CONTROL OF PORT-INTERCONNECTED DRIFTLESS SYSTEMS

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Abstract. We use a recently reported extended Matrosov's theorem to establish sufficient and necessary conditions to stabilize a chain of driftless systems. This benchmark can be regarded as a problem of controlling a communication channel and is a generalization of the well known chained-form non holonomic systems benchmark. Roughly speaking our extended Matrosov's theorem is an extension of La Salle's invariance principle to the case of non autonomous systems.

1 Introduction

Many nonlinear control algorithms rely heavily on analysis tools that establish convergence to the origin for trajectories of a time-varying nonlinear system having a uniformly stable origin. For time-invariant problems, the typical analysis tool used is the Krasovski/LaSalle invariance principle. It is the key result that leads to the so-called Jurjevic/Quinn control algorithm for open-loop stable nonlinear control systems. When the closed-loop is time-varing, one tool that is often used is Barbalat's lemma. In adaptive control, Barbalat's lemma is frequently relied upon to establish convergence to zero of part of the state. Barbalat's lemma has also been used to establish convergence to the origin for a class of non holonomic systems controlled by smooth time-varying feedback. For time-varying systems, another tool that has been used, but more sparingly, is Matrosov's theorem [6, 11]. It was used e.g. in [9] to establish one of the first results on uniform global asymptotic stability (UGAS) of robot manipulators in closed loop with a tracking controller. It also appears in the context of adaptive control in [5] and output feedback control in [8].

In the recent paper [4] we presented a result which is an extension in certain directions, of Matrosov's theorem. In particular, our result relies on the ability of finding a (non a priori fixed) number of auxiliary functions which do not have sign-definite derivatives. Also, as in [9] the bounds on these functions derivatives are allowed to be time-varying. In this paper we present a case-study which illustrates the utility of the main result in [4] as a tool to aid control design. In particular, we will address the problem of stabilizing a chain of port-interconnected and port-controlled driftless systems and establish sufficient and necessary conditions for uniform global asymptotic stability of the closed loop system. As we will see, this class of systems include as particular case, the non holonomic systems in chained form.

The rest of the paper is organized as follows. In next section we present some definitions and recall our extended Matrosov theorem from [4]. In Section 3 we present our main result. We conclude in Section 4 with some remarks.

2 Preliminaries

For two constants $\Delta \geq \delta \geq 0$ we define $\mathcal{B}(\Delta) := \{x \in \mathbb{R}^n : |x| \leq \Delta\}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\gamma \in \mathcal{K}_\infty$ if in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}$ if it is continuous and positive definite. We denote by $x(\cdot, t_\circ, x_\circ)$, the solutions of the differential equation

$$\dot{x} = f(t, x) \tag{1}$$

with initial conditions $(t_\circ, x_\circ)$. For a function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we define its derivative in the direction $(1, f(t,x)^T)^T$ as $\dot{V}(t,x) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$. This abuse of notation is reasonable because of (1). Furthermore, with an abuse of notation we will use the same definition to express the derivative of locally Lipschitz functions. For the latter, $\dot{V}(t,x)$ is defined everywhere except on a set of measure zero where the gradient of V is not defined, i.e., almost everywhere. When clear from the context and to simplify the notation we will also use $\nabla f(x)$ to denote $\frac{\partial f(x)}{\partial x}$.

Definition 1 (Uniform global stability) The origin of the system (1) is said to be uniformly globally stable (UGS) if there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R} \times \mathbb{R}^n$ each solution $x(\cdot, t_\circ, x_\circ)$ satisfies

$$|x(t, t_\circ, x_\circ)| \leq \gamma(|x_\circ|) \qquad \forall\, t \geq t_\circ \; . \tag{2}$$

Definition 2 (Uniform global attractivity) The origin of the system (1) is said to be uniformly globally attractive if for each r, $\sigma > 0$ there exists $T > 0$ such that

$$|x_\circ| \leq r \implies \|x(t, t_\circ, x_\circ)\| \leq \sigma \qquad \forall\, t \geq t_\circ + T \; . \tag{3}$$

Furthermore, we say that the (origin of the) system is uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive.

The following theorem is a generalization of the so-called Matrosov theorem which combines an auxiliary function with a Lyapunov function that establishes UGS. See [6] and the more recent expositions [11, Theorem 5.5, p.58] and [10, Theorem 2.5, p. 62]. This is the fundamental tool that we use to prove our main result.

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Theorem 1 [4] Under the following assumptions the origin of system (1) is UGAS.

Assumption 1 The origin of system (1) is UGS.

