Fixed Poles for the Disturbance Rejection by Measurement Feedback: the case without any controllability assumption.

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Abstract

The fixed poles for the problem of Disturbance Rejection by Measurement Feedback are geometrically and structurally characterized (in terms of invariant zeros and non controllable poles) for systems which are globally observable but without any controllability assumption and the dual case, system globally controllable but without any observability assumption.

Keywords: Linear multivariable systems, Geometric Control, Disturbance Rejection, Fixed Dynamics

1 Introduction

This paper deals with the Disturbance Rejection problem by (dynamic) Measurement Feedback, here noted as DRMF (see for instance [11] and [13]) in the case when the system can be uncontrollable but under some observability asumptions. As shown in [6] and[2], there exists a set of Fixed Poles, here noted as FP, for this problem, namely poles which are present in any output feedback closed loop system that rejects the disturbance, whatever be the way used to find the compensator. In [6], the authors consider just the poles in the transfer function matrix from the control input to the measured output. In [2], a more general notion of poles is used: the set of all the internal dynamics present in any feedback system solution through the overall state representation, but considering state descriptions which are globally minimal(i.e.controllable and observable) with respect to both input signals (control input and disturbance) and both output signals (controlled output and measurement).

In this paper, we propose geometric and structural (in terms of invariant zeros and uncontrollable poles of some subsystems) characterizations of the DRMF fixed poles in the more general case of (possibly) non controllability but considering global observability from the measurement and controlled output. Those results can easily be dualized to the case of (possibly) non observability but considering controllability restriction from the joint control and disturbance inputs. We also propose "optimal" solutions to the DRMF problem in terms of the FP, i.e., solutions for which all the poles, other than the FP, can be freely assigned. The characterization of "optimal" solutions in terms of pole placement is a stronger result than the previous ones concerning internal stability and pole shifting (see [13] and [7]). Indeed, solutions insuring internal stability, in general are not optimal in the sense of pole placement. The key notions behind our results are those of self-bounded and self-hidden invariant subspaces as introduced by [1].

The paper is organized as follows. Section 2 is devoted to some basic notions. In Section 3 the problem is stated and Section 4 gathers our main new result. The last section is devoted to concluding remarks.

2 Notation

We consider here linear time-invariant systems (A, B, C, D, E) described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ z(t) = Ex(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x(t) \in \mathcal{X} \approx \mathsf{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathsf{R}^m$ is the control input, $h(t) \in \mathcal{H} \approx \mathsf{R}^q$ is the disturbance input, $z(t) \in \mathcal{Z} \approx \mathsf{R}^r$ is the output to be controlled and $y(t) \in \mathcal{Y} \approx \mathsf{R}^p$ is the measured output. The same notation is used for maps and their matrix representations in particular bases $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$, $D : \mathcal{H} \to \mathcal{X}$, and $E : \mathcal{X} \to \mathcal{Z}$. We shall denote $\mathcal{B}$ the image of B, $\mathcal{D}$ the image of D, $\mathcal{C}$ the kernel of C, $\mathcal{E}$ the kernel of E, $\langle A \mid \mathcal{B}\rangle$ the controllable space of (A, B) and $\langle \mathcal{C} \mid A\rangle$ the unobservable space of (C, A).

We shall use in the sequel the following notation. sp(s). will stand for subspace(s). Given two sps. $\mathcal{T}$ and $\mathcal{L} \subset \mathcal{X}$, we shall note (see Wonham [14] and Basile & Marro [1]):

$\mathcal{V}^*_{(\mathcal{T},\mathcal{L})}$:=the supremal $(A, \mathcal{T})$-invariant subspace (inv.sp.) contained in $\mathcal{L}$,

$\mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$:=the infimal $(\mathcal{L}, A)$-inv.sp. containing $\mathcal{T}$,

$\mathcal{R}^*_{(\mathcal{T},\mathcal{L})} := \mathcal{V}^*_{(\mathcal{T},\mathcal{L})} \cap \mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$ = the supremal $(A, \mathcal{T})$-controllability sp. contained in $\mathcal{L}$,

$\mathcal{N}^*_{(\mathcal{L},\mathcal{T})} := \mathcal{V}^*_{(\mathcal{T},\mathcal{L})} + \mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$ =the infimal $(\mathcal{L}, A)$-complementary observability sp. containing $\mathcal{T}$.

