OPTIMAL INPUT SIGNAL DESIGN FOR IDENTIFICATION OF MAX–PLUS–LINEAR SYSTEMS

Gernot Schullerus†, Volker Krebs†, Bart De Schutter‡, Ton van den Boom‡

† Institut für Regelungs- und Steuerungssysteme, Universität Karlsruhe
Kaiserstraße 12, D–76131 Karlsruhe, Germany
phone: +49 721 608 3179, fax: +49 721 608 2707
e-mail: {schullerus,krebs}@irs.uni-karlsruhe.de
http://www.irs.uni-karlsruhe.de

‡ Delft Center for Systems and Control, Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands
phone:+31 15 278 51 13, fax: +31 15 278 66 79
e-mail: {b.deschutter,t.j.j.vandenboom}@dcsc.tudelft.nl
http://www.dcsc.tudelft.nl/

Keywords: max–plus–linear systems, parameter estimation, input signal design, model predictive control, discrete event systems.

Abstract

The present contribution addresses the problem of designing an adequate persistent excitation for state space identification of max–plus–linear systems. The persistent excitation is designed using the same techniques that have recently been developed for model predictive control for max–plus–linear systems. The application of this method for input signal design allows to incorporate additional objectives which are desirable for the input signals and the resulting process behaviour such that an optimal persistent excitation is obtained.

1 Introduction

When considering processes from manufacturing or chemical engineering, their behaviour can often be adequately represented by a discrete event model [2] accounting for the typically discrete sensor and actuator equipment of such processes. In addition, the behaviour of these processes is often adequately described by a sequence of transitions between discrete process states. The focus of this contribution is on a particular class of such discrete event systems where synchronization but no concurrency occurs. This system class has gained significant attention in recent years due to the fact that the sequences of event times for such processes can be described by equations which are linear in a particular algebra, the so called max–plus algebra [1]. The resulting equations exhibit a structural equivalence to system descriptions from conventional control engineering such as transfer functions or state space models. Thus, a system theory for these max–plus–linear systems has been developed [1, 3], and various concepts well known from control engineering have been adapted to this system class in control design [8, 12] and diagnosis [14]. The application of any of these methods requires a process model which can be obtained by theoretical modeling or identification algorithms.

The identification problem by parameter estimation for max–plus–linear systems has been considered in several publications e.g. by estimating the parameters of an ARMA model [10] or impulse response [12], by determining state space models using either the system's Markov parameters in [5] or minimizing a prediction error based on input output data [9]. As shown in [10, 12], the given methods will in general overestimate the true system parameters. This issue of identifiability [11], that is, the convergence of the estimated parameters to their corresponding true values is addressed and solved in [15] by applying certain input signals to the system. However, the choice of appropriate input signals used in [15] may be difficult to handle in engineering applications. Therefore, we propose an alternative way to compute such input signals using a model–predictive–control–like approach as has been developed in [8]. The application of this method for input signal design allows to incorporate additional objectives which are desirable for the input signals and the resulting process behaviour, thus obtaining an optimal persistent excitation.

This paper is organized as follows. The following section briefly reviews the basic notions of max–plus–linear systems. The parameter estimation algorithm and the issue of identifiability of parameters is then discussed followed by a presentation of a procedure for designing a persistently exciting input sequence. Finally, the overall identification procedure is illustrated in an example.

2 Max–plus–linear systems

We consider in the sequel discrete event systems where the evolution of the events is governed by synchronization effects and where no structural alternatives occur. The behaviour of these systems is completely specified if the occurrence times of each event are known. Thus, the time instant when event e_i occurs for the k–th time is denoted by the "dater" $x_i(k)$. Similarly, the input event times are given by $u_j(k)$. The evolution of the event times $x(k) \in \mathbf{R}_{\max}^{n}$ depending on the input event times $u(k) \in \mathbf{R}_{\max}^{n_u}$, where $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ is then given by the following model, which is structurally equivalent to a conventional discrete–time linear state space model [1]

$$x(k+1) \;=\; A \otimes x(k) \oplus B \otimes u(k+1) \;, \tag{1}$$

where $A \in \mathbf{R}_{\max}^{n\times n}$, $B \in \mathbf{R}_{\max}^{n\times n_u}$. The operators $\oplus$ and $\otimes$ are the addition and multiplication operators of the max–plus algebra and are defined by

$$x \oplus y = \max(x, y) \;, \quad x \otimes y = x + y \;, \quad \forall\, x, y \in \mathbf{R}_{\max} \;.$$