Assumption 2 There exist integers j, $m > 0$ and for each $\Delta > 0$ there exist: 1) a number $\mu > 0$, 2) a locally Lipschitz continuous functions $V_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, j\}$, 3) a continuous function $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, 4) a continuous functions $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \{1, \ldots, j\}$ such that, for almost all $(t, x) \in \mathbb{R} \times \mathcal{B}(\Delta)$,

$$\max\{|V_i(t,x)|\,,\ |\Phi(t,x)|\} \le \mu, \tag{4}$$
$$\dot{V}_i(t,x) \le Y_i(x, \Phi(t,x))\,. \tag{5}$$

Assumption 3 For each integer[2] $k \in \{1, \cdots, j\}$

$$\text{(A): } \{\ (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ ,\ Y_i(z,\psi) = 0 \quad \forall i \in \{1, \ldots, k-1\}\ \ \}$$

implies (B): $\{\ Y_k(z,\psi) \le 0\ \}$.

Assumption 4 We have that

$$\text{(A): } \{\ (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ , \quad Y_i(z,\psi) = 0 \quad \forall i \in \{1, \ldots, j\}\ \}$$

implies (B): $\{\ z = 0\ \}$. □

3 Control over communication channels

We present now our main result. We will address the problem of stabilizing by smooth feedback, a series of port-interconnected driftless systems as illustrated in Figure 1. This control problem covers that of time-varying smooth feedback stabilization of chained-form nonholonomic systems (cf. [1, 7]) and we will solve it by appealing to Theorem 1. In Figure 1, each block contains a bank of integrators with nonlinearities at the input and output of the integrator. The dynamics of each block is is given by

$$\Sigma_i : \quad \begin{cases} \dot{x}_i &= B_i(x_i) u_i \\ y_i &= h_i(x_i) = B_i(x_i)^T \nabla W_i(x_i) \end{cases} \tag{6}$$

where $i \le n$ y_i, $u_i \in \mathbb{R}^{p_i}$, $x_i \in \mathbb{R}^{n_i}$ and the functions in the dynamics above satisfy the following conditions

$$\nabla h(x_i) B_i(x_i) \ \ge\ c_i I\,, \qquad c_i > 0 \tag{7}$$
$$|B_i(x_i)| \ \le\ \rho_{B_i}(|x_i|)\,, \qquad \rho_{B_i} \in \mathcal{N} \tag{8}$$
$$W_i(x_i) \ \ge\ \alpha_i(|x_i|)\,, \qquad \alpha_i \in \mathcal{N} \tag{9}$$
$$\nabla W_i(x_i) \ \le\ \rho_{W_i}(|x_i|)\,, \qquad \rho_{W_i} \in \mathcal{K}_\infty \tag{10}$$
$$\theta_i(|x_i|) \ge |y_i| \ \ge\ \kappa_i(|x_i|)\,, \qquad \kappa_i \in \mathcal{PD}\,, \theta_i \in \mathcal{N}\,. \tag{11}$$

The nonlinear integrator blocks are interconnected via static, nonlinear, time-varying "communication channels". In particular, the connection between blocks i and $i+1$ is modeled by the nonlinear gain function g_i which takes values in $\mathbb{R}$ and may depend, in general, on any of the states, time, and perhaps some additional states from outside of the network. This means that the input from the left to the ith block, denoted $u_{i,\ell}$, and the input from the right to the ith block, denoted $u_{i,r}$, are

$$\begin{aligned} u_{i,\ell} &= g_{i-1} \cdot y_{i-1} \\ u_{i,r} &= g_i \cdot y_{i+1}\,. \end{aligned}$$

The blocks are such that the input to the nonlinear integrator is given by

$$u_i = u_{i,r} - u_{i,\ell}\ .$$

We assume that the communication channel gains have the functional form

$$\dot{z}_i = -z_i + \widetilde{g}_{i,a}(t,x) \tag{12a}$$
$$\widetilde{g}_i(t,x,z) = -z_i + \widetilde{g}_{i,a}(t,x) \tag{12b}$$
$$g_i(t,x,z) = \prod_{j=i}^{n-1} \widetilde{g}_i(t,x,z) \tag{12c}$$

for $i \le n-1$ and where the functions $\widetilde{g}_{i,a}$ are continuous and Lipschitz in x uniformly in t.

The control problem is to attach a system Σ_{n+1} to the right of Σ_n, and give necessary and sufficient conditions on the communication channel gains to guarantee that the origin for the closed-loop system is uniformly globally asymptotically stable. For the controller, we will use any static strict "first and third sector" nonlinearity $\sigma(\cdot)$, and the connection to Σ_n will be made with a reliable communication channel, e.g., $g_n \equiv 1$. In particular, we have

$$\Sigma_{n+1} : \quad \begin{cases} y_{n+1} &= \sigma(u_{n+1}) \\ u_{n+1}^T \sigma(u_{n+1}) &\ge \rho(|u_{n+1}|) \qquad \rho \in \mathcal{PD} \\ u_{n+1} &= y_n \\ y_{n+1} &= u_n\ . \end{cases} \tag{13}$$

With this controller architecture and functional form for the communication channel gains indicated in Figure 1, we ask the question:

> *What are necessary and sufficient conditions on the communication channel gains to guarantee uniform asymptotic stability of the origin for the system (6)-(12)?*

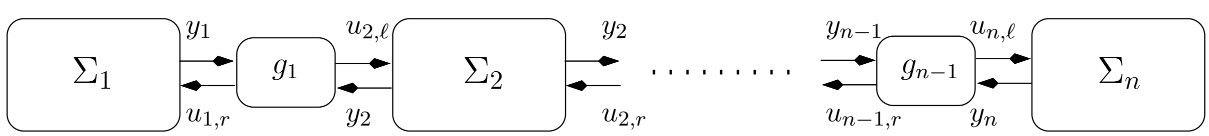

Figure 1: Communication channels.

[2] For the case that $k = 1$ one should read that $Y_1(z,\psi) \le 0$ for all $(z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)$.

The answer will be expressed in terms of the notion of "uniform δ-persistency of excitation" (see [3]) which is defined next for completeness.

Definition 3 (Uniform δ-persistency of excitation (Uδ-PE)) The function $(t,\xi)\mapsto\varphi(t,\xi)\in\mathbb{R}^p$ is said to be uniformly δ persistently exciting if for each pair of strictly positive real numbers $\delta\le\Delta$ there exist $T>0$ and $\mu>0$ such that

$$t\in\mathbb{R}\,,\quad \delta\le|\xi|\le\Delta \qquad\Longrightarrow\qquad \mu\le\int_t^{t+T}|\varphi(\xi,\tau)|d\tau\,.$$

□

We will also make use of the following observation.

Fact 1 For locally Lipschitz functions, uniformly in t, and such that $\phi_i(t,0)\equiv 0$ if the product $\prod_{i=1}^n\phi_i(t,x)$ is Uδ-PE then, necessarily each function $\phi_i(t,x)$ is Uδ-PE.

Note that in the definition above, x is a constant parameter hence, the necessary and sufficient conditions for stability will be expressed in terms of the state x, being constant. To that end note that when x is constant the z_i subsystems in (12) are time-invariant linear systems with time-varying inputs. To better see this, let $x_n\equiv 0$, let $x=\bar{x}$ with $\bar{x}$ a constant vector and call $\bar{z}$ the new state of the *linear* system (12) in this setting. Then, defining

$$\tilde{g}^\circ_{i,a}(t,x):=\tilde{g}_{i,a}(t,x)\mid_{x_n=0} \qquad (14)$$

it is direct to show that the ith communication channel gain in (12) satisfies

$$\begin{aligned}\dot{\tilde{g}}^\circ_i(t,\bar{x},\bar{z}) &= -\bar{z}_i+\tilde{g}^\circ_{i,a}(t,\bar{x})+\frac{d\tilde{g}^\circ_{i,a}}{dt} &(15)\\ \dot{\tilde{g}}^\circ_i(t,\bar{x},\bar{z}) &= -\tilde{g}^\circ_i(t,\bar{x},\bar{z})+\frac{\partial\tilde{g}^\circ_{i,a}}{\partial t} &(16)\end{aligned}$$

and we note that the steady-state solution of (16) is given by

$$\omega_i(t,\bar{x}):=\int_{-\infty}^t e^{-(t-\tau)}\psi_i(\tau,\bar{x})d\tau \qquad i\in[1,\ldots,n-1] \qquad (17)$$

where

$$\psi_i(t,\bar{x}):=\frac{\partial\tilde{g}^\circ_{i,a}(t,\bar{x})}{\partial t}\,. \qquad (18)$$

That is, $\omega_i(t,\bar{x})$ is the steady-state value of the ith communication channel gain and correspondingly, the steady-state value of the gain of the first communication channel can be computed to be the product of all the $\omega_i(t,\bar{x})$'s for all $i\le n-1$. Based on these observations, we are now ready to present our main result.

Theorem 2 Suppose the function $g_{i,a}(\cdot,\cdot)$ is continuous and locally Lipschitz in x uniformly in t. The origin of the system (6)-(12) is uniformly globally asymptotically stable (UGAS) if $\tilde{g}_{i,a}(t,0)\equiv 0$ and the function

$$(t,x_1,\ldots,x_{n-1})\mapsto\prod_{i=1}^{n-1}\omega_i(t,x)\Bigg|_{x_n=0} \qquad (19)$$

is *Uδ*-PE. Moreover, the origin is UGAS only if $\tilde{g}^\circ_{i,a}(t,x)$ is Uδ-PE with respect to $\xi:=\ \mathrm{col}[x_1,\ldots,x_{n-1}]$. □

Roughly speaking, the sufficient condition for UGAS is that the first communication channel gain be Uδ-PE, at least when $x_n=0$. From the structure of the gains in (12) and Fact 1, this implies that each of the communication channel gains is Uδ-PE when $x_n=0$. In asking that the first communication channel gain be Uδ-PE we are not asking that this channel is always functioning but rather that each communication channel is functioning on average. Moreover, this average should be uniform in time. However, the average does not need to be uniform in the state. For example, the quality of the communication channel could possibly degrade as the "power" of the transmission signals, perhaps encoded by the size of the states x_i, decreases to zero. This idea is captured by the notion of Uδ-PE.