Let $\mathcal{V}$ be an $(A, \mathcal{B})$-inv.sp. Then $\mathcal{F}(\mathcal{V})$ (also noted as $F_\mathcal{V}$) denotes the set of matrices F that satisfy: $(A + BF)\mathcal{V} \subset \mathcal{V}$. This set is also called "the friends of $\mathcal{V}$". If $\mathcal{V} \subset \mathcal{E}$ also satisfies $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{B} \subset \mathcal{V}$, then $\mathcal{V}$ is called $(A, \mathcal{B})$-self bounded with respect to $\mathcal{E}$. The set of all the sps. $(A, \mathcal{B})$-self bounded with respect to $\mathcal{E}$ is closed under addition and intersection and the infimum is $\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}$. Let $\mathcal{S}$ be a $(\mathcal{C}, A)$-inv.sp. Then, in a dual way, $\mathcal{G}(\mathcal{S})$ (or $G_\mathcal{S}$) are well defined and also the set of all sps. $(\mathcal{C}, A)$-self hidden with respect to $\mathcal{D}$ and the supremum $\mathcal{N}^*_{(\mathcal{C},\mathcal{D})}$ [1].

$\sigma(F_\mathcal{L} \mid \mathcal{L}/\mathcal{M})$ denotes the spectrum of the map induced by $(A + BF_\mathcal{L})$ in the quotient space $\frac{\mathcal{L}}{\mathcal{M}}$, where $\mathcal{M} \subset \mathcal{L}$ and both are $(A + BF_\mathcal{L})$-inv.sps. The spectrum of $(A + BF_\mathcal{V})$ can be decomposed (in connection with the $(A, \mathcal{B})$-inv.sp. $\mathcal{V}$) into fixed and free parts (see [12] or in a more general case [9], Lemma A.2, pp. 353). The fixed part (called the fixed spectrum of $\mathcal{V}$) is given by:

$$\sigma_{fix}(\mathcal{V}) := \sigma(F_\mathcal{V} \mid \mathcal{X}/\langle A \mid \mathcal{B}\rangle + \mathcal{V}) \,\dot{\cup}\, \sigma\left(F_\mathcal{V} \mid \mathcal{V}/\mathcal{R}^*_{(\mathcal{B},\mathcal{V})}\right) \tag{2}$$

for any $F_\mathcal{V}$ and where $\dot{\cup}$ stands for the union of sets with common elements repeated. The set $\sigma\left(F_\mathcal{V} \mid \mathcal{V}/\mathcal{R}^*_{(\mathcal{B},\mathcal{V})}\right)$ is called the internal fixed spectrum of $\mathcal{V}$ and $\sigma(F_\mathcal{V} \mid \mathcal{X}/\langle A \mid \mathcal{B}\rangle + \mathcal{V})$ the external fixed spectrum of $\mathcal{V}$. In a dual way, $\sigma(G_\mathcal{M} \mid \mathcal{L}/\mathcal{M})$, the spectrum of the map induced by $(A + G_\mathcal{M}C)$ in the quotient space $\frac{\mathcal{L}}{\mathcal{M}}$ is defined and can be decomposed (in connection with the $(\mathcal{C}, A)$-inv.sp. $\mathcal{S}$) into fixed and free parts in order to get $\sigma_{fix}(\mathcal{S})$.The fixed spectra of $\mathcal{V}$ and $\mathcal{S}$ can also be written in the following way:

Property 1:

$$\sigma_{fix}(\mathcal{V}) := \sigma\left(F_\mathcal{V} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle}\right) \dot{\cup}\, \sigma\left(F_\mathcal{V} \mid \frac{\mathcal{V} \cap \langle A \mid \mathcal{B}\rangle}{\mathcal{R}^*_{(\mathcal{B},\mathcal{V})}}\right)$$

$$\sigma_{fix}(\mathcal{S}) := \sigma\left(G_\mathcal{S} \mid \frac{\mathcal{N}^*_{(\mathcal{C},\mathcal{S})}}{\mathcal{S} + \langle \mathcal{C} \mid A\rangle}\right) \dot{\cup}\, \sigma\left(G_\mathcal{S} \mid \frac{\langle \mathcal{C} \mid A\rangle}{\{0\}}\right)$$

Property 2: The internal unassignable eigenvalues of $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ are equal to the external unassignable eigenvalues of $\mathcal{S}^*_{(\mathcal{E},\mathcal{B})}$ and correspond to the so-called invariant zeros of (A, B, E) (see [1]), i.e.:

$$\mathcal{Z}(A, B, E) := \sigma\left(F_{\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}} \mid \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}\right)$$
$$= \sigma\left(G_{\mathcal{S}^*_{(\mathcal{E},\mathcal{B})}} \mid \mathcal{N}^*_{(\mathcal{E},\mathcal{B})}/\mathcal{S}^*_{(\mathcal{E},\mathcal{B})}\right)$$

Note that in the same way, the invariant zeros of the systems (A, B, C), (A, D, C), (A, D, E), $(A, \left[B \vdots D\right], E)$, ... are well defined. A pair of sps. of $\mathcal{X}$, say $(\mathcal{S},\mathcal{V})$, is called a $(\mathcal{C}, A, \mathcal{B})$-pair if $\mathcal{S}$ is a $(\mathcal{C}, A)$-inv.sp., $\mathcal{V}$ is an $(A, \mathcal{B})$-inv.sp. and $\mathcal{S} \subset \mathcal{V}$[11].

3 Preliminaries

In particular, in this section we shall denote: $\mathcal{R}^*_c := \mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}$ and $\mathcal{N}^*_c := \mathcal{N}^*_{(\mathcal{C}\cap\mathcal{E},\mathcal{D})}$. Let us recall the definition of the DRMF problem. First of all, consider the general measurement feedback processor:

$$\begin{cases} \dot{w}(t) = Nw(t) + My(t) \\ u(t) = Lw(t) + Ky(t) \end{cases} \tag{3}$$

where $w(t) \in \mathcal{W} \approx \mathsf{R}^v$ is the state of the compensator. Then, the composite system (1) with the output feedback compensator (3) in the extended state space $\mathcal{X}_w = \mathcal{X} \oplus \mathcal{W}$ is described by:

$$\begin{cases} \dot{x}_w(t) = A_w x_w(t) + D_w h(t) \\ z(t) = E_w x_w(t) \end{cases} \tag{4}$$

where $A_w = \begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix}$; $D_w = \begin{bmatrix} D \\ 0 \end{bmatrix}$ and $E_w = \begin{bmatrix} E & 0 \end{bmatrix}$.

DRMF problem formulation: Find, if possible, a feedback processor for (1) of the type given by (3) such that for the closed loop system (4), the transfer function matrix from h to z be identically zero. Equivalently: find, if possible, an extension space $\mathcal{W}$ and a compensator matrix A_w such that an A_w-inv.sp. $\mathcal{V}_w \subset \mathcal{X}_w$ exists with $\mathrm{im} D_w \subset \mathcal{V}_w \subset \ker E_w$.

The basic geometric solvability condition for the DRMF problem ([3], [11] and [4]) states that the DRMF problem is solvable if and only if $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subset \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$, or equivalently, if and only if there exists a $(\mathcal{C}, A, \mathcal{B})$-pair, say $(\mathcal{S},\mathcal{V})$ ($(\mathcal{C}, A, \mathcal{B})$-pair solution), such that $\mathcal{D} \subset \mathcal{S} \subset \mathcal{V} \subset \mathcal{E}$.