The neutral elements of max–plus addition and max–plus multiplication are $-\infty = \varepsilon$ and 0, respectively. Note that ε is absorbing with respect to $\otimes$. The matrix addition and multiplication are defined similarly to the conventional algebra:

$$\begin{aligned} \forall P, Q \in \mathbf{R}_{\max}^{n\times p} \;:\; (P \oplus Q)_{ij} &= P_{ij} \oplus Q_{ij} \;, \\ \begin{matrix} \forall P \in \mathbf{R}_{\max}^{n\times p} \\ \forall Q \in \mathbf{R}_{\max}^{p\times q} \end{matrix} \;:\; (P \otimes Q)_{ij} &= \bigoplus_{k=1}^{p} (P_{ik} \otimes Q_{kj}) \;. \end{aligned}$$

The structural equivalence of the model (1) with the discrete time state space equation makes it possible to adapt well known concepts from system theory to this particular system class, provided the model and its parameter can be determined. The following section describes an identification procedure that, given the model structure, allows the determination of the model parameters from event time measurements.

3 Parameter estimation by minimization of a prediction error

In this paper, the following parameter estimation problem is considered:

Given the system model

$$x(k+1) \;=\; A \otimes x(k) \oplus B \otimes u(k+1) \;,$$

the input event times $u(k),\; k = 1, \ldots, N$ and the measurements $x(k),\; k = 0, \ldots, N$, determine estimates $\hat{A}$ and $\hat{B}$ for the system matrices A and B such that the prediction error

$$\begin{aligned} \xi(k+1) &= x(k+1) - \Big(\hat{A} \otimes x(k) \oplus \hat{B} \otimes u(k+1)\Big) \qquad (2) \\ &= x(k+1) - \underbrace{\begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}}_{=:\,\widehat{\Theta}} \otimes \underbrace{\begin{bmatrix} x(k) \\ u(k+1) \end{bmatrix}}_{=:\,m(k+1)} \\ &= x(k+1) - \widehat{\Theta} \otimes m(k+1) \end{aligned}$$

is minimized and the estimated parameters $\hat{A}_{ij}$ and $\hat{B}_{ij}$ are as close as possible to the true system parameters given by A_{ij} and B_{ij}.

It is assumed that no noise is present and the internal structure (also called the ε–structure) of the system is known so that we know which entries of the systems matrices are equal to ε and which are not.

Remark 3.1 *The ε–structure of the system is determined by the layout and the internal connection between different subparts of the system (see, e.g., [1]). For most discrete event systems the internal structure is known so that this assumption is not restrictive. Hence, we may without loss of generality assume that all entries of $\widehat{\Theta}$ are different from ε (we can remove the ε entries from $\widehat{\Theta}$ or put $\hat{A}_{ij} = \varepsilon$ and $\hat{B}_{i'j'} = \varepsilon$ for all index pairs (i, j) and (i', j') that define the ε–structure of respectively A and B and only consider the finite entries of $\widehat{\Theta}$).*

Remark 3.2 *We assume that all event times are measurable as it is the case for most discrete event systems such as, e.g., manufacturing systems, where one can usually measure all the starting times of the various production units. Note however that the results derived below can be extended to also include an output equation since this equation can be dealt with in the same way as the state update equation (1).*

Remark 3.3 *We assume that the real system belongs to the model class, i.e., that its input–state behaviour can indeed be described by a model of the form (1). Note that for discrete event systems the assumption that no noise (or measurement errors) are present is not as restrictive as for conventional continuous-time or discrete–time (non)linear systems since measurements of occurrence times of events are in general not as susceptible to noise and measurement errors as measurements of continuous–time signals.*

To solve the parameter estimation problem defined above, first an estimate for the system parameters is determined. Considering the given measurements of $x(k+1)$ and $m(k+1)$, the prediction error matrix results in

$$\begin{aligned} &\begin{bmatrix} \xi(N) & \ldots & \xi(1) \end{bmatrix} = \\ &\quad= \begin{bmatrix} x(N) & \ldots & x(1) \end{bmatrix} - \widehat{\Theta} \otimes \begin{bmatrix} m(N) & \ldots & m(1) \end{bmatrix} \\ &\quad= X - \widehat{\Theta} \otimes M \;. \end{aligned}$$

The data matrices X and M contain the event times, whereas $\widehat{\Theta} = [\hat{A} \;\; \hat{B}]$ denotes the matrix of estimated parameters determined from the given measurements of x and m.