Remark 1 It is worth mentioning that the system architecture above covers the so-called "skew-symmetric" systems considered in [12, 2]. We may see this if we let $\tilde{g}_i=u_1$ for all i where u_1 is one of the two control inputs in those references (in particular, the controller which is required to be Uδ-PE in [2]), $y_i=x_i$ for each $2\le i\le n$ we replace z_i by[3] x_1 and finally, we relate the function $\tilde{g}_{i,a}$ to the function whose second derivative in [2] is required to be Uδ-PE or, to the "heat function" in [12]. □

3.1 Proof of Theorem 2

We first point out the following identities.

$$\forall\, i\in[1,\ldots,n-2],\quad g_i=\tilde{g}_i\,g_{i+1},\quad g_{n-1}=\tilde{g}_{n-1},\quad g_n>0\,. \qquad (20)$$

Furthermore, using these identities one can show that for all $i\in[1,\ldots,n-2]$,

$$\begin{aligned} g_i &= [g_{n-1}\cdots g_i]^{1/n-i}\,\tilde{g}_{n-2}^{1/n-i}\cdots\tilde{g}_{i+j}^{n-i-j-1/n-i}\cdots\\ &\quad\times\tilde{g}_i^{n-i-1/n-i} &(21a)\\ g_{n-1} &= \tilde{g}_{n-1}\,. &(21b)\end{aligned}$$

$$\begin{aligned}|g_i\,y_i| &= \underbrace{|[g_{n-1}\cdots g_i]|^{1/n-i}\;|y_i|^{1/n-i}}_{\phi_i^{1/n-i}}\;|y_i|^{-1+n-i/n-i}\\ &\quad\times\left|\tilde{g}_{n-2}^{1/n-i}\cdots\tilde{g}_i^{n-i-1/n-i}\right| &(22a)\\ |g_{n-1}|\,|y_{n-1}| &= \phi_{n-1}\,. &(22b)\end{aligned}$$

3.1.1 Necessity: It follows applying [3, Theorem 1] which states that if the origin of $\dot{x}=F(t,x)$ with $F(\cdot,\cdot)$ locally Lipschitz in x uniformly in t, is UGAS then, necessarily $F(\cdot,\cdot)$ is Uδ-PE. Then, defining $F(t,x)$ as the right hand side of the closed loop system, and since $B_i(x_i)$ is uniformly bounded for each $\Delta>0$ and all $x\in\mathcal{B}(\Delta)$ we have that $F(\cdot,\cdot)$ is Uδ-PE with respect to ξ only if $g_i(t,x,z)^2$ is Uδ-PE with respect to ξ for any i. Hence, in view of (12) we have that each $\tilde{g}_i(t,x,z)$ is Uδ-PE with respect to ξ which in turn implies that $\tilde{g}^\circ_{i,a}(t,x)$ is Uδ-PE with respect to ξ for all i.

[3]Here, the index i does not make sense because we have only one z-state, this is because all the gains $\tilde{g}_i=u_1$.

3.1.2 Sufficiency: Proof of UGS: The Lyapunov function

$$V_1(x) := \sum_{i=1}^{n} W_i(x_i)\ , \tag{23}$$

which is positive definite and proper with respect to x, has the property that

$$\dot V_1(x) \le -\rho(|y_n|) \le 0 \tag{24}$$

regardless of the properties of the communication channels. So, with the guarantee of local existence of solutions, we have that

$$|x(t)| \le \gamma(|x_\circ|) \qquad \forall t \ge t_\circ\ ,\ x_\circ \in \mathbb{R}^n\ . \tag{25}$$

Technically, we only have this bound on the maximal interval of definition. But with x bounded on the maximal interval of definition and the properties of g_i, it follows that

$$|z_i(t)| \le |z_{i,\circ}| + \gamma(||x||_\infty) \tag{26}$$

on the maximal interval of definition. Thus, solutions are defined for all time and, in fact, the origin is UGS.

