Δ, and Δ_v occupies the lower right quadrant of Δ as in equation (12) . Keeping this arrangement in mind, the following bound on skew μ can be stated:

Theorem 2 *The skew μ upper bound ν can be calculated from the generalized eigenvalue problem*

$$\left(\left[\begin{array}{cc|c} M_{11}M_{11}^H - I_f & M_{11}M_{21}^H & 0 \\ M_{21}M_{11}^H & M_{21}M_{21}^H & I \\ \hline 0 & I & 0 \end{array}\right] + \frac{1}{\nu}\left[\begin{array}{cc|c} M_{12}M_{12}^H & M_{12}M_{22}^H & 0 \\ M_{22}M_{12}^H & M_{22}M_{22}^H & 0 \\ \hline 0 & 0 & I \end{array}\right]\right)\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = 0. \tag{18}$$

Proof: From theorem (1), the requirement that

$$\overline{\sigma}(\sqrt{S^{-1}}M\sqrt{S^{-1}}) \leq 1 \tag{19}$$

is evident. This can be expanded to

$$\overline{\sigma}(\sqrt{S^{-1}}M\sqrt{S^{-1}}) = \bar{\lambda}([\sqrt{S^{-1}}M\sqrt{S^{-1}}\sqrt{S^{-1}}^H M\sqrt{S^{-1}}^H), \tag{20}$$

where $\bar{\lambda}$ indicates the magnitude of the largest eigenvalue. Note that $S^H = S$ and $\sqrt{S^{-1}}^H = \sqrt{S^{-1}}$, which implies that

$$\sqrt{S^{-1}}MS^{-1}M^H\sqrt{S^{-1}} \leq I, \quad or \quad MS^{-1}M^H \leq S. \tag{21}$$

From previous definitions,

$$S = \left[\begin{array}{cc} I_f & 0 \\ 0 & \nu \cdot I_v \end{array}\right] \quad and \quad S^{-1} = \left[\begin{array}{cc} I_f & 0 \\ 0 & \frac{1}{\nu} \cdot I_v \end{array}\right]. \tag{22}$$

Partition S, such that

$$S = X + \nu Y = \left[\begin{array}{cc} I_f & 0 \\ 0 & 0 \end{array}\right] + \nu\left[\begin{array}{cc} 0 & 0 \\ 0 & I_v \end{array}\right] \tag{23}$$

and

$$S^{-1} = X + \frac{1}{\nu}Y = \left[\begin{array}{cc} I_f & 0 \\ 0 & 0 \end{array}\right] + \frac{1}{\nu}\left[\begin{array}{cc} 0 & 0 \\ 0 & I_v \end{array}\right]. \tag{24}$$

Substituting these definitions into equation (21) and rearranging gives

$$MXM^H + \frac{1}{\nu}MYM^H - X - \nu Y \leq 0. \tag{25}$$

Examine the terms of equation (25) to get

$$MXM^H = \left[\begin{array}{cc} M_{11}M_{11}^H - I_f & M_{11}M_{21}^H \\ M_{21}M_{11}^H & M_{21}M_{21}^H \end{array}\right] = \left[\begin{array}{cc} U & W \\ W^H & V \end{array}\right] \tag{26}$$

and

$$MYM^H = \begin{bmatrix} M_{12}M_{12}^H & M_{12}M_{22}^H \\ M_{22}M_{12}^H & M_{22}M_{22}^H \end{bmatrix} = \begin{bmatrix} E & G \\ G^H & F \end{bmatrix}. \quad (27)$$

At this point, one notes that it is possible to construct a general, augmented problem,

$$\left(\left[\begin{array}{cc|c} U & W & 0 \\ W^H & V & I \\ \hline 0 & I & 0 \end{array} \right] + \frac{1}{\nu} \left[\begin{array}{cc|c} E & G & 0 \\ G^H & F & 0 \\ \hline 0 & 0 & I \end{array} \right] \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \quad (28)$$

One also notes that this problem is of the form $Ax = \lambda Bx$, a generalized eigenvalue problem, which can be readily solved.