A solution that minimizes the prediction error is obtained [1, 3, 12] by computing the greatest solution of the inequality

$$X \;\geq\; \Theta \otimes M$$

which is given by

$$\begin{aligned} \widehat{\Theta} &= X \otimes' \left(-M^T\right) \;, \\ \widehat{\Theta}_{ij} &= \bigoplus_{k=1}^{N}{}' \big(x_i(k) - m_j(k)\big) \end{aligned} \tag{3}$$

where the operators "$\oplus'$" and "$\otimes'$" of the min–plus algebra [3] correspond to conventional minimization and addition, respectively. As shown in [12], the solution determined by (3) has two particular properties[1]:

$$X = \widehat{\Theta} \otimes M \,, \tag{4}$$
$$\widehat{\Theta} \geq \Theta \,. \tag{5}$$

From (4) it immediately follows that the prediction error $\xi(k) = 0$ for $k = 1, \ldots, N$. However, the property (5) shows that an estimated parameter value will in general differ from the true parameter value. This issue is addressed in [15]. There, it is shown that the estimated values are equal to the true values only for those trajectories of x and m that are informative enough to obtain estimates that are equal to the true system parameters. Sufficient conditions for x and m are now discussed using a result from [15] given in the following theorem:

Theorem 3.1 *[15] The parameter Θ_{ij} is correctly identified by (3) based on the given data $m(k+1), x(k+1),\ \ k = 0, \ldots, N-1$, i.e. $\widehat{\Theta}_{ij} = \Theta_{ij}$ if and only if*

$$\exists k \in \{0, \ldots, N-1\} \text{ s. t. } x_i(k+1) = \Theta_{ij} \otimes m_j(k+1) \,.$$

Let us now assume that we want to obtain an estimate of the (finite) parameter Θ_{ij} for a given i, j. Recalling that in

$$x_i(k+1) = \bigoplus_{r=1}^{n+n_u} \Theta_{ir} \otimes m_r(k+1)$$

$\oplus$ represents conventional maximization, a *necessary and sufficient* condition for the condition of theorem 3.1 is given by

$$\Theta_{ij} \otimes m_j(k+1) \geq \bigoplus_{\substack{r=1 \\ r \neq j}}^{n+n_u} \Theta_{ir} \otimes m_r(k+1) \quad \text{for some } k. \tag{6}$$

Although the true system parameters are unknown, (6) can be still evaluated replacing Θ_{ij} by its lower bound[2] 0 and Θ_{ir} by its upper bound $\widehat{\Theta}_{ir}$ which is obtained from an estimation with arbitrary input signals (cf. (5)). Thus, if the condition

$$m_j(k+1) \geq \bigoplus_{\substack{r=1 \\ r \neq j}}^{n+n_u} \widehat{\Theta}_{ir} \otimes m_r(k+1) \tag{7}$$

is satisfied for at least one data point in the measurements then (6) is also satisfied and the correct estimation of Θ_{ij} by (3) is ensured. This behaviour can be obtained by designing particular input signals that persistently excite the system. The focus of the following section is on the computation of such signals by formulating the determination of the feasible solutions set as an Extended Linear Complementarity Problem (ELCP).

4 Designing persistent excitation signals using prediction models

We will now discuss an approach to identify the (non–ε) parameters of the model (1). We will identify each parameter Θ_{ij} separately in an iterative way. So every, say, N_i event steps one parameter will be identified using the persistent excitation approach. In this section we will illustrate how input signals satisfying condition (7) can be designed. We will first give a short introduction to the Extended Linear Complementarity Problem. Next, we will show how this mathematical programming problem can be used to obtain accurate parameter estimations based on the persistent excitation condition.

4.1 The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) arose from our research on discrete event systems and hybrid systems, and is defined as follows [4]:

> Given $P \in \mathbf{R}^{n_p \times n_z}$, $Q \in \mathbf{R}^{n_q \times n_z}$, $p \in \mathbf{R}^{n_p}$, $q \in \mathbf{R}^{n_q}$ and $\phi_1, \ldots, \phi_m \subseteq \{1, \ldots, n_p\}$, find $z \in \mathbf{R}^{n_z}$ such that
>
> $$Pz \geq p \tag{8}$$
> $$Qz = q \tag{9}$$
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pz - p)_i = 0 \,. \tag{10}$$