Proof of UGA (with a family of sign-indefinite Lyapunov functions): We will define $3n-3$ functions which may be classified in 3 groups of $n-1$ functions. The first group of functions is defined as follows

$$\begin{aligned} V_2(t,x,z) &:= y_n \cdot g_{n-1} \cdot y_{n-1} && (27)\\ V_3(t,x,z) &:= y_{n-1} \cdot g_{n-2}\, g_{n-1}^2 \cdot y_{n-2} && (28) \end{aligned}$$

where '$\cdot$' denotes the scalar product and for $i = 4, \cdots, n$,

$$V_i(t,x,z) := y_{n-i+2} \cdot g_{n-i+1}\, g_{n-i+2}^2 \cdots g_{n-1}^2 \cdot y_{n-i+1}\,. \tag{29}$$

For clarity we recall that $y_{n-i+2} \in \mathbb{R}^{p_i}$ and the gains g_k are scalar functions. We proceed to compute some bounds for the time derivative of $V_i(t,x,z)$. To that end we first notice first that

$$\dot y_j = \nabla h_j(x_j) B_j(x_j) \left[g_j y_{j+1} - g_{j-1} y_{j-1} \right], \qquad j \in [2, \ldots, n]\,. \tag{30}$$

Long but straightforward calculations which involve the use of the bounds (7)–(11) and show that for each $\Delta > 0$ there exists of $\nu > 0$ such that for all $(x,z) \in \mathcal{B}(\Delta)^2$, we have

$$\begin{aligned} \dot V_2(t,x,z) &\le -c_n |g_{n-1} y_{n-1}|^2 + \nu(|y_n| + |\sigma(y_n)|) && (31a)\\ \dot V_3(t,x,z) &\le -c_{n-1} |g_{n-2} g_{n-1} y_{n-2}|^2 \\ &\quad + \nu\left(|y_n| + |g_{n-1} y_{n-1}|\right) && (31b)\\ \dot V_i(t,x,z) &\le -c_{n-i+2} |g_{n-i+1} \cdots g_{n-1} y_{n-i+1}|^2 && (31c)\\ &\quad + \nu\left(|g_{n-i+2} \cdots g_{n-1} y_{n-i+2}| + |g_{n-i+3} \cdots g_{n-1} y_{n-i+3}|\right) \end{aligned}$$

for all $i \in [4, \ldots, n]$ and where the coefficients c_i come from (7).

Then, using (20) and (22) successively we obtain that for all $i \in [2 \ldots n-2]$,

$$\begin{aligned} |g_i y_{i+1}| &\le |[g_{n-1} \cdots g_{i+1}]|^{1/n-i-1}\, |y_{i+1}|^{1/n-i-1} \times \\ &\quad |y_{i+1}|^{n-i-2/n-i-1} \left| \widetilde g_{n-2}^{\,1/n-i-1} \cdots \widetilde g_{i+1}^{\,n-i-2/n-i-1} \widetilde g_i \right| \end{aligned} \tag{32}$$

so defining the following $n-1$ functions for all $i \in [3, \ldots, n]$,

$$\begin{aligned} \phi_{n-1}(t,x,z) &:= |g_{n-1}(t,x,z)|\, |h_{n-1}(x_{n-1})| && (33a)\\ \phi_{n-i+1}(t,x,z) &:= |g_{n-i+1}(t,x,z) \cdots g_{n-1}(t,x,z)| \times \\ &\quad\ |h_{n-i+1}(x_{n-i+1})| && (33b) \end{aligned}$$

we can rewrite the inequalities in (31) as

$$\begin{aligned} \dot V_2(t,x,z) &\le -c_n \phi_{n-1}^2 + \nu(|y_n| + |\sigma(y_n)|) && (34a)\\ \dot V_3(t,x,z) &\le -c_{n-1} \phi_{n-2}^2 + \nu\left(|y_n| + \phi_{n-1}\right) && (34b)\\ \dot V_i(t,x,z) &\le -c_{n-i+2} \phi_{n-i+1}^2 + \nu\left(\phi_{n-i+2} + \phi_{n-i+3}\right). && (34c) \end{aligned}$$

Remark 2 We wish to emphasize the way the functions V_1 to V_n defined so far, are ordered. Intuitively, from $\dot V_1$ we may think (following Barbalat's lemma) that $y_n \to 0$ and therefore, $x_n \to 0$ asymptotically. To make this precise, and for the use of our Matrosov's theorem what is important to observe is that $\dot V_2 \le 0$ on the set where the bound on $\dot V_1$ is zero, that is when $y_n \equiv 0$. Accordingly, each of the bounds on the succeeding $\dot V_i$'s contain three essential terms: the first is a negative term of ϕ_{n-i+1}, the next two correspond to a number ν times ϕ_{n-i+2} and ϕ_{n-i+3} which appear squared and with sign '-' in the previous two derivatives $\dot V_{i-1}$ and $\dot V_{i-2}$ respectively. This hints at the idea that one should be able to recursively show that if $y_n \to 0$ then, so does ϕ_{n-1} hence also ϕ_{n-2}, etc.[4] Then, the Uδ-PE assumption will be used essentially to imply that if $\phi_i \to 0$, necessarily $x_i \to 0$. □

