COMPUTER ALGEBRA ALGORITHMS FOR THE TEST ON ACCESSIBILITY AND OBSERVABILITY FOR IMPLICIT DYNAMICAL SYSTEMS

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Abstract

This contribution is devoted to computer algebra based algorithms for the analysis of systems of implicit ordinary differential equations. These systems are identified with submanifolds in a suitable jet space. It is outlined how the accessibility and observability test for implicit ordinary differential equations can be solved via an approach utilizing transformation groups, which has been already published. Since this approach requires systems in formally integrable form, an algorithm for the derivation of this form is presented and it is shown, how Groebner bases can be successfully applied in the case of implicit systems with polynomial nonlinearities. In addition, the algorithms for the tests on observability and accessibility according to the proposed approach are sketched.

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1 Introduction

When modeling finite dimensional mechanical or electrical systems the DAE (differential algebraic equations) approach turns out to be a very natural one. Especially object oriented modeling or the modeling via port connections lead to systems of this form. A DAE system is a set of n_x ordinary differential equations together with n_s purely algebraic equations (constraints) of the form

$$\begin{array}{rcllll} \dot{w}^{\alpha_x} & = & f^{\alpha_x}(w,v) & , & \alpha_x = 1,\ldots,n_x & \\ 0 & = & g^{\alpha_s}(w,v) & , & \alpha_s = 1,\ldots,n_s \ , & \end{array} \tag{1}$$

in the dependent variables w and v. Here $\dot{w}$ denotes the derivative of w with respect to the single independent variable t. For control purposes, the treatment of DAEs is not straightforward, since a DAE may contain hidden constraints which lead to the index problem (see e.g. [2]). One possible way is the calculation of a corresponding explicit system. Once this step is accomplished, one can use numerous design tools, see e.g. [6], [5], [10]. However, to find the corresponding explicit system to a given DAE is a task where no simple solution exists. Therefore, we propose a different approach, investigating the equations (1) themselves, without transforming them to an explicit system. We do this by using a geometric description of the dynamical system, such that it is independent of the chosen representation. With the help of this framework we consider systems of the form

$$f^{\alpha_e}(t,z,\dot{z}) = 0 \ , \ \alpha_e = 1,\ldots,n_e \text{ with } z = (z^{\alpha_z}) \tag{2}$$

in the n_z dependent variables z^{α_z} and show how the accessibility and observability problem can be solved for this system class.

This contribution is organized as follows: In Section 2 we summarize some mathematical preliminaries and notation we will need in the sequel. We introduce jet bundles and show how one can identify a dynamical system as a submanifold of a suitable jet space. The key concept for all further analysis is formal integrability, which we will show in Section 2.2. An algorithm to derive the formally integrable form is presented and in Section 2.3 the issue of equivalence to an explicit system is treated. In Section 2.4 we consider systems (2), in which the f^{i_e} are polynomials, and use the prior defined geometric correspondence and tools from algebraic geometry to present a more efficient algorithm for this subclass. Afterwards we will outline a test on observability and accessibility for formally integrable implicit systems in Section 3 by applying transformation groups and give algorithms for the implementation. We apply these algorithms in Section 4 to the planar VTOL model in order to show their applicability. We finish this contribution with some conclusions.

2 Preliminaries

2.1 Jet-Bundles

In the sequel we will use the notion of jet manifolds for our investigations. For an introduction into the theory and concepts of jet manifolds the reader is referred to [8], [11]. In this Section we mainly want to summarize the notation.

We will use (smooth) bundles as our geometric framework, a special type of fibred manifolds given by a triple $(\mathcal{E},\pi,\mathcal{B})$ where $\mathcal{E}$ and $\mathcal{B}$ are smooth manifolds and the projection $\pi : \mathcal{E} \to \mathcal{B}$ is a surjective submersion, which assigns each point of the total manifold $\mathcal{E}$ the corresponding point in the base $\mathcal{B}$. The manifold $\mathcal{F}_p = \pi^{-1}(p)$ with $p \in \mathcal{B}$ is called the *fibre* over p. If all fibres are diffeomorphic to a typical fibre, we call $(\mathcal{E},\pi,\mathcal{B})$ a (fibre-) bundle. However, we will write $\mathcal{E}$ instead of $(\mathcal{E},\pi,\mathcal{B})$, whenever the projection π and the base manifold $\mathcal{B}$

follow from the context. A map $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ is called a *section* σ of $\mathcal{E}$ if $\pi \circ \sigma = \mathrm{id}_{\mathcal{B}}$ is met, where $\mathrm{id}_{\mathcal{B}}$ denotes the identity map on $\mathcal{B}$. The set of all sections of $\mathcal{E}$ is denoted with $\Gamma(\mathcal{E})$. On a bundle, one can always, at least locally, introduce adapted coordinates in the form $\left(x^i, u^\alpha\right)$, $i = 1, \ldots, n_x$, $\alpha = 1, \ldots, n_u$ with the independent coordinates x^i on $\mathcal{B}$ and the u^α called the dependent ones. Given two bundles $(\mathcal{E}, \pi, \mathcal{B})$ and $\left(\bar{\mathcal{E}}, \pi, \bar{\mathcal{B}}\right)$, a bundle morphism is a transformation $(f_{\mathcal{B}}, f_{\mathcal{E}})$ which preserves the bundle structure, i.e. $\bar{\pi} \circ f_{\mathcal{E}} = f_{\mathcal{B}} \circ \pi$ holds. Important examples are the tangent and the cotangent bundle, which are vector bundles, i.e. $f_{\mathcal{E}}(x, u)$ is linear in u, denoted by $(\mathcal{T}\mathcal{B}, \tau, \mathcal{B})$ and $(\mathcal{T}^*\mathcal{B}, \tau^*, \mathcal{B})$, respectively. Using the Einstein convention for sums, we write $v = v^i \partial_i$ for $v \in \Gamma(\mathcal{T}\mathcal{B})$ and $\omega = \omega_i \mathrm{d}x^i$ for $\omega \in \Gamma(\mathcal{T}^*\mathcal{B})$ with $v^i, \omega_i \in C^\infty(\mathcal{B})$, where $C^\infty(\mathcal{B})$ denotes the set of smooth functions on $\mathcal{B}$. With $v(f)$ we denote the Lie derivative of $f \in C^\infty(\mathcal{B})$ along $v \in \Gamma(\mathcal{T}\mathcal{B})$.