Condition (10) represents the complementarity condition of the ELCP and can be interpreted as follows. Since $Pz \geq p$, all the terms in (10) are nonnegative. Hence, (10) is equivalent to $\prod_{i \in \phi_j} (Pz - p)_i = 0$ for $j = 1, \ldots, m$. So each set ϕ_j corresponds to a group of inequalities in $Pz \geq p$, and in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

In [4] we have developed an algorithm to find a parametric representation of the *entire* solution set of an ELCP. The computation time and the memory storage requirements of this algorithm increase exponentially as the size of the ELCP increases, which makes this approach intractable even for small–scale ELCPs. However, in [7] we have recently developed an approach to efficiently solve ELCPs with a bounded feasible set $\{z | Pz \geq p, Qz = q\}$ that is based on mixed integer linear programming, and that allows us to solve much larger instances of the ELCP.

4.2 Persistent excitation signal design

Suppose that we are at event step k_0 and that we want to identify Θ_{ij} within the next N_i event steps. Assume, that a first estimate $\widehat{\Theta}^{(k_0)} = [\hat{A}^{(k_0)}\ \hat{B}^{(k_0)}]$ from a previous identification step, possibly based on arbitrary input signals, has been determined from the measurements up to k_0. Then, the parameters of the system are known to stay within certain bounds given by

$$0 \leq \Theta_{ij} \leq \widehat{\Theta}_{ij}^{(k_0)} \,.$$

Now an input/state trajectory $\{m(k_0+1), \ldots, m(k_0+N_\text{i})\}$[3] should be determined which satisfies (7) and for which the fol-

[1] Recall that we assume that no noise is present (cf. Remark 3.3).

[2] Since we only consider finite entries in Θ (cf. Remark 3.1) and since these finite entries correspond to processing times, transportation times, and so on they are always nonnegative.

[3] Note that $x(k_0)$ is included in $m(k_0+1)$ (cf. (2)) so that the corresponding components of $m(k_0+1)$ are assumed to be fixed as $x(k_0)$ is assumed to be known at event step $k = k_0$.

lowing prediction model equation holds:

$$\begin{aligned} x(k+1) &= \hat{A}^{(k_0)} \otimes x(k) \oplus \hat{B}^{(k_0)} \otimes u(k+1) \qquad (11) \\ &\text{for } k = k_0, \dots, k_0 + N_\mathrm{i} - 2, \end{aligned}$$

where $\hat{A}^{(k_0)}$ and $\hat{B}^{(k_0)}$ correspond to the parameter estimates $\widehat{\Theta}^{(k_0)}$ obtained using the previous identification run or based on arbitrary input signals[4]. Note that (11) predicts the relation between the u part and the x part of two subsequent m vectors.

Now we have the following proposition.

Proposition 4.1 *For a given index pair (i, j) the condition that there exists an index $k \in \{k_0, \dots, k_0 + N_\mathrm{i} - 1\}$ such that (7) and (11) holds can be rewritten as an ELCP.*

Proof: First we consider (7) for a fixed $k = k_0 + \ell$ with $\ell \in \{0, \dots, N_\mathrm{i} - 1\}$. Recalling that $\oplus$ and $\otimes$ represent conventional maximization and addition, respectively, it is easy to verify that (7) can be rewritten as

$$\begin{aligned} m_j(k+1) - m_r(k+1) &\geq \widehat{\Theta}_{ir} \\ \text{for } r = 1, \dots, n + n_u,\ r \neq j, \end{aligned}$$

or equivalently

$$P^{(\ell)} \tilde{m} \geq q^{(\ell)}$$

for an appropriately defined matrix $P^{(\ell)}$ and vector $q^{(\ell)}$, where $\tilde{m} = \left[\, m^T(k_0+1) \ \ \dots \ \ m^T(k_0 + N_\mathrm{i}) \,\right]^T$.

In order to express that (7) should hold for $k = k_0$ **or** $k = k_0 + 1$ **or** ... **or** $k = k_0 + N_\mathrm{i} - 1$, we introduce binary variables $\delta_0, \dots, \delta_{N_\mathrm{i}-1} \in \{0, 1\}$ such that if $\delta_\ell = 1$ then (7) holds for $k = k_0 + \ell$. So we should have

$$\delta_\ell = 1 \quad \Rightarrow \quad P^{(\ell)} \tilde{m} \geq q^{(\ell)} \qquad (12)$$

$$\delta_\ell = 0 \quad \Rightarrow \quad \tilde{m} \text{ is arbitrary.} \qquad (13)$$