The aim of this paper is then, assuming that the DRMF problem is solvable and considering that the system is $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observable:

- To show that there exists a maximal set of poles that are fixed and present in the closed loop system (A_w, D_w, E_w) for any compensator solution: these are called the DRMF fixed poles.
- To give geometric and invariant-zero characterizations of the DRMF fixed poles.
- To characterize "optimal" compensators solution to the DRMF problem, i.e., those for which all the poles are freely placed (modulo the compulsory symmetry with respect to the real axis) in the closed loop system except the DRMF fixed poles.

Let us define a class of compensators (Definition 1) and show (Lemma 1) how we can associate with any $(\mathcal{C}, A, \mathcal{B})$-pair solution to the DRMF problem, say $(\mathcal{S},\mathcal{V})$, one such compensator which is a solution to the problem and for which all the poles can be freely placed except the set $\sigma_{fix}(\mathcal{S}, \mathcal{V})$.

Definition 1 We shall call S-V based compensators, those which are designed from an associated $(\mathcal{C}, A, \mathcal{B})$-pair, say $(\mathcal{S},\mathcal{V})$, with a full order compensator structure, by taking $N = A + G_{\mathcal{S}}C + BF_{\mathcal{V}}L_2$; $M = -G_{\mathcal{S}} + BF_{\mathcal{V}}L_1$; $L = F_{\mathcal{V}}L_2$; and $K = F_{\mathcal{V}}L_1$ in (3), where $F_{\mathcal{V}} \in \mathcal{F}(\mathcal{V})$, $G_{\mathcal{S}} \in \mathcal{G}(\mathcal{S})$ and with L_1, L_2 such that: $L_1C + L_2 = I_n$ and $\ker L_2 \oplus (\mathcal{S} \cap \mathcal{C}) = \mathcal{S}$ (for the fact that such L_1 and L_2 always exist and for details about this kind of compensators see [1]).

Lemma 1 Consider a S-V based compensator designed from a $(\mathcal{C}, A, \mathcal{B})$-pair $(\mathcal{S},\mathcal{V})$, solution to the DRMF problem. Then, the S-V based compensator rejects the disturbance and the poles of the corresponding compensated system, $\sigma(A_w)$, can be freely placed by an adequate choice of friends of $\mathcal{V}$ and $\mathcal{S}$, except a set of poles that is fixed for any $G_{\mathcal{S}} \in \mathcal{G}(\mathcal{S})$ and any $F_{\mathcal{V}} \in \mathcal{F}(\mathcal{V})$, called the FP of $(\mathcal{S}, \mathcal{V})$, and which is given by: $\sigma_{fix}(\mathcal{S}, \mathcal{V}) := \sigma_{fix}(\mathcal{S}) \;\dot{\cup}\; \sigma_{fix}(\mathcal{V})$

Proof. The proof is exactly the same as in [2], Lemma 2 (where the asumptions $(A, [BD])$ controllable and $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observable were taken) , as the non controllable and non observable poles will always be included in $\sigma(A_w)$,i.e.: $\sigma(F_{\mathcal{V}} \mid \mathcal{X}/\langle A \mid \mathcal{B}\rangle + \mathcal{V}) \subset \sigma(A_w)$ and $\sigma(G_{\mathcal{S}} \mid \langle \mathcal{C} \mid A\rangle \cap \mathcal{S}) \subset \sigma(A_w)$. ■

Definition 1 and Lemma 1 will allow us to focus our search for particular solutions on the use of some $(\mathcal{C}, A, \mathcal{B})$-pair $(\mathcal{S},\mathcal{V})$, just keeping in mind that the S-V based compensators are naturally deduced from the chosen $(\mathcal{C}, A, \mathcal{B})$-pair.