The next step is to construct a group of functions to conclude on the behavior of the gains, more precisely to show that the communication channel gains converge to their steady state solution as it was briefly discussed above. To that end, we introduce the function

$$\zeta_i(t,x,z) := z_i - \widetilde g_{i,a}^{\circ}(t,x) + \omega_i(t,x)\,, \qquad i \in [1, \ldots, n-1] \tag{35}$$

which can be regarded to some extent as the error between the ith channel communication gain and its steady state solution. Then, the next group of $n-1$ functions that we introduce is defined by

$$V_{n+i}(t,x,z) := \zeta_i(t,x,z)^2\,. \tag{36}$$

The aim is to bound (on compact sets of the states x and z) the total derivatives of these functions with terms of the type $-\zeta_i(t,x,z)^2$ plus terms involving the $|\phi_i(t,x,z)|$'s defined in the previous group of functions. To that end we need to introduce some useful identities and bounds on certain functions. In the sequel whenever convenient and clear from the context we will drop the arguments.

First, we immediately see that

$$\dot\zeta_i = -z_i + \widetilde g_{i,a} - \frac{\partial \widetilde g_{i,a}^{\circ}}{\partial t} - \frac{\partial \widetilde g_{i,a}^{\circ}}{\partial x}\dot x + \frac{\partial \omega_i}{\partial t} + \frac{\partial \omega_i}{\partial x}\dot x \tag{37}$$

[4]This reasoning is similar to the arguments employed in [12] to prove (non uniform) convergence for skew-symmetric systems.

where $x = \text{col}[\,B_i(g_i y_{i+1} - g_{i-1} y_{i-1})\,]$ and

$$
\begin{aligned}
-\frac{\partial \widetilde{g}^{\circ}_{i,a}(t,x)}{\partial t} &= -\psi_i(t,x) &(38)\\
\frac{\partial \omega_i(t,x)}{\partial x} &= \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)} \frac{\partial \psi_i}{\partial x}(\tau,x)d\tau &(39)\\
\frac{\partial \omega_i(t,x)}{\partial t} &= \psi_i(t,x) - \omega_i(t,x) &(40)
\end{aligned}
$$

so adding and substracting $\widetilde{g}^{\circ}_{i,a}(t,x)$ in (37) we get that

$$
\dot{\zeta}_i = -\zeta_i + \widetilde{g}_{i,a} - \widetilde{g}^{\circ}_{i,a} + \left[\int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)} \frac{\partial \psi_i}{\partial x}(\tau,x)d\tau - \frac{\partial \widetilde{g}^{\circ}_{i,a}}{\partial x}\right]\cdot \dot{x}\,. \tag{41}
$$

At this point we observe that the terms in brackets multiplying $\dot{x}$ are uniformly bounded in t by a continuous function of the norm of the state and hence, it is bounded by a number $\nu > 0$ for all $x \in \mathcal{B}(\Delta)$. To find suitable bounds on $\dot{x}$ we recall that

$$
\begin{aligned}
\dot{x}_1 &= B_1 g_1 y_2\\
\dot{x}_j &= B_j(x_j)\,[g_j y_{j+1} - g_{j-1} y_{j-1}]\,, \quad j \in [2,\ldots,n-2]\\
\dot{x}_{n-1} &= B_{n-1}[g_{n-1} y_n - g_{n-2} y_{n-2}]\\
\dot{x}_n &= B_n[g_n \sigma(y_n) - g_{n-1} y_{n-1}]\,.
\end{aligned}
$$

so we use (20), (21) and (6) and the fact that all the gains g_i and $\widetilde{g}_i$ and the functions B_i are uniformly bounded in t on compact sets $\mathcal{B}(\Delta)$ and proceed as we did for (30) to find that for all $(x,z) \in \mathcal{B}(\Delta)^2$ and all $i \in [2,\cdots,n-2]$,

$$
\begin{aligned}
|\dot{x}| \;\leq\; \nu \Big[& \phi_{i+1}^{1/n-i-1} + \phi_{i-1}^{1/n-i+1} + \phi_2^{1/n-2} + |y_n| + |\sigma(y_n)|\\
& + \phi_{n-2}^{1/2} + \phi_{n-1} \Big]\,.
\end{aligned} \tag{42}
$$

We use now this inequality to find a bound on the total time derivative (at the points of existence) of (36) along the trajectories of (41). For this purpose we also use the Lipschitz property of $\widetilde{g}_{i,a}(t,\cdot)$ to see that there exists $L > 0$ such that $\left|\widetilde{g}_{i,a}(t,x) - \widetilde{g}^{\circ}_{i,a}(t,x)\right| \leq L\,|x_n|$, and we observe that $|\zeta_i(t,x,z)| \leq \nu$ for almost all $(t,x,z) \in \mathbb{R}\times\mathcal{B}(\Delta)^2$ to obtain finally that

$$
\dot{V}_{n+i}(t,x,z) \leq -\zeta_i(t,x,z)^2 + \nu\left(|x_n| + |y_n| + \sum_{j=1}^{n-1} \phi_j^{1/n-j}\right) \tag{43}
$$

for all $i \in [1,\ldots,n-1]$ which is what we were seeking for. Notice that the positive terms above appear with negative signs in the bounds on the derivatives of the previous group of auxiliary functions.