Given a section $\sigma \in \Gamma(\mathcal{E})$, its partial derivatives are denoted by $\partial_i \sigma = \frac{\partial}{\partial x^i}\sigma$ or $\frac{\partial^k}{\partial_1^{j_1} \cdots \partial_p^{j_p}} f = \partial_J f$, with the ordered multi-index $J = (j_1, \ldots, j_p)$, and $k = \# J = \sum_{i=1}^p j_i$. The special index $J = j_1, \ldots, j_p$, $j_i = \delta_{ik}$ will be denoted by 1_k and $J + 1_k$ is a shortcut for $j_i + \delta_{ik}$ with the Kronecker symbol δ_{ik}. The *first jet manifold* of $\mathcal{E}$ (see, e.g., [4], [11] and [7]) is denoted by $J^1\mathcal{E}$. Roughly speaking $J^1\mathcal{E}$ is a container for prolongations $j^1\sigma$ of sections $\sigma \in \Gamma(\mathcal{E})$, which in adapted coordinates take the form $j^1\sigma : x \rightarrow \left(x^i, \sigma(x), \partial_i\sigma(x)\right)$. An adapted coordinate system on $J^1\mathcal{E}$ is given by $\left(x^i, u^\alpha, u^\alpha_{1_i}\right)$ with the $n_i n_\alpha$ new coordinates $u^\alpha_{1_i}$. The two natural projections of $J^1\mathcal{E}$ are denoted by $\pi : J^1\mathcal{E} \rightarrow \mathcal{B}$ and $\pi^1_0 : J^1\mathcal{E} \rightarrow \mathcal{E}$ with $\pi\left(j^1\sigma(x)\right) = x$ and $\pi^1_0\left(j^1\sigma(x)\right) = \sigma(x)$ for the section $\sigma \in \Gamma(\mathcal{E})$ and $j^1\sigma \in \Gamma\left(J^1\mathcal{E}\right)$. Analogously, the n^{th}-jet manifold of $\mathcal{E}$ is denoted by $J^n\mathcal{E}$. We write $u^{(n)} = u^\alpha_J$, $\alpha = 1, \ldots, n_u$, $\# J = 0, \ldots, n$ for all jet variables of u up to the order n. The n^{th}-order total derivatives with respect to the independent coordinates x^i are denoted with d_i. In adapted coordinates $\left(x^i, u^\alpha\right)$ the $d_i \in \mathcal{T}J^{n+1}\mathcal{E}$ take the form

$$d_i = \partial_i + u^\alpha_{J+1_i}\partial^J_\alpha \text{ , with } \partial^J_\alpha = \frac{\partial}{\partial u^\alpha_J} \text{ , } \# J = 0, \ldots, n. \quad (3)$$

Given a section $\sigma \in \mathcal{E}$ and a smooth real valued function $f \in C^\infty\left(J^k\mathcal{E}\right)$, the d_i connect $j^k\sigma$ and $j^{k+1}\sigma$ in the form $d_i f \circ j^{k+1}\sigma = \partial_i f\left(j^k\sigma\right)$. The dual objects to the fields $d_i \in \mathcal{T}J^{n+1}\mathcal{E}$ are the 1-forms $\omega^\alpha_J \in \mathcal{T}^*J^{n+1}\mathcal{E}$, called contact forms,

$$\omega^\alpha_J = \mathrm{d}u^\alpha_J - u^\alpha_{J+1_i}\mathrm{d}x^i \text{ , } \quad \# J = 0, \ldots, n. \quad (4)$$

A bundle morphism $(f_{\mathcal{B}}, f_{\mathcal{E}})$ defined between bundles $(\mathcal{E}, \pi, \mathcal{B})$, $\left(\bar{\mathcal{E}}, \bar{\pi}, \bar{\mathcal{B}}\right)$, can be prolonged to a map $j^1 f_{\mathcal{E}} : J\mathcal{E} \rightarrow J\bar{\mathcal{E}}$ in the form $j^1 f_{\mathcal{E}} \circ j^1\sigma = j^1\left(f_{\mathcal{E},*}\sigma\right) f_{\mathcal{B}}$ with $f_{\mathcal{E},*}\sigma = f_{\mathcal{E}} \circ \sigma \circ f_{\mathcal{B}}$, iff $f_{\mathcal{B}}$ is a diffeomorphism.