Now show that this general case applies to the specific problem in equation (25),

Note that

$$x_3 I = -\nu x_2 I \quad (29)$$

in equation (28). Thus reduced equations can be written from equation (28) as

$$\begin{array}{l} (U + \frac{1}{\nu}E)x_1 + (W + \frac{1}{\nu}G)x_2 = 0 \\ (W^H + \frac{1}{\nu}G^H)x_1 + (V - \nu I + \frac{1}{\nu}F)x_2 = 0. \end{array} \quad (30)$$

Reforming equation (30) into a matrix form,

$$\begin{bmatrix} U + \frac{1}{\nu}E & W + \frac{1}{\nu}G \\ W^H + \frac{1}{\nu}G^H & V - \nu I + \frac{1}{\nu}F \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad (31)$$

Now, substituting in the individual components one gets

$$\begin{bmatrix} M_{11}M_{11}^H - I_f + \frac{1}{\nu}M_{12}M_{12}^H & M_{11}M_{21}^H + \frac{1}{\nu}M_{12}M_{22}^H \\ M_{21}M_{11}^H + \frac{1}{\nu}M_{22}M_{12}^H & M_{21}M_{21}^H - \nu I_v + \frac{1}{\nu}M_{22}M_{22}^H \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad (32)$$

At this point, one notes that this is easily reduced to the equation of interest, (25). Thus equation (28) can be readily solved for the desired value ν. This value is found at the equality. The inequality was originally constructed because the desire was to find an upper bound on the value of skew μ.

The problem has been solved for a ν_i that sets $\sigma_i(S^{-1}M) = 1$. This does not guarantee that $\sigma_i(S^{-1}M) = \overline{\sigma}(S^{-1}M)$. Now, suppose that another ν_b is found such that solves equation (28), such that $\nu_b < \nu_i$. However, if $\nu_b < \nu_i$, then $\sigma_b(S^{-1}M) < \sigma_i(S^{-1}M)$, which means that $\sigma_b(S^{-1}M) < 1$, a contradiction on the requirements of the solution. Thus if ν_i solves the equations, then $\sigma_i(S^{-1}M) = \overline{\sigma}(S^{-1}M)$. □

Remarks: A numerical solution to equation (18) can typically be found using an generalized eigenvalue solver like those in Matlab®.

5 An LMI Methodology for the Skew μ Upper Bound

5.1 Skew μ LMI Upper Bound Derivation

It is possible to derive an expression that provides for acquiring the D-scales as well as skew μ in terms of an LMI. There are several sources where a basic form for this LMI can be found, in particular [1], [4], and [5]. In [5] the actual skew μ upper bound is given without derivation. Within LMI terminology, this problem is commonly referred to as the generalized eigenvalue problem (GEVP).

To obtain an understanding of the GEVP, start from theorem 1, where it was shown that

$$\mu(M_s) \leq 1.$$

Consequently, from μ theory it can be stated that

$$\overline{\sigma}(D_L M_s D_R^{-1}) \leq 1 \quad or \quad (D_L M_s D_R^{-1})^H (D_L M_s D_R^{-1}) \leq I \quad (33)$$

Noting that the matrix S^{-1} commutes with D, the previous equation can be written as

$$(D_L M D_R^{-1} S^{-1})^H (D_L M D_R^{-1} S^{-1}) \leq I. \quad (34)$$

Expanding and manipulating this equation gives

$$M^H D_L^H D_L M \leq D_R^H D_R S^2. \quad (35)$$

Note that since full block complex perturbations ($\Delta_{m_C}^C$ in equation (2)) need not be square, M need not be be square and D-scales need not be the same size on each side of M. However, each D-scale will be square. Theoretically, it causes no complications, but it needs to be pointed out occasionally. In the midst of software programming, it can be a terribly annoying discovery.

Now let $P_L \triangleq D_L^H D_L$ and $P_R \triangleq D_R^H D_R$, which have a Hermitian matrix structure [7], then

$$M^H P_L M - \begin{bmatrix} I_f & 0 \\ 0 & \nu^2 I_v \end{bmatrix} P_R \leq 0. \quad (36)$$

Partitioning along the lines of the fixed and varying components, letting $\lambda_v = \nu^2$, and manipulating gives,

$$M^H P_L M - \begin{bmatrix} I_f & 0 \\ 0 & 0 \end{bmatrix} P_R \leq \lambda_v \begin{bmatrix} 0 & 0 \\ 0 & I_v \end{bmatrix} P_R. \quad (37)$$

Note that equation (37) is approximately in the form of an eigenvalue problem where λ_v is the square of the value sought. The desired value to be found is $\sqrt{\lambda_v}$.