The condition $\delta_\ell \in \{0, 1\}$ is equivalent to the ELCP

$$0 \leq \delta_\ell \leq 1 \quad \text{and} \quad \delta_\ell (1 - \delta_\ell) = 0 \ . \qquad (14)$$

To express that at least one δ_ℓ should be equal to 1, we add the condition

$$\delta_0 + \ldots + \delta_{N_\mathrm{i}-1} \geq 1 \ . \qquad (15)$$

As the components of $\tilde{m}$ correspond to inputs and states and as we only look a finite number (N_i) of event steps ahead, we may assume with loss of generality that the components of $\tilde{m}$ are bounded, i.e., $\tilde{m} \in \mathcal{M}$ with $\mathcal{M}$ a bounded set. As a consequence, the number

$$M_P = \min_{\ell = 0, \dots, N_\mathrm{i}-1} \ \min_{\tilde{m} \in \mathcal{M}} \ \min_{j = 1, \dots, n_{P^{(\ell)}}} (P^{(\ell)} \tilde{m})_j \qquad (16)$$

where $n_{P^{(\ell)}}$ is the number of rows of $P^{(\ell)}$, is finite, and we have $P^{(\ell)} \tilde{m} \geq M_P$ for all $\tilde{m} \in \mathcal{M}$. Hence, condition (12)–(13) is equivalent to

$$P^{(\ell)} \tilde{m} \geq \delta_\ell q^{(\ell)} + M_P (1 - \delta_\ell) \mathbf{1} \ . \qquad (17)$$

So, the condition that (7) should hold for at least one index $k \in \{k_0, \dots, k_0 + N_\mathrm{i} - 1\}$ is equivalent to the ELCP (14), (15), (17).

Now we consider the prediction equation (11). By repeated substitution this results in a max–plus–linear equation of the form

$$\tilde{m}_{(x)} = \tilde{C} \otimes x(k_0) \oplus \tilde{D} \otimes \tilde{m}_{(u)} \qquad (18)$$

where $\tilde{m}_{(x)} = \left[\, x^T(k_0+1) \ \ \dots \ \ x^T(k_0 + N_\mathrm{i} - 1) \,\right]^T$ and $\tilde{m}_{(u)} = \left[\, u^T(k_0+1) \ \ \dots \ \ u^T(k_0 + N_\mathrm{i} - 1) \,\right]^T$. Now we can make use of the fact that a system of max–plus–linear equations can be rewritten as an ELCP[5] [6].

Since the merge of two ELCPs is again an ELCP, we can now merge the ELCP (14), (15), (17) and the ELCP corresponding to (18) into one ELCP. □

Remark 4.1 *Note that using the ELCP approach we can also easily include a constraint of the form*

$$d_{\min} \leq u(k+1) - u(k) \leq d_{\max} \quad \text{for } k = k_0, \dots, k_0 + N_\mathrm{i} - 1,$$

which bounds the input rate between $d_{\min}$ and $d_{\max}$ (this may sometimes be necessary to ensure safe operation of the system or to prevent buffer overflows). In general, we can accommodate any linear constraint of the form

$$D_1 m(k_0+1) + \ldots + D_{N_\mathrm{i}} m(k_0 + N_\mathrm{i}) \geq d \ . \qquad (19)$$

The main advantage of the ELCP approach is that — using the ELCP algorithm of [4] — we can find *all* sequences $\{m(k_0+1), \dots, m(k_0 + N_\mathrm{i})\}$ that guarantee correct estimation of the parameter Θ_{ij}. This enables us to perform an extra optimization within this set of persistent excitation signals, e.g., for control purposes.

5 Example

Consider now a manufacturing cell shown in figure 1, where parts are delivered to the machine by a conveyor, machined and released to an additional conveyor. The capacity of each conveyor is limited to one part. The machine can process one part at the same time.

Let $x_1(k)$ be the time instant when the k–th part is loaded onto the conveyor 1. After τ_{21} time units, this part is ready to enter the machine. The dater $x_2(k)$ denotes the time when the k–th part enters the machine. After the machining operation which takes τ_{32} time units, the part is released to an additional conveyor 2 at time $x_3(k)$ reaching a final position after τ_{43} time units. From this final position, the part is picked up at time $x_4(k)$. Conveyor 1 and 3 can receive a new part after τ_{12} and τ_{34}, respectively, whereas the machine must be prepared for a new operation for τ_{23} time units. The input event times $u_1(k)$

[4]Note that by repeatedly performing the sweeping identification process we will get better estimates, and thus also better predictions, as the process goes on.