In order to use algebraic tools for the description of submanifolds, let us recall the following fact. Given a manifold $\mathcal{M}$, if we find d functions $f^i \in C^\infty(\mathcal{M})$ which are functionally independent then the variety $\mathcal{N} = \left\{x \in \mathcal{M}, f^i(x) = 0, i = 1, \ldots, d\right\}$ is regular submanifold. Let us make some further observations. Obviously the set $C^\infty(\mathcal{M})$ has the structure of a local commutative ring (with the operations addition and multiplication). If we denote with $\mathcal{S}_p$ a regular submanifold in the neighborhood of a point $p \in \mathcal{M}$, then the set $\{f \in C^\infty(\mathcal{M}), f(\mathcal{S}_p) = 0\}$ generates an ideal, which we denote by $I(\mathcal{S}_p)$. In [7] it is shown, that any functionally independent set of functions $B = \left\{f^i | i = 1, \ldots, d\right\}$ generate an ideal denoted by $\left\langle f^i | i = 1, \ldots, d\right\rangle$. If now $f^i(\mathcal{S}_p) = 0$ holds then $I(\mathcal{S}_p) = \left\langle f^i | i = 1, \ldots, d\right\rangle$ follows. These observations lead us to a coordinate independent representation of a dynamical system.

2.2 Dynamical Systems and Formal Integrability

In the sequel we restrict our attention to systems of ordinary differential and algebraic equations of the form

$$0 = f^{\alpha_e}(t, z^{\alpha_z}, z^{\alpha_z}_1) \text{ , } \alpha_e = 1, \ldots, n_e \quad (5)$$

in the single independent variable $t \in \mathcal{T} \subset \mathbb{R}$ which involves jet coordinates $z^{(1)}$ of the dependent variables z up to order 1 with $z \in \mathcal{Z} \subset \mathbb{R}^{n_z}$. $\mathcal{T}$ and $\mathcal{Z}$ are smooth manifolds. The total manifold is given by $\mathcal{E} = \mathcal{T} \times \mathcal{Z}$ where we use the coordinates (t, z) locally. Within the jet bundle formalism, introduced in the last section, z_1 is just the name of a coordinate and (5) is an equation on $J^1\mathcal{E}$ and we assume that (5) defines a regular submanifold $\mathcal{S}^1$ of $J^1\mathcal{E}$, at least locally. Additionally, if the jacobian of f has full rank on the solution set $\mathcal{S}^1$ of (5) then $\mathcal{S}^1$ defines a regular submanifold of $J^1\mathcal{E}$.

For a system like (5) there exist two natural operations, its prolongation and its projection [7]. The projection π^n_m allows us to transfer objects defined on $J^n\mathcal{E}$ to objects on $J^m\mathcal{E}$. The projection $\pi^n_m(\mathcal{S}^n)$ with $n > m$ of $\mathcal{S}^n$ is simply obtained by elimination of all variables u^α_J, $m < \# J \leq n$. The geometric picture of this operation is straightforward due to the bundle structure of $J^n\mathcal{E}$, which incorporates the natural projection $\pi^n_m : J^n\mathcal{E} \rightarrow J^m\mathcal{E}$. The prolongation of an object allows us to transfer an object of $J^m\mathcal{E}$ to $J^n\mathcal{E}$ with $n \geq m$. Given the submanifold $\mathcal{S}^m \subset J^m(\mathcal{E})$, then the s-order prolongation of $\mathcal{S}^m$ is given by $\mathcal{S}^{m+s} = \left(\pi^{m+s}_m\right)^{-1}(\mathcal{S}^m) \cap J^{m+s}(\mathcal{E})$. With the total derivative d_1 from (3) the first prolongation $\mathcal{S}^{1+1} = \mathcal{S}^2 \subset J^2\mathcal{E}$ of the submanifold $\mathcal{S}^1$ is given by $\left\{f^{i_e}, d_1 f^{i_e}\right\}$. In a straightforward manner we can define the r-th prolongations of $\mathcal{S}^n$, which is denoted by $\mathcal{S}^{n+r}$. Repeated prolongation and projection of a system like (5) generate new systems such that the inclusion $\pi^{n+r+s}_{n+r}\left(\mathcal{S}^{n+r+s}\right) \subset \mathcal{S}^{n+r}$ is met. Iff $\mathcal{S}^{n+r}$ are regular submanifolds of $J^{n+r}\mathcal{E}$ and $\pi^{n+r+s}_{n+r}\left(\mathcal{S}^{n+r+s}\right) = \mathcal{S}^{n+r}$ is met for all $r, s \geq 0$, then we call the system $\mathcal{S}^{n+r}$ formally integrable. In general, the projection task can be very difficult, if the equations (5) incorporate nonlinearities in the highest derivatives. However, especially if the projection step follows a prolongation, the prolonged equations are affine in the highest order derivatives and only linear algebra is needed for the elimination process.

In contrast to systems involving more independent variables, i.e. PDEs, for (5) we can state an algorithm, that terminates after a finite number of steps, and computes its formally integrable system. To do this, let us assume that the functions f^{α_e} are linearly independent with respect to $z^{(1)}$. Taking the first prolongation $\mathcal{S}^{1+1}$ leads to the new equations

$$d_1 f^{\alpha_e} = \partial^1_{\alpha_z} f^{\alpha_e}\ z_2^{\alpha_z} + \partial_{\alpha_z} f^{\alpha_e}\ z_1^{\alpha_z} + \partial_t f^{\alpha_e} = 0\,. \quad (6)$$