By partitioning all matrices along the fixed and varying perturbations and rearranging, the following can be arrived at (notation change $P^L \triangleq P_L$ and $P^R \triangleq P_R$ to help reduce equation clutter),

$$\begin{array}{c} \begin{bmatrix} M_{11}^H & M_{21}^H \\ M_{12}^H & M_{22}^H \end{bmatrix} \begin{bmatrix} P_{11}^L & 0 \\ 0 & P_{22}^L \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \\ - \begin{bmatrix} P_{11}^R & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & \lambda_v \cdot P_{22}^R \end{bmatrix} \end{array} \quad (38)$$

To formulate this as an LMI compatible with the Matlab ® LMI Controls toolbox [4], equation (38) is broken down in the following manner,

$$\left[\begin{array}{l} M_{11}^H P_{11}^L M_{11} + M_{21}^H P_{22}^L M_{21} - P_{11}^R \\ M_{12}^H P_{11}^L M_{11} + M_{22}^H P_{22}^L M_{21} \end{array}\right.$$

$$\left.\begin{array}{r} M_{11}^H P_{11}^L M_{12} + M_{21}^H P_{22}^L M_{22} \\ M_{12}^H P_{11}^L M_{12} + M_{22}^H P_{22}^L M_{22} - \lambda_\nu \cdot P_{22}^R \end{array}\right] \leq 0. \tag{40}$$

Here, 0 is an appropriately size $n \times n$ matrix.

Note that this method is better suited to cases where the perturbations are predominantly complex in nature. Real perturbations may be included; however, depending on the nature of the problem, the bound may be overly conservative. An addition to this method which improves the bound for real perturbations is given in section 5.2.

From this derivation, the following can be said about skew μ and the matrix M:

$$\mu_s(M) \leq \nu, \tag{41}$$

and

$$\begin{array}{r} \mu_s(M) \leq \inf_{D \in \mathcal{D}_{\mathcal{K}_f, \mathcal{K}_v}} \min_{0 \leq \nu \in \mathbb{R}} \left\{ \nu : M^H D_L^H D_L M \right. \\ \left. - \left[\begin{array}{cc} I_f & 0 \\ 0 & \nu^2 \cdot I_v \end{array}\right] D_R^H D_R \leq 0 \right\}. \end{array} \tag{42}$$

Where λ_ν has been replaced by ν^2 to be consistent with other works in this area. A brief reference to this can be found in [5].

Care must be given to setting up the variable structure of these matrices in the LMI toolbox code. A note of importance is that the number of variables can grow tremendously as the size of the problem increases. What may appear to be a reasonable size problem can easily generate over 5000 variables, enough to overburden the computational horsepower of many PC's.

5.2 Inclusion of G-Scales

Real parameter variations take place along the real axis (oddly enough), whereas complex parameter variations are assumed to vary on a disk in the complex plane with a radius of the specified Δ. This extra piece of *phase* information about real perturbations allows for the construction of disks which cover the uncertainties, but which are not necessarily centered at the operation point in the complex plane. Ultimately this yields a less conservative upper bound on μ.

To obtain this less conservative bound in the case of skew μ, first consider a set of scaling matrices constructed similar to those in [2].

The G-scales associated with the fixed range perturbations are defined as

$$\begin{array}{c} \mathcal{G}_{\mathcal{K}_f} = \{ block\ diag(G_1, \ldots, G_{m_{r_f}}, 0_{k_{m_{r_f+1}}}, \ldots, 0_{k_m}) : \\ G_i = G_i^H \in \mathbb{C}^{k_i \times k_i} \} \end{array} \tag{43}$$

and the G-scales associated with the varying range perturbations are defined as

$$\begin{array}{c} \mathcal{G}_{\mathcal{K}_v} = \{ block\ diag(G_1, \ldots, G_{m_{r_v}}, 0_{k_{m_{r_v+1}}}, \ldots, 0_{k_m}) : \\ G_i = G_i^H \in \mathbb{C}^{k_i \times k_i} \}. \end{array} \tag{44}$$

The combined G-scales for skew μ may then be written as

$$\mathcal{G}_{\mathcal{K}_f, \mathcal{K}_v} = \{ G = block\ diag(G_f, G_v) \}. \tag{45}$$

The subscript m_r indicates the number of real perturbation blocks, and the subscript m_{r+1} to k_m indicates the number of complex block and full block perturbations. Note that the matrix G only affects the parts of M subject to real perturbations, not the complex or full block perturbations, and is Hermitian in structure. This structure corresponds to the structure set forth in section 3 where the complex perturbation elements have been set to 0.