4 Main Results

The characterization of the DRMF fixed poles has already been done in [2], but considering $(A, [BD])$ controllability and $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observability. We shall follow almost the same way to get our DRMF fixed poles but without any controllability consideration. We shall proceed as follows :

- First, we show that, starting from any compensated system solution there exists a particular $(\mathcal{C}, A, \mathcal{B})$-pair solution $(\mathcal{S}, \mathcal{V})$ which fixed spectrum $\sigma_{fix}(\mathcal{S}, \mathcal{V})$ is contained in the closed loop spectrum (Lemma 2). This result is valid independently on the way the compensator was found and is the key of the generality of our main results.
- In Lemma 3 we find a "better" $(\mathcal{C}, A, \mathcal{B})$-pair solution $(\overline{\mathcal{S}}, \overline{\mathcal{S}} + \mathcal{R}_c^*)$, with $\overline{\mathcal{S}} := \mathcal{S} + (\mathcal{N}_c^* \cap \mathcal{R}_c^*)$, leading to a set of FP which is included in $\sigma_{fix}(\mathcal{S}, \mathcal{V})$.
- Finally, in Lemma 4, we prove that this $(\mathcal{C}, A, \mathcal{B})$-pair solution belongs to a set of solutions $(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$ where the particular element $(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*)$ has the nice characteristic: $\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) \subset \sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$ and which is characterized independently on the particular values of the initial $(\mathcal{C}, A, \mathcal{B})$-pair solution $(\mathcal{S}, \mathcal{V})$. The proof that $\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*)$ is the set of DRMF FP follows directly.

Lemma 2 Consider that the DRMF problem is solvable. Let (4) be the compensated system with any particular measurement feedback solution of type (3), with $\sigma(A_w)$ the poles of the corresponding compensated system. Then, there exists a $(\mathcal{C}, A, \mathcal{B})$-pair solution $(\mathcal{S}, \mathcal{V})$ which satisfies $\sigma_{fix}(\mathcal{S}, \mathcal{V}) \subset \sigma(A_w)$, with $\mathcal{V}$ $(A, \mathcal{B})$-self bounded with respect to $\mathcal{E}$ and $\mathcal{S}$ $(\mathcal{C}, A)$-self hidden with respect to $\mathcal{D}$.

proof: The proof is the same that in the controllable-observable case (see [2], Lemma 3), just note that $\sigma(F \mid \mathcal{X}/\langle A \mid \mathcal{B}\rangle) \subset \sigma_{fix}(\mathcal{V})$ and $\sigma(G \mid \langle \mathcal{C} \mid A\rangle) \subset \sigma_{fix}(\mathcal{S})$, for any F and G.¥

Lemma 3 Consider that the DRMF problem is solvable. Let $(\mathcal{S}, \mathcal{V})$ be any $(\mathcal{C}, A, \mathcal{B})$-pair solution such that $\mathcal{S}$ is $(\mathcal{C}, A)$-self hidden with respect to $\mathcal{D}$ and $\mathcal{V}$ is $(A, \mathcal{B})$-self bounded with respect to $\mathcal{E}$. Let $\overline{\mathcal{S}} := \mathcal{S} + (\mathcal{N}_c^* \cap \mathcal{R}_c^*)$. Then $(\overline{\mathcal{S}}, \overline{\mathcal{S}} + \mathcal{R}_c^*)$ is a $(\mathcal{C}, A, \mathcal{B})$-pair solution that moreover satisfies $\sigma_{fix}(\overline{\mathcal{S}}, \overline{\mathcal{S}} + \mathcal{R}_c^*) \subset \sigma_{fix}(\mathcal{S}, \mathcal{V})$. Note that, as $\mathcal{V}$ contains $\mathcal{D}$, and $\mathcal{R}_c^*$ is the infimal $(A, \mathcal{B})$-self bounded sp. with respect to $\mathcal{E}$ containing $\mathcal{D}$ then $\mathcal{R}_c^* \subset \mathcal{V}$, and by duality $\mathcal{S} \subset \mathcal{N}_c^*$.