Now, if linear combinations of the form $\lambda_{\alpha_e}\partial^1_{\alpha_z} f^{\alpha_e}\ z_2^{\alpha_z} = 0$ can be found, then it is obvious that the system

$$\lambda_{\alpha_e}\left(\partial_{\alpha_x} f^{\alpha_e}\ x_1^{\alpha_x} + \partial_{\alpha_u} f^{\alpha_e}\ u_1^{\alpha_u} + \partial_t f^{\alpha_e}\right) \overset{\mathcal{S}^1}{=} 0 \quad (7)$$

is independent of z_2 on $\mathcal{S}^1$. Thus, the system (7) defines an additional constraint $\lambda_{\alpha_e} f^{\alpha_e}$. This can be seen by looking at the equation

$$\partial^1_{\alpha_z}\left(\lambda_{\alpha_e} f^{\alpha_e}\right) = \partial^1_{\alpha_z}\left(\lambda_{\alpha_e}\right) f^{\alpha_e} + \lambda_{\alpha_e}\partial^1_{\alpha_z}\left(f^{\alpha_e}\right) \overset{\mathcal{S}^1}{=} 0\,. \quad (8)$$

The first term vanishes on the solution set of f^{α_e} and the second term due to the condition for λ_{α_e}. It is worth mentioning here, that in general it is not straightforward to symbolically eliminate the variables $z_1^{\alpha_z}$ from the equations $\lambda_{\alpha_e} f^{\alpha_e}$, although we know the elimination is possible in principle, see also Remark 2. In order to state the algorithm, let us rewrite the system (5) in the form

$$\begin{aligned} 0 &= f^{\alpha_x}\left(t, z^{\alpha_z}, z_1^{\alpha_z}\right),\ \alpha_x = 1, ..., n_x && \text{(9a)}\\ 0 &= f^{\alpha_s}\left(t, z^{\alpha_z}\right),\ \alpha_s = n_x + 1, ..., n_x + n_s && \text{(9b)} \end{aligned}$$

such that the sets of functions $\{f^{\alpha_s}\}$ and $\{d_1 f^{\alpha_s}, f^{\alpha_x}\}$ are functionally independent on $\mathcal{S}^1$ with respect to $z^{(1)}$. We say a system is *well-posed*, if it is formally integrable, i.e., no constraints of the form (7) exist. With the following algorithm every system of this form (9a)-(9b) can be transformed to a well-posed one.

Algorithm 1 [`wellpose`] *Formal Integrability*
Input: $\{f^{\alpha_x}\}, \{f^{\alpha_s}\}, [t, z^{\alpha_z}]$
Initialization: $F_1 = \{f^{\alpha_x}\},\ F_0 = \{f^{\alpha_s}\} = \left\{f^{\alpha_s^0}\right\}$
Eliminate a minimal number of equations of F_0, s.t. the $f^{\alpha_s^0}$ are functionally independent.
Iteration (k):
(1) Prolong F_1 and $F_{0,k}$ in the form $F_1^{(1)} = d_1(F_1)$ and $F_{0,k}^{(2)} = (d_1)^2(F_{0,k})$. Find the set of all constraints $C_k = \pi_1^2\left(F_1^{(1)} \cup F_{0,k}^{(2)}\right)$, with $\pi_1^2 : J^2\mathcal{E} \to J^1\mathcal{E}$, as described above, and derive $F_{0,k+1} = F_{0,k} \cup C_k$.
(2) Eliminate a minimal number of functions from $F_{0,k+1} = \left\{f^{\alpha_s^{k+1}}\right\}$, such that the remaining are functionally independent. If $\dim(F_{0,k}) < \dim(F_{0,k+1})$ then set $k = k+1$ and goto (1).
(3) Eliminate a minimal number of equations of F_1, such that no more linear combinations of the form (8) exist ($F_{1,end} \subset F_1$).
Result: $[F_{1,end}, F_{0,k}]$

Remark 2 *For systems, linear in the highest order derivatives, the projection π_1^2 in step (1) just needs linear algebra in order to carry out the required elimination. For general implicit systems, symbolic simplifiers provided by the user can help in this step, however, it may occur, that we know an additional invariant exists, but we cannot derive the symbolic expression, in which the variables z_1 disappear. In Section 2.4 we show how π_1^2 can be implemented in the case of polynomial nonlinearities.*

2.3 Equivalence to Explicit System

In [13] it has been shown, that for a formally integrable system in the form (9a) and (9b), there exists a coordinate transformation (10)

$$\begin{aligned} \bar{z}^{\alpha_x} &= \varphi^{\alpha_x}(t, z),\ \bar{z}^{\alpha_s} = \varphi^{\alpha_s}(t, z), && (10)\\ \bar{z}^{\alpha_u} &= \varphi^{\alpha_u}(t, z), \end{aligned}$$

$\alpha_x = 1, ..., n_x,\ \alpha_s = n_x + 1, ..., n_e,\ \alpha_u = n_e + 1, ..., n_z$, with $\varphi^{\alpha_s}(t, z) = f^{\alpha_s}(t, z)$ from (9b) and $z^{\alpha_z} = \psi^{\alpha_z}(t, \bar{z})$, such that in the new coordinates $\bar{z}$ the system takes the form

$$\begin{aligned} \bar{z}_1^{\alpha_x} &= \bar{f}^{\alpha_x}\left(t, \bar{z}^{\alpha_x}, \bar{z}^{\alpha_x}, \bar{z}^{\alpha_u}, \bar{z}_1^{\alpha_u}, \bar{z}^{\alpha_s}\right), && (11)\\ \bar{z}_1^{\alpha_s} &= 0,\ \bar{z}^{\alpha_s} = 0, && (12) \end{aligned}$$