Roughly speaking, from the following group of functions we will be able to get terms that allow us to conlude on the convergence of the states to zero provided that the ϕ_i's converge to zero and that the gains are persistently exciting in the specific way we imposed. So we define now

$$
\begin{aligned}
V_{2n-1+i}(t,x) &:= -\int_{t}^{\infty} \mathrm{e}^{t-\tau}\,|\Omega_i(\tau,x)h(x_i)|^2\,d\tau &(44)\\
\Omega_i(t,x) &:= \prod_{j=i}^{n-1} \omega_j(t,x)\,, \quad i \in [1,\ldots,n-1] &(45)
\end{aligned}
$$

The total derivative of $V_{2n-1+i}(t,x)$ can be easily computed to find

$$
\dot{V}_{2n-1+i}(t,x) = V_{2n-1+i}(t,x) + |\Omega_i(\tau,x)h_i(x_i)|^2 + \frac{\partial V_{2n-1+i}}{\partial x}\dot{x} \tag{46}
$$

almost everywhere. Concerning the last term on the right hand side, we may proceed bounding them on compact sets of the states as we did before and use the local Lipschitz property of $V_{2n-1+i}(t,x)$ to bound the gradient on compact sets of the state. The first two terms require more consideration.

From the results in [3] we have that defining for each $\Omega_i(t,x)$ which is Uδ-PE with respect to $\xi = \text{col}[x_1,\ldots,x_{n-1},0]$, there exists $\gamma_i \in \mathcal{K}$ such that

$$
V_{2n-1+i}(t,x) \leq -\gamma_i(|\xi|)\,|h_i(x_i)|^2\,. \tag{47}
$$

Concerning the term $|\Omega_i(\tau,x)h_i(x_i)|^2$ in (46) we will derive a bound (as before on compact sets of the states) involving $|\phi_i|$, $|\zeta_j|$ and $|y_n|$ which appear with negative sign in the bounds on the previous functions' derivatives. To that end, we use (35) and (12) to obtain a more convenient expression for ω_i and substitute in (45) to see that

$$
|\Omega_i|^2 = \prod_{j=i}^{n-1}\,[\,\widetilde{g}_j + \zeta_j - (\widetilde{g}_{j,a} - \widetilde{g}^{\circ}_{j,a})\,]^2\,.
$$

Then, we use the Lipschitz property of the gains $\widetilde{g}_{j,a}(t,\cdot)$ to obtain that for all $(t,x,z) \in \mathbb{R}\times\mathcal{B}(\Delta)^2$,

$$
[\,\widetilde{g}_j + \zeta_j - (\widetilde{g}_{j,a} - \widetilde{g}^{\circ}_{j,a})\,]^2 \leq \widetilde{g}_j^{\,2} + \nu(|\zeta_j| + |x_n|)\,.
$$

Furthermore, using once more the uniform boundedness in t of $\widetilde{g}_j(t,\cdot,\cdot)$ and arguing as above, we have that for any integer $j \in [i,\ldots,n-1]$,

$$
\begin{aligned}
&[\,\widetilde{g}_j + \zeta_j - (\widetilde{g}_{j,a} - \widetilde{g}^{\circ}_{j,a})\,]^2[\,\widetilde{g}_{j+1} + \zeta_{j+1} - (\widetilde{g}_{j+1,a} - \widetilde{g}^{\circ}_{j+1,a})\,]^2\\
&\qquad \leq \widetilde{g}_j^{\,2}\widetilde{g}_{j+1}^{\,2} + \nu(|\zeta_j| + |\zeta_{j+1}| + |x_n|)\,.
\end{aligned}
$$

It follows that for all $(t,x,z) \in \mathbb{R}\times\mathcal{B}(\Delta)^2$,

$$
|\Omega_i y_i|^2 \leq \prod_{j=i}^{n-1} \widetilde{g}_j^{\,2} y_i^2 + \nu\left(\sum_{j=i}^{n-1}|\zeta_j| + |x_n|\right)
$$

where we have also used (11). Notice that from (12) the bound above is exactly the same as

$$
|\Omega_i y_i|^2 \leq g_i^2 y_i^2 + \nu\left(\sum_{j=i}^{n-1}|\zeta_j| + |x_n|\right)
$$

and using the identities (22) involving $g_i\,y_i$ we finally obtain that for all $(t,x,z) \in \mathbb{R}\times\mathcal{B}(\Delta)^2$, and all $i \in [1,\ldots,n-1]$

$$
|\Omega_i y_i|^2 \leq \nu(\,\phi_i^{2/n-i} + \sum_{j=i}^{n-1}|\zeta_j| + |x_n|\,) \tag{48}
$$