proof: The proof is the same that in the controllable-observable case (see [2], Lemma 3), just note that $\sigma(F \mid \mathcal{X}/\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^*) \subset \sigma_{fix}(\mathcal{V})$ and $\sigma(G \mid \langle \mathcal{C} \mid A\rangle \cap \mathcal{N}_c^*) \subset \sigma_{fix}(\mathcal{S})$, for any F and G.¥

Lemma 4 Assume that the system is $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observable and that the DRMF problem is solvable. Let us define: $\sigma^* := \sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*)$. Then, for any $(\mathcal{C}, A)$-inv.sp. $\mathcal{S}_i$ such that $\mathcal{N}_c^* \cap \mathcal{R}_c^* \subset \mathcal{S}_i \subset \mathcal{N}_c^*$, $(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$ is a (C, A, B)-pair solution which moreover satisfies: $\sigma^* \subset \sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$, and similarily, for any $(A, \mathcal{B})$-inv.sp. $\mathcal{V}_i$ such that $\mathcal{R}_c^* \subset \mathcal{V}_i \subset \mathcal{N}_c^* + \mathcal{R}_c^*$, $(\mathcal{V}_i \cap \mathcal{N}_c^*, \mathcal{V}_i)$ is a (C, A, B)-pair solution which moreover satisfies: $\sigma^* \subset \sigma_{fix}(\mathcal{V}_i \cap \mathcal{N}_c^*, \mathcal{V}_i)$.

Proof: Assume that $\mathcal{S}_{(\mathcal{C},\mathcal{D})}^* \subset \mathcal{V}_{(\mathcal{B},\mathcal{E})}^*$. Consider any $(\mathcal{C}, A)$-inv.sp. $\mathcal{S}_i$ such that $\mathcal{R}_c^* \cap \mathcal{N}_c^* \subset \mathcal{S}_i \subset \mathcal{N}_c^*$. By [2], Property A.4, the sp. $\mathcal{S}_i + \mathcal{R}_c^*$ is $(A, \mathcal{B})$-invariant included in $\mathcal{E}$, then the pair $(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$ is a $(\mathcal{C}, A, \mathcal{B})$-pair solution. The fixed spectrum for this $(\mathcal{C}, A, \mathcal{B})$-pair is: $\sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*) = \sigma_{fix}(\mathcal{S}_i) \ \dot{\cup} \ \sigma_{fix}(\mathcal{S}_i + \mathcal{R}_c^*)$.

From the dual version of [2] Property A.5: $\langle \mathcal{C} \cap \mathcal{E} \mid A\rangle = \langle \mathcal{C} \mid A\rangle \cap \mathcal{N}_c^*$, and from the observability hypothesis we obtain $\langle \mathcal{C} \cap \mathcal{E} \mid A\rangle = \langle \mathcal{C} \mid A\rangle \cap \mathcal{N}_c^* = 0$. Then, $\mathcal{S}_i \cap \langle \mathcal{C} \mid A\rangle = 0$ and we get $\sigma\left(G_{\mathcal{S}_i} \mid \frac{\mathcal{S}_i \cap \langle \mathcal{C} \mid A\rangle}{\{0\}}\right) = 0$. Then: $\sigma_{fix}(\mathcal{S}_i) = \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{S}_i)}^*/\mathcal{S}_i\right) = \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{D})}^*/\mathcal{S}_i\right) = \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{D})}^*/\mathcal{N}_c^*\right) \dot{\cup} \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_c^*/\mathcal{S}_i\right)$. From [2] Lemma A.3, by duality, we know that: $\sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / (\mathcal{S}_i + \mathcal{R}_c^*)) = \sigma(G_{\mathcal{S}_i} \mid \mathcal{N}_c^*/\mathcal{S}_i)$, with $F \in \mathcal{F}(\mathcal{R}_c^* + \mathcal{N}_c^*)$ and $G \in \mathcal{G}(\mathcal{S}_i)$. Then: $\sigma_{fix}(\mathcal{S}_i) = \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{D})}^*/\mathcal{N}_c^*\right) \dot{\cup} \sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / (\mathcal{S}_i + \mathcal{R}_c^*))$.