which is an explicit system in the variables $\bar{z}^{\alpha_x}$ and $\bar{z}^{\alpha_s}$. Here the n_x variables $\bar{z}^{\alpha_x}$ denote the state of the system and the variables $\bar{z}^{\alpha_u}$ the input, i.e. the functions that can be chosen freely. By substituting $\bar{z}^{\alpha_s} = 0$ we obtain an explicit differential equation in the coordinates $\bar{z}^{\alpha_x}$. However we can do more and ask whether also the input derivatives $\bar{z}_1^{\alpha_u}$ can be eliminated in (11), i.e. a classic state space representation is possible, by considering the ideal generated by the functions f^{α_x}, f^{α_s} in (9). If we state the test for equivalence as an ideal membership test (see [13] for details) we get the compatibility condition

$$\bigwedge_{\alpha_s} \mathrm{d}\varphi^{\alpha_s} \wedge \bigwedge_{\beta_z} \partial^1_{\alpha_z} f^{\beta_x}\mathrm{d}z^{\alpha_z} \wedge \mathrm{d}\left(\partial^1_{\alpha_z} f^{\beta_x}\mathrm{d}z^{\alpha_z}\right) \wedge \mathrm{d}t \overset{\mathcal{S}^1}{=} 0 \quad (13)$$

which can be checked via computer algebra in a straightforward way.

2.4 Algebraic Geometry

We have seen in Section 2.1 that by introducing jet coordinates, we are able to use algebraic tools for the analysis. The field of algebraic geometry provides very efficient algorithms for the investigation into ideals generated by polynomials. In order to state an improved Algorithm 1 for this subclass, we first introduce some terminology. We refer to e.g. [3] or [1] for an introduction and further issues of algebraic geometry. A polynomial f in the coordinates x^i with coefficients in k is a finite linear combination of monomials of the form $f = a_J x^J$, $x = (x_1, \ldots, x_n)$, $J = (j_1, \ldots, j_n)$, $j_i \in \mathbb{N}_0$, $a_J \in k$. We denote the set of polynomials with coefficients $a_J \in k$

with $k\left[x^1,\ldots,x^n\right]$, which has the structure of a commutative ring. The submanifold given by the set of solutions for the polynomial system $f^\alpha(x)=0$, $\alpha=1,\ldots,s$, is called the variety generated by the functions f^α. We denote it with $V(f^\alpha,\alpha=1,\ldots,s)$. Again here, in order to link the algebraic description and its geometric picture we consider ideals, generated by polynomial functions, in our notation $\langle f^1,\ldots,f^s\rangle = \langle f^\alpha\rangle$, and ideals corresponding to a given variety, denoted by $I(V)$. In the sequel we will deal with regular varieties only, i.e., we require $\operatorname{rank}(\partial_i f^\alpha_{rad})$ to be constant, with $\langle f^\alpha_{rad}\rangle = \sqrt{\langle f^\alpha\rangle}$ being the radical ideal of $\langle f^\alpha\rangle$. The Hilbert basis theorem now states, that any ideal in $k\left[x^1,\ldots,x^n\right]$, with k algebraically closed, can be represented by a finite basis. One possible basis for a given polynomial ideal is the Groebner basis. Given a monomial (well-)ordering $>$, the Groebner basis G corresponding to $\langle f^\alpha\rangle$ is unique. One example for a monomial ordering is the lexicographic ordering $>_{lex}$, which we will need later on. A Groebner basis has some very pleasing properties. Let us just mention the elimination property here, which is important for our applications. Given a Groebner basis with respect to $>_{lex}$-order $G=\{g^\alpha, j=1,\ldots,s\}$ of a polynomial ideal I, the k-th elimination ideal I_k is defined as $I_k = I\cap k\left[x^{k+1},\ldots,x^n\right]$. Now, the Groebner basis G_k for I_k is given by $G_k = G\cap k\left[x^{k+1},\ldots,x^n\right]$ and there exists a subset of G in the form $\left\{g^{j_k}, j_k=1,\ldots,s_k\leq s\right\}$ such that $G_k=\{g^{\alpha_k}\}$ holds. This property tells us, that once we have computed the Groebner basis for I, we also have found the Groebner basis G_k for I_k by just taking the corresponding subset of G.

With this notation set up, we can state an improved algorithm for systems (9a, 9b) with $f^{\alpha_x}, f^{\alpha_s}\in\mathbb{R}\left[t,z^{\alpha_z},z_1^{\alpha_z}\right]$.

Algorithm 3 [`wellpose`] *Formal Integrability for functions* $f^{\alpha_x}, f^{\alpha_s}\in\mathbb{R}\left[t,z^{\alpha_z},z_1^{\alpha_z}\right]$
Input: $\{f^{\alpha_x}\},\{f^{\alpha_s}\},[t,z^{\alpha_z}]$
Initialization: $F_1=\{f^{\alpha_x}\},\ F_{0,0}=\{f^{\alpha_s}\}=\left\{f^{\alpha_s^0}\right\}$
Eliminate a minimal number of equations of $F_{0,0}$, s.t. the $f^{\alpha_s^0}$ are functionally independent.
Define $>_{lex}$ on $\left(t,z^{(1)}\right)$ in the form $z_1^{\alpha_z}>z^{\alpha_z}>t$.
Iteration (k):
(1) Use $>_{lex}$ to calculate the Groebner basis corresponding to $\left\{f^{\alpha_x}(t,z,z_1)=0,\ d_1 f^{\alpha_s^0}(t,z)=0\right\}$. This results in an equation system of the form $\left\{g^{i_1}(t,z,z_1)=0,\ g^{i_0}(t,z)=0\right\}$, with n_1 equations g^{i_1} of order one and n_0 equations g^{i_0} of order zero. Add these equations to the set of restrictions, i.e. set $F_{0,k+1}=F_{0,k}\cup\left\{g^{i_0}\right\}$.
(2) Eliminate a minimal number of equations from $F_{0,k+1}=\left\{f^{\alpha_s^{k+1}}\right\}$, such that the $f^{\alpha_s^{k+1}}$ are functionally independent. If $\dim(F_{0,k})<\dim(F_{0,k+1})$ then set $k=k+1$ and goto (1).
(3) Eliminate a minimal number of equations of F_1, such that no more linear combinations of the form (7) exist ($F_{1,end}\subset F_1$).
Result: $[F_{1,end},F_{0,k}]$