Summarizing, we have from (46), (47) and (48) that

$$
\begin{aligned}
\dot{V}_{2n+i-1} \;\leq\; & -\gamma_i(|\xi|)\,|h_i(x_i)|^2 + \nu\Big(|x_n| + \phi_i^{2/n-i} + |y_n| +\\
& \sum_{\ell=1}^{n-1}\phi_\ell^{1/n-\ell} + |\zeta_\ell|\Big) \qquad \text{a.e.}
\end{aligned} \tag{49}
$$

where the last two terms come from using (42) to bound $\dot{x}_i$ in the last two terms of (46), similarly as we did to obtain (43), and from a uniform bound (for all $(t, x, z) \in \mathbb{R} \times \mathcal{B}(\Delta)^2$) on the partial derivatives of V_{2n+i-1} and h_i. The final group of functions is simply

$$V_{3n-2+i}(z) = z_i^2, \qquad \forall\, i \in [1, \ldots, n-1] \tag{50}$$

whose total derivative along the trajectories of the first equation in (12) yields for all $(t, x, z) \in \mathbb{R} \times \mathcal{B}(\Delta)^2$,

$$\dot{V}_{3n-2+i}(z) \le -z_i^2 + \nu\,|x| \,. \tag{51}$$

With the aim at applying Theorem (1) define

$$X := \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}, \quad \Phi(t, X) := \begin{bmatrix} \phi_1(t, x, z) \\ \vdots \\ \phi_{n-1}(t, x, z) \\ \zeta_1(t, x, z) \\ \vdots \\ \zeta_{n-1}(t, x, z) \end{bmatrix}. \tag{52}$$

Then, we have from (24), (31), (33), (43), (49) and (51) that for all $i \le n-1$,

$$\begin{aligned}
\dot{V}_1 &\le -\rho \circ \kappa_n(|X_n|) \\
\dot{V}_2 &\le -c_n\,|\Phi_{n-1}|^2 + \nu[\,\theta_n(|X_n|) + \sigma \circ \theta_n(|X_n|)\,] \\
\dot{V}_3 &\le -c_{n-1}\,|\Phi_{n-2}|^2 + \nu[\,|\Phi_{n-1}| + \theta_n(|X_n|)\,] \\
&\vdots \\
\dot{V}_{i+3} &\le -c_{n-i-1}\,|\Phi_{n-i-2}(t,X)|^2 + \nu\,(|\Phi_{n-i-1}(t,X)| \\
&\quad + |\Phi_{n-i}(t,X)|) \\
&\vdots \\
\dot{V}_n &\le -c_2\,|\Phi_1(t,X)|^2 + \nu\,(|\Phi_2(t,X)| + |\Phi_3(t,X)|) \\
&\vdots \\
\dot{V}_{n+i} &\le -\,|\Phi_{n+i-1}(t,X)|^2 + \nu\,(\theta_n(|X_n|) + |X_n| + \\
&\quad \sum_{\ell=1}^{n-1} |\Phi_\ell(t,X)|^{1/n-\ell} \Big) \\
&\vdots \\
\dot{V}_{2n+i-1} &\le -\gamma_i(|X_{1\to n-1}|)\,|h_i(X_i)|^2 + \\
&\quad \nu\,\Big(\,\theta_n(|X_n|) + |X_n| + |\Phi_i(t,X)|^{2/n-i} + \\
&\quad \sum_{\ell=1}^{n-1} |\Phi_\ell(t,X)|^{1/n-\ell} + |\Phi_{n+\ell-1}(t,X)|\,\Big) \\
&\vdots \\
\dot{V}_{3n+i-2} &\le -X_{n+i}^2 + \nu\,|X_{1\to n}| \,.
\end{aligned}$$

Letting each of the bounds above be $Y_k(X, \Phi(t, X))$ with $k \in [1, \ldots, 4n-3]$ we see that each of these functions is bounded on compact sets (i.e. Assumption 2 holds), is non-positive on the sets where all the previous are identically zero (i.e., Assumption 3 holds). Moreover, in view of (6), in particular since $\kappa \in \mathcal{PD}$ and also $\gamma_i \in \mathcal{K}$ we see that the set where $Y_k(0, 0) \equiv 0$ is the origin, $\{X = 0\}$ (i.e., Assumption 4). UGAS follows from Theorem 1. ■

4 Conclusions

In this paper we have addressed a case-study in control design which covers the popular exmaple of nonholonomic systems. Our approach relies on a property known as δ-persistency of excitation and a new tool for stability analysis which can be regarded as an extension of the well known Matrosov's theorem. This tool involves the use of an arbitrary finite number of auxiliary functions whose derivatives are simultaneously zero only at the origin. Hence, it can also be regarded as a generalization to the case of non autonomous systems, of the celebrated La Salle's invariance principle (when the latter is used to conclude stability of the origin).

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