[5]Basically, the proof of this statement boils down to the fact that an equation of the form $\alpha = \max(\beta, \gamma)$ for some scalar variables or expressions α, β, γ can be rewritten as the ELCP $\alpha \geq \beta$, $\alpha \geq \gamma$, $(\alpha - \beta) \cdot (\alpha - \gamma) = 0$, as the latter of these equations in combination with the first two implies that $\alpha = \beta \geq \gamma$ or $\alpha = \gamma \geq \beta$.

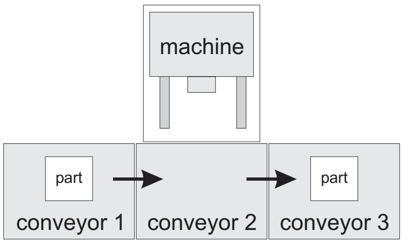

Figure 1: Manufacturing cell.

k	1	2	3	4	5	6	7	8	9	10	11
$x(k)$	3	7	11	15	19	23	29	34	43	46	50
	1	5	9	13	17	21	25	32	36	44	48
	4	8	12	16	20	24	31	35	39	47	51
	2	6	10	14	18	22	30	33	37	41	49
$u(k)$	0	1	3	4	5	6	29	30	43	44	45
	0	2	3	4	5	7	30	31	36	37	38

Table 1: The input signals and the corresponding states values used in the worked example for $k = 1, \dots, 11$.

and $u_2(k)$ correspond to the time instants when a new part is available to be delivered for conveyor 1 or removed from conveyor 3, respectively. As shown in figure 1, in the initial state the manufacturing cell already contains one part on conveyor 1 and 3 which are assumed to have entered at time $x_1(0)$ and $x_3(0)$, respectively (see [13] for more details). Using the above reasoning and the initial state of the system, the following equations hold for $x(k)$:

$$
\begin{aligned}
x_1(k+1) &= \tau_{12} \otimes x_2(k+1) \oplus u_1(k+1) && (20)\\
x_2(k+1) &= \tau_{21} \otimes x_1(k) \oplus \tau_{23} \otimes x_3(k) && (21)\\
x_3(k+1) &= \tau_{32} \otimes x_2(k+1) \oplus \tau_{34} \otimes x_4(k+1) && (22)\\
x_4(k+1) &= \tau_{43} \otimes x_3(k) \oplus u_2(k+1) && (23)
\end{aligned}
$$

Inserting equation (21) into (20) and (21) and (23) into (22) yields the max–plus–linear system

$$
x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1)
$$
$$
A = \begin{bmatrix} a_{11} & \varepsilon & a_{13} & \varepsilon \\ a_{21} & \varepsilon & a_{23} & \varepsilon \\ a_{31} & \varepsilon & a_{33} & \varepsilon \\ \varepsilon & \varepsilon & a_{43} & \varepsilon \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & b_{32} \\ \varepsilon & b_{42} \end{bmatrix}, \quad (24)
$$

with the true system parameters given by

$$
\begin{aligned}
&a_{11} = 3\,, \quad a_{21} = 1\,, \quad a_{31} = 4\,, \\
&a_{13} = 3\,, \quad a_{23} = 1\,, \quad a_{33} = 4\,, \quad a_{43} = 2\,, \\
&b_{11} = 0\,, \quad b_{32} = 1\,, \quad b_{42} = 0\,.
\end{aligned}
$$

Let us now assume that the ε–structure of the system matrices is known and let us determine the unknown parameters a_{ij} and b_{ij} from event time measurements.

First, the estimation is carried out using a set of arbitrary input signals for $k = 1, \dots, 6$. These signals and the corresponding states are listed in table 1. This yields the following estimates (/upper bounds) for the system parameters (where we have not included the initial state in this estimation as we assume it to be unknown):

$$
\begin{aligned}
&\hat{a}_{11} = 4\,, \quad \hat{a}_{21} = 2\,, \quad \hat{a}_{31} = 5\,, \\
&\hat{a}_{13} = 3\,, \quad \hat{a}_{23} = 1\,, \quad \hat{a}_{33} = 4\,, \quad \hat{a}_{43} = 2\,, \\
&\hat{b}_{11} = 6\,, \quad \hat{b}_{32} = 6\,, \quad \hat{b}_{42} = 4\,.
\end{aligned}
$$