Remark 4 *The equations g^{i_0} of step (1) constitute a basis (namely a Groebner basis) for the elimination ideal $I_{n_z}=I\cap\mathbb{R}\left[t,z^{\alpha_z}\right]$.*

Remark 5 *The efficiency of the algorithm can be improved by using a different ordering for $\left(t,z^{(1)}\right)$, however, no investigations were made into this direction so far.*

3 Applications

In the following we want to show, how the observability and accessibility analysis for formally integrable systems (9a), (9b) with output $y\in\mathcal{Y}\subseteq R^{n_y}$

$$y^{\alpha_y}=c^{\alpha_y}(t,z),\ \alpha_y=1,\ldots,n_y \tag{14}$$

can be approached with the tools set up in the foregoing. We again assume, that equation (14) defines a regular submanifold $\mathcal{S}^1\subset J^1\mathcal{E}$. Since we want to focus on the computational implementation, we mainly summarize the results here and refer for more details on this approach via Lie groups to [12].

3.1 The Observability Test for Implicit Systems

Let us consider a 1-parameter transformation (Lie) group $\Phi_\varepsilon:(t,z)\to(t,\bar{z})$ that acts on the solutions of (9). Its infinitesimal generator v and first prolongation $j(v)$ (see e.g. [7] or [9]) has the form

$$v=Z^{\alpha_z}\partial_{\alpha_z}\ ,\quad j(v)=v+d_1\left(Z^{\alpha_z}\right)\partial^1_{\alpha_z}\ . \tag{15}$$

If such a Lie group exists, that transforms solutions of (9) to other solutions, and leaves the output (14) invariant, i.e., $j(v)$ of (15) satisfys the conditions

$$j(v)\left(f^{\alpha_x}\right)\ =\ 0\ ,\ v\left(c^{\alpha_y}\right)=0\ , \tag{16a}$$
$$v\left(\varphi^{\alpha_s}\right)\ =\ 0\ ,\ v\left(\varphi^{\alpha_u}\right)=0\ , \tag{16b}$$

with the functions $\varphi^{\alpha_s},\varphi^{\alpha_u}$ from (10), then we say the system is not observable. To test, whether non trivial solutions for v exist we have to calculate the formally integrable system of (16) and (9) according to Algorithm 1. With this approach the check for observability is reduced to a matrix rank condition. For explicit systems of the form

$$x_1^{\alpha_x}\ =\ f^{\alpha_x}(x,u)\ ,\quad \alpha_x=1,\ldots,n_x \tag{17}$$
$$y^{\alpha_y}\ =\ c^{\alpha_y}(x,u)\ ,\quad \alpha_y=1,\ldots,n_y \tag{18}$$

test is equivalent to the well known criteria for observability one can find in text books like [5] and [6].

Algorithm 6 *Observability test for system (14).*
Input: $\left\{\bar{f}^{\alpha_x}\right\},\left\{\bar{f}^{\alpha_s}\right\},[t,z^{\alpha_z}],c^{\alpha_y},\varphi^{\alpha_u}$
(1) Calculate the system in formal integrable form $[\{f^{\alpha_x}\},\{f^{\alpha_s}\}]=\texttt{wellpose}\left(\left\{\bar{f}^{\alpha_x}\right\},\left\{\bar{f}^{\alpha_s}\right\},[t,z^{\alpha_z}]\right)$.
(2) Set up the vector field $j(v)=Z^{\alpha_z}\partial_{\alpha_z}+Z_1^{\alpha_z}\partial^1_{\alpha_z}$.
(3) Calculate $F_1=\{\langle j(v),\mathrm{d}f^{\alpha_x}(t,z,z_1)\rangle\}$,

$F_{0,0} = \left\{ \langle \mathrm{d}c^{\alpha_y}, j(v) \rangle, \langle \mathrm{d}f^{i_s}, j(v) \rangle, \langle \mathrm{d}\varphi^{\alpha_u}, j(v) \rangle \right\}$.
(4) Set $f_1 = \{ f^{\alpha_x}, d_1(f^{\alpha_s}) \}$.
(5) Define the projection $pr_2 : \mathcal{JTE} \to \mathcal{TE}$ *of the form* $(t, z, Z, z_1, Z_1) \to (t, z, Z)$ *given by elimination via the functions* F_1 *and* f_1.
Iteration (k):
(1) Calculate the prolongation $F_{0,k}^{(1)} = d_1(F_{0,k})$. *Find the set of all constraints* C_k *via the projection* π_0^1 *according to* $C_k = \pi_0^1\left(F_1 \cup F_{0,k}^{(1)}\right)$, *and add them to* $F_{0,k}$ *to dervive* $F_{0,k+1} = F_{0,k} \cup C_k$.
(2) Eliminate a minimal number of equations $F_{0,k+1} = \{g^j\}$, *such that the* g^j *are functionally independent.*
If $\dim(F_{0,k}) < \dim(F_{0,k+1})$ *then set* $k = k+1$ *and goto (1).*
Result:
If $\dim(F_{0,k}) = n_x$ *then the system is observable.*
Else the system is not observable.