5.2.1 G-Scale Derivation

Starting with the basic eigenvalue form $(I - \frac{1}{\nu}\Delta M)x = 0$, the case for G scales can be derived. For $G \in \mathcal{G}$, the following set of equations holds true, assuming $\nu \neq 0$. If there exists a Δ and x such that

$$\Delta M x = \nu x, \tag{46}$$

then

$$\begin{array}{c} x = \frac{1}{\nu}\Delta M x \\ x^H G M x = \frac{1}{\nu}(x^H M^H \Delta^H) G M x \\ x^H G M x = x^H M^H G^H \frac{1}{\nu} \Delta M x \\ x^H G M x = x^H M^H G^H x \\ x^H (GM - M^H G^H) x = 0. \end{array} \tag{47}$$

Using properties of Hermitian matrices [7], $(GM - M^H G^H)$ is skew-Hermitian, $j(GM - M^H G^H)$ is Hermitian. Equation (47) is in the form of an LMI with the exception of the equality sign. Equation (47) can be used in addition to the previous upper bound LMI derivation in equation (38) to achieve a less conservative upper bound.

5.2.2 Upper Bound including G-Scales

Using the derivation of the G-scales from the previous section, the following construction of the upper bound can be created (see [2] , [13] or [12] for more background). This is the same form as equation (42) with the exception of the addition of G-scales.

$$\begin{array}{r} \mu_s(M) \leq \inf_{\substack{D \in \mathcal{D}_{\mathcal{K}_f, \mathcal{K}_v}, \\ G \in \mathcal{G}_{\mathcal{K}_f, \mathcal{K}_v}}} \min_{0 \leq \nu \in \mathbb{R}} \left\{ \nu : M^H D_L^H D_L M \right. \\ \left. + j(GM - M^H G) - \left[\begin{array}{cc} I_f & 0 \\ 0 & \nu^2 \cdot I_v \end{array}\right] D_R^H D_R \leq 0 \right\} \end{array} \tag{48}$$

Note that if $G = 0$ in equation (48), then the standard complex ν upper bound is recovered.

Since the matrix G only affects the parts of M subject to real perturbations, not the complex or full block perturbations, it only affects the singular values associated with the real perturbations.

5.2.3 The Specifics of a Skew μ Upper Bound Including G-Scales

The specifics of the skew μ LMI equations are laid out for clarity. As with the original skew μ LMI, care needs to be taken when setting up the variables for solution because of their Hermitian nature.

Referencing equations (37) and (39), the following LMI's including G-scales are achieved

$$M^H P^L M + j(GM - M^H G^H) - \begin{bmatrix} I_f & 0 \\ 0 & \nu^2 \cdot I_v \end{bmatrix} P^R \leq 0. \quad (49)$$

This structure is similar to the structure set forth in section 3 where the fixed and varying complex perturbation elements have been set to 0.

Partitioning the G-scales in a similar fashion to the LMI's of section 5.1, the expression for $j(GM - M^H G^H)$ is achieved,

$$j\left(\begin{bmatrix} G_{11}M_{11} - M_{11}^H G_{11}^H & G_{11}M_{12} - M_{21}^H G_{22}^H \\ G_{22}M_{21} - M_{12}^H G_{11}^H & G_{22}M_{22} - M_{22}^H G_{22}^H \end{bmatrix}\right). \quad (50)$$

Finally, inserting the detailed G-scale expression into the LMI upper bound expression, equation (39),

$$\begin{bmatrix} M_{11}^H P_{11}^L M_{11} + M_{21}^H P_{22}^L M_{21} + j(G_{11}M_{11} - M_{11}^H G_{11}^H) - P_{11}^R \\ M_{12}^H P_{11}^L M_{11} + M_{22}^H P_{22}^L M_{21} + j(G_{22}M_{21} - M_{12}^H G_{11}^H) \end{bmatrix}$$

$$\begin{bmatrix} M_{11}^H P_{11}^L M_{12} + M_{21}^H P_{22}^L M_{22} + j(G_{11}M_{12} - M_{21}^H G_{22}^H) \\ M_{12}^H P_{11}^L M_{12} + M_{22}^H P_{22}^L M_{22} + j(G_{22}M_{22} - M_{22}^H G_{22}^H) - \lambda_v P_{22}^R \end{bmatrix} \leq 0. \quad (52)$$

This expression is in the proper format for implementation via Matlab® LMI Toolbox.

6 Conclusion

An extension of the μ upper bound methodology is devised to provide for the solution of the potentially more common skew μ problem. This method is superior to previous skew μ calculation methods because it does not include repetitive calculations of μ to find skew μ [3]. These previous methods are computationally expensive in comparison to both μ and the proposed skew μ technique.

In addition to theoretical development of two skew μ upper bounds, methodologies for calculating the skew μ upper bound via software algorithms are outlined.

Associated issues such as conversion of scaling sets between the two types of upper bounds and extended algorithms have been dealt with, to bring the state of skew μ development near to that of μ. Details of these issues and expanded discussion of the developments and software algorithms described here can be found in [6].

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