Based on these system parameters, it is easy to satisfy (7) for the parameters b_{11}, b_{32} and b_{42} for $k = 6$ by choosing $u(k+1) = u(7)$ such that

$$
\begin{aligned}
b_{11}:\ u_1(7) &\geq (\hat{A}^{(6)} \otimes x(6))_1 = 27\,, \\
b_{32}:\ u_2(7) &\geq (\hat{A}^{(6)} \otimes x(6))_3 = 28\,, \\
b_{42}:\ u_2(7) &\geq (\hat{A}^{(6)} \otimes x(6))_4 = 26\,.
\end{aligned}
$$

To be on the safe side we select $u(k+1) = u(7) = \begin{bmatrix} 29 & 30 \end{bmatrix}^T$. Furthermore, we — arbitrarily — take $u(8) = 1 \otimes u(7)$. The corresponding states $x(7)$ and $x(8)$ are listed in table 1. From the resulting measurements one now obtains the true parameter values for the entries of B while $\hat{A}$ remains unchanged.

Let us now consider the estimation of the parameter a_{31}. We use the ELCP approach to determine a persistent excitation input sequence[6] $\{u(k_0+1), u(k_0+2), u(k_0+3)\} = \{u(9), u(10), u(11)\}$. For a_{31} condition (7) is satisfied if :

$$
\begin{aligned}
x_1(k) &\geq 4 \otimes x_3(k) && (25)\\
x_1(k) &\geq 1 \otimes u_2(k+1) && (26)
\end{aligned}
$$

holds for some $k \geq k_0$, i.e., for $k = 8$ or $k = 9$ or $k = 10$ as we have $N_{\mathrm{i}} = 3$. As (25) does not hold for $k = 8$ anyway, in order to satisfy (7) we should have

$$
\begin{matrix} x_1(9) - x_3(9) \geq 4 \\ x_1(9) - u_2(10) \geq 1 \end{matrix} \quad \textbf{or} \quad \begin{matrix} x_1(10) - x_3(10) \geq 4 \\ x_1(10) - u_2(11) \geq 1 \end{matrix} \qquad (27)
$$

If we assume that the components of $\tilde{m}$ are between 0 and 100, we get $M_P = -100$ (cf. (16)). Following the lines of the proof of Proposition 4.1 we obtain then the following ELCP[7]

$$
\begin{aligned}
x_1(9) - x_3(9) - 104\delta_2 &\geq -100 && (28)\\
x_1(9) - u_2(10) - 101\delta_2 &\geq -100 && (29)\\
x_1(10) - x_3(10) - 104\delta_3 &\geq -100 && (30)\\
x_1(10) - u_2(11) - 101\delta_3 &\geq -100 && (31)\\
\delta_2 + \delta_3 &\geq 1 && (32)\\
\delta_2, \delta_3 \geq 0, \qquad 1 - \delta_2, 1 - \delta_3 &\geq 0 && (33)\\
\delta_2(1-\delta_2) + \delta_3(1-\delta_3) &= 0 && (34)\\
x_1(9) &\geq 38 && (35)\\
x_1(9) - u_1(9) &\geq 0 && (36)\\
x_3(9) &\geq 39 && (37)
\end{aligned}
$$

[6] We have selected a small value $N_{\mathrm{i}} = 3$ to keep the number of equations limited, so that all the equations of the ELCP can be listed explicitly.

[7] As $x_2(k)$ and $x_4(k)$ do not appear in (27) and as they do not directly influence the values of $x_1(k)$ and $x_3(k)$, we have — for the sake of brevity — omitted the equations corresponding to $x_2(k)$ and $x_4(k)$. The same holds for the prediction equation for $x(11)$ as $x(11)$ does not appear in (27). Note that $\delta_1 = 0$ as (25) does not hold for $k = k_0 = 8$.

$$x_3(9) - u_2(9) \geq 1 \quad (38)$$
$$x_1(10) - x_1(9) \geq 4 \quad (39)$$
$$x_1(10) - x_3(9) \geq 3 \quad (40)$$
$$x_1(10) - u_1(10) \geq 0 \quad (41)$$
$$x_3(10) - x_1(9) \geq 5 \quad (42)$$
$$x_3(10) - x_3(9) \geq 4 \quad (43)$$
$$x_3(10) - u_2(10) \geq 1 \quad (44)$$
$$\mathrm{d}(35)\mathrm{d}(36) + \mathrm{d}(37)\mathrm{d}(38) + \mathrm{d}(39)\mathrm{d}(40)\mathrm{d}(41) + \mathrm{d}(42)\mathrm{d}(43)\mathrm{d}(44) = 0, \quad (45)$$