Remark 7 *Again we can improve this algorithm for systems, where the functions* f^{α_x} *and* f^{α_s} *are polynomials in* $\left(t, z^{(1)}\right)$. *The following algorithm shows, how* π_0^1 *of step (1) can be implemented with the help of Groebner bases.*

Algorithm 8 *Observability test for system (14) with* $\bar{f}^{\alpha_x}, \bar{f}^{\alpha_s} \in \mathbb{R}[t, z^{\alpha_z}, z_1^{\alpha_z}]$.
Input: $\{\bar{f}^{\alpha_x}\}, \{\bar{f}^{\alpha_s}\}, [t, z^{\alpha_z}], c^{\alpha_y}, \varphi^{\alpha_u}$
(1)-(4) like in Algorithm 6.
(5) Define a lexicographic ordering $>_{lex}$ *in the form* $z_1^{\alpha_z} > Z_1^{\alpha_z} > z^{\alpha_z} > Z^{\alpha_z} > t$.
Iteration (k):
(1) Use $>_{lex}$ *of step (5) to calculate the Groebner basis corresponding to the set* $\{F_1, F_{0,k}, f_1\}$. *This results in an equation system of the form* $\left\{g^{i_1}(t, Z, z, Z_1, z_1) = 0,\ g^{i_0}(t, Z, z) = 0\right\}$, *with the* n_1 *equations* g^{i_1} *of order one and* n_0 *equations* g^{i_0} *of order zero. Add these equations to the set of restrictions, i.e. set* $F_{0,k+1} = F_{0,k} \cup \{g^{i_0}\}$.
(2) Eliminate a minimal number of equations $F_{0,k+1} = \{g^j\}$, *such that the* g^j *are functionally independent. If* $\dim(F_{0,k}) < \dim(F_{0,k+1})$ *then set* $k = k+1$ *and goto (1).*
Result:
If $\dim(F_{0,k}) = n_x$ *then the system is observable.*
Else the system is not observable.

3.2 The Accessibility Test for Implicit Systems

For the accessibility analysis we consider transformation groups $\Phi_\varepsilon : (t, z) \to (t, \bar{z})$ acting on the variables z, their infinitesimal generators v having the form (15), and ask whether there exist common invariants I of all possible v satisfying

$$
\begin{aligned}
v(I) &= \langle \omega, v \rangle = 0\,, \quad \mathrm{d}I = \omega + \partial_1 I \mathrm{d}t\,, && (19)\\
\omega &= \omega_{\beta_z} \mathrm{d}z^{\beta_z}\,. && (20)
\end{aligned}
$$

Like in the last section, we have to formulate appropriate conditions for v. Of course, the group action has to leave the system equations of the formally integrable system (9) invariant, which can be stated in the form

$$
\begin{aligned}
0 &= j(v)(f^{\alpha_x})\,, && (21a)\\
0 &= v(f^{\alpha_s}) = v(\varphi^{\alpha_s}) = j(v)(d_1\varphi^{\alpha_s})\,. && (21b)
\end{aligned}
$$

If non trivial invariants I of the form (19) exist for all solutions v satisfying (21), we say the system is not accessible. By rearranging the (Z^{α_z}) of v into $(Z^{\alpha_x}, Z^{\alpha_s}, Z^{\alpha_u})$ such that we can solve the linear equation (21b) for Z^{α_s} in the form $Z^{\alpha_s} = c_{\alpha_x}^{\alpha_s} Z^{\alpha_x} + c_{\alpha_u}^{\alpha_s} Z^{\alpha_u}$ and eliminate the variables Z^{α_s}, $d_1 Z^{\alpha_s}$ in the equations (21a) and (19) we get

$$
\begin{aligned}
j(v)(f^{\alpha_x}) &= a_{\gamma_x}^{\alpha_x} d_1 Z^{\gamma_x} + a_{\gamma_u}^{\alpha_x} d_1 Z^{\gamma_u} \\
&\quad + b_{\gamma_x}^{\alpha_x} Z^{\gamma_x} + b_{\gamma_u}^{\alpha_x} Z^{\gamma_u} && (22)\\
0 = d_1(v(I)) &= d_1\bar{\omega}_{\gamma_x} Z^{\gamma_x} + d_1\bar{\omega}_{\gamma_u} Z^{\gamma_u} \\
&\quad + \bar{\omega}_{\gamma_x} d_1 Z^{\gamma_x} + \bar{\omega}_{\gamma_u} d_1 Z^{\gamma_u} && (23)
\end{aligned}
$$

with $\bar{\omega}_{\gamma_i} = \omega_{\gamma_i} + c_{\gamma_i}^{\alpha_s}\omega_{\alpha_s}$, $i \in \{x, s\}$. Due to the formal integrability of (9) we can solve (22) for $d_1 Z^{\gamma_x}$ and eliminate these variables from (23), which has to vanish independently from the choice of Z^{γ_x}, Z^{γ_u} and $d_1 Z^{\gamma_u}$. From this claim, we directly get (with $\hat{a}_{\alpha_x}^{\gamma_x} a_{\beta_x}^{\alpha_x} = \delta_{\beta_x}^{\gamma_x}$)

$$
\begin{aligned}
d_1\bar{\omega}_{\gamma_x} &= \bar{\omega}_{\gamma_x} \hat{a}_{\alpha_x}^{\gamma_x} b_{\gamma_x}^{\alpha_x}\,, && (24a)\\
d_1\bar{\omega}_{\gamma_u} &= \bar{\omega}_{\gamma_x} \hat{a}_{\alpha_x}^{\gamma_x} b_{\gamma_u}^{\alpha_x}\,,\ \bar{\omega}_{\gamma_u} = \bar{\omega}_{\gamma_x} \hat{a}_{\alpha_x}^{\gamma_x}\,, && (24b)
\end{aligned}
$$

the partial differential equations (24) defining the solution space for ω. Now, we have to check by calculating the formally integrable form for (24a, 24b, 9) whether this solution space is non trivial. Again, for explicit systems of the form (17) this result coincides with the well known criteria one can find in textbooks like [5], [6].

Remark 9 *It is worth mentioning, that the invariants* I *in (19) may depend on the input. This might sound a bit strange, however, one can easily state even very simple examples of explicit systems of the form (11) which admit such invariants. We just mention the system* $\dot{x} = \dot{u}$ *here. If we are not interested in this type of invariants, only slight modifications to the presented algorithms are necessary (see also [12]).*

For the accessibility test algorithms, analogous to Algorithms 6 and 8 for the test on observability can be stated, however, due to lack of space we confine ourselves with the provided sketch of an implementation for the accessibility test in this section.

4 An example: The PVTOL-Aircraft

According to [10] a mathematical model for a planar vertical take-off and landing aircraft (PVTOL) has the form

$$
\begin{aligned}
0 &= z_1^1 - z^2 && (25a)\\
0 &= z_1^2 - \left(-z^7 \sin\left(z^5\right) + \varepsilon z^8 \cos\left(z^5\right)\right) && (25b)\\
0 &= z_1^3 - z^4 && (25c)\\
0 &= z_1^4 - \left(z^7 \cos\left(z^5\right) + \varepsilon z^8 \cos\left(z^5\right) - 1\right) && (25d)\\
0 &= z_1^5 - z^6 && (25e)\\
0 &= z_1^6 - z^8 && (25f)\\
0 &= f(t) - \left(z^1 - \varepsilon \sin\left(z^5\right)\right)\,. && (25g)
\end{aligned}
$$

We added the equation (25g), which represents a trajectory profile, the airplane is intended to follow. In order to test the restricted mathematical model (25) on accessibility and observability and to show, how Groebner bases can be successfully applied, we substitute $\cos\left(z^5\right) = cz^5$, $\sin\left(z^5\right) = sz^5$ and add the equation $\left(sz^5\right)^2 + \left(cs^5\right)^2 = 1$. The formally integrable form for (25) is given by the equations

$$
\begin{aligned}
0 &= z_1^3 - z^4 && \text{(26a)}\\
0 &= z_1^4 - \left(z^7 cz^5 + \varepsilon z^8 cz^5 - 1\right) && \text{(26b)}\\
0 &= z_1^5 - z^6 && \text{(26c)}\\
0 &= z_1^6 - z^8 && \text{(26d)}
\end{aligned}
$$

and the algebraic restrictions

$$
\begin{aligned}
0 &= \left(sz^5\right)^2 + \left(cs^5\right)^2 - 1 && \text{(27a)}\\
0 &= f\left(t\right) - \left(z^1 - \varepsilon sz^5\right) && \text{(27b)}\\
0 &= -z^2 + \varepsilon cz^5 z^6 + f_1(t), && \text{(27c)}\\
0 &= z^7 sz^5 - \varepsilon sz^5 \left(z^6\right)^2 + f_2(t)\,. && \text{(27d)}
\end{aligned}
$$

So we see that, by adding the restriction (25g), two more hidden constraints confine the system dynamics and the state dimension reduces to 4. For the accessibility test we set up the partial differential equations in the components $\bar{\omega}$ according to (24) and derive a formally integrable form with 4 (functionally independent) algebraic restrictions in addition to the 4 restrictions implied by (27). This means, there is no non trivial solution for ω, no invariant I according to (19) exists, and the equations (25) are accessible. For the observability test with respect to the output $y = z^3$ we derive for the formally integrable form of (16) 4 additional restrictions in addition to the 4 implied by (27), thus the system is observable from the output $y = z^3$. The choise $y = z^1$ leads to a formally integrable form of (16) with a total of 6 algebraic restrictions and 2 partial differential equations in the components of v in the form

$$\left\{d_1 Z^3 - Z^4,\; d_1 Z^4 - Z^7 cz^5 + \left(z^7 sz^5 - \varepsilon z^8 cz^5\right) Z^5\right\}.$$

These equations admit a non trivial solution for Z^3 and Z^4 and we conclude, that system (25) is non observable from the output $y = z^1$.

5 Conclusions

This contribution deals with the application of computer algebra methods in the analysis of systems of implicit ordinary differential equations. A system description, independent of the chosen coordinate system and representation in form of equations is set up, by identification of the system with its corresponding submanifold in a suitable jet space. To do this, a corresponding system in formally integrable form has to be derived and an algorithm which performs the required steps has been presented.

For systems non-linear in the highest order derivatives the elimination tasks, which have to be performed, are not straightforward symbolically. However, if the nonlinearities in f^{α_x} are polynomials we can use Algorithm 3 for the computation, which utilizes Groebner bases to perform the elimination. As applications of this approach, algorithms to solve the observability and accessibility problem for formally integrable implicit systems, based on transformation groups, have been sketched. The presented algorithms are implemented in a package DAEEXTALG for the computer algebra system MAPLE. An example namely the application of the package to the planar VTOL model has been shown.

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