where $\mathrm{d}(i)$ denotes the difference between the left–hand side and the right–hand side of equation (i). Note that (35)–(45) correspond to the prediction equation (11). We also add the condition that all variables should lie in the interval $[0, 100]$ (cf. the determination of M_P). In order to guarantee a minimal separation of input times (cf. (19)), we also the condition

$$u(9) - u(8) \geq 1, \;\; u(10) - u(9) \geq 1, \;\; u(11) - u(10) \geq 1 \; .$$

Finally, as we use the measurements of the state $x(8)$ as a starting point for the estimation, we have to wait at least until $t = \max_i(x_i(8)) = 35$ before we can start the estimation, which implies that, e.g., $u(9) \geq 36$. The solution set of the resulting ELCP consists of a polytope with 188 vertices. If we (arbitrarily) select the vertex with $\delta_2 = 1$, $\delta_3 = 0$ closest to the origin, we obtain the input sequence and corresponding state sequence listed in the last 3 columns of table 1. If we now use the total input–state sequence to make new estimates of the system matrices, we obtain the correct value $\hat{a}_{31} = 4 = a_{31}$.

6 Conclusions

The focus of the present contribution is on input signal design methods that are required for an accurate parameter estimation of max–plus–linear systems. Based on an already existing parameter estimation method and a condition for the determination of the true system parameters, a new input signal design method is developed and illustrated in an example. The method constitutes an improvement with respect to the already existing approaches in the sense that the set of all possible solutions can be characterized. Furthermore, additional requirements on the input signals can be incorporated in the design procedure, leading to an optimal input signal design.

Acknowledgements

This work has been partially supported by the Deutsche Forschungsgemeinschaft (DFG) under contract number Kr 949/9, by the Dutch Technology Foundation STW project “MPC for hybrid systems” (DMR.5675), and by the European IST project “SICONOS”(IST–2001–37172).

References

[1] F. Baccelli, G. Cohen, G. J. Olsder, and J.–P. Quadrat. *Synchronization and Linearity*. John Wiley & Sons, Chichester, 1992.

[2] C. Cassandras and S. Lafortune. *Introduction to Discrete Event Systems*. Kluwer Academic, Boston, 1999.

[3] R. Cuninghame–Green. *Minimax Algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer–Verlag, Berlin, 1979.

[4] B. De Schutter and B. De Moor. The extended linear complementarity problem. *Mathematical Programming*, 71(3):289–325, 1995.

[5] B. De Schutter and B. De Moor. Minimal realization in the max algebra is an extended linear complementarity problem. *Systems & Control Letters*, 25(2):103–111, 1995.

[6] B. De Schutter and B. De Moor. A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra. *Discrete Event Dynamic Systems*, 2(2):115–138, 1996.

[7] B. De Schutter, W.P.M.H. Heemels, and A. Bemporad. On the equivalence of linear complementarity problems. *Operations Research Letters*, 30(4):211–222, 2002.

[8] B. De Schutter and T. van den Boom. Model predictive control for max–plus–linear discrete event systems. *Automatica*, 37(7):1049–1056, July 2001.

[9] B. De Schutter, T.J.J. van den Boom, and V. Verdult. State space identification of max–plus–linear discrete event systems from input–output data. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 4024–4029, Las Vegas, 2002.

[10] F. Gallot, J. L. Boimond, and L. Hardouin. Identification of simple elements in max–algebra: Application to SISO discrete event systems modelisation. In *Proceedings of the 4th European Control Conference*, paper 488, Brussels, 1997.

[11] L. Ljung. *System Identification : Theory for the User*. Prentice Hall, Upper Saddle River, NJ, 2nd edition, 1999.

[12] E. Menguy, J. L. Boimond, L. Hardouin, and J. L. Ferrier. A first step towards adaptive control for linear systems in max algebra. *Discrete Event Dynamic Systems*, 10(4):347–368, 2000.

[13] G. Schullerus, J. Fox, and V. Krebs. The influence of measurement noise on the parameter estimation of max–plus–linear systems. In *Proceedings of the 7th European Control Conference*, Cambridge, UK, 2003.

[14] G. Schullerus and V. Krebs. Diagnosis of batch processes based on parameter estimation of discrete event models. In *Proceedings of the 6th European Control Conference*, pages 1612–1617, Porto, 2001.

[15] G. Schullerus and V. Krebs. A method for estimating the holding times in timed event graphs. In *Proceedings of the 6th International Workshop on Discrete Event Systems*, pages 209–216, Zaragoza, 2002.