Let us define $\sigma_{\overline{c}} := \sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle + \mathcal{S}_i + \mathcal{R}_c^*}\right)$. Then:

$$\sigma_{fix}(\mathcal{S}_i + \mathcal{R}_c^*) =$$
$$= \sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid (\mathcal{S}_i + \mathcal{R}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{S}_i + \mathcal{R}_c^*)}^*\right) \dot{\cup} \sigma_{\overline{c}}$$
$$= \sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid (\mathcal{S}_i + \mathcal{R}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{E})}^*\right) \cup \sigma_{\overline{c}}$$

Then:

$$\sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*) = \sigma_{fix}(\mathcal{S}_i) \dot{\cup} \sigma_{fix}(\mathcal{S}_i + \mathcal{R}_c^*) =$$
$$\sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{D})}^*/\mathcal{N}_c^*\right) \dot{\cup} \sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / (\mathcal{S}_i + \mathcal{R}_c^*))$$
$$\dot{\cup} \sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid (\mathcal{S}_i + \mathcal{R}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{E})}^*\right) \cup \sigma_{\overline{c}}$$

and, since $\sigma_{fix}(\mathcal{N}_c^*) = \sigma\left(G_{\mathcal{S}_i} \mid \mathcal{N}_{(\mathcal{C},\mathcal{D})}^*/\mathcal{N}_c^*\right)$, due to the dual version of [2] Property A.5, $\sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*) = \sigma_{fix}(\mathcal{N}_c^*) \dot{\cup} \sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{E})}^*) \cup \sigma_{\overline{c}}$

On the other hand,

$$\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) = \quad (5)$$
$$\sigma_{fix}(\mathcal{N}_c^*) \dot{\cup} \sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{E})}^*) \cup \sigma_{\overline{c}} = \quad (6)$$
$$\sigma_{fix}(\mathcal{N}_c^*) \ \dot{\cup} \ \sigma(F \mid (\mathcal{R}_c^* + \mathcal{N}_c^*) / \mathcal{R}_{(\mathcal{B},\mathcal{E})}^*) \dot{\cup}$$
$$\sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^* + \mathcal{N}_c^*}\right) \quad (7)$$

as

$$\sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^* + \mathcal{N}_c^*}\right) \subset \sigma_{\overline{c}}$$

it is clear that

$$\sigma^* = \sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) \subset \sigma_{fix}(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$$

A similar procedure can easily be followed to get $\sigma^* \subset \sigma_{fix}(\mathcal{V}_i \cap \mathcal{N}_c^*, \mathcal{V}_i)$¥

We can now state our main result, which characterizes the DRMF Fixed Poles (DRMF FP):

Theorem 5 Assume that the system is $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observable and that the DRMF problem is solvable. Then:

- DRMF FP $= \sigma_{fix}(\mathcal{N}_c^*, \mathcal{N}_c^* + \mathcal{R}_c^*)$.
- $(\mathcal{N}_c^*, \mathcal{N}_c^* + \mathcal{R}_c^*)$ is a (C, A, B)-pair solution which leads to an optimal solution (to the DRMF problem) in the sense of maximal pole placement, i.e., from which all the poles can be freely placed (modulo symmetry/real axis) in the closed loop system, except the DRMF FP.

The proof is simply the connection between the different previous results, noting that $(\mathcal{N}_c^*, \mathcal{N}_c^* + \mathcal{R}_c^*)$ is himself a (C, A, B)-pair solution member of the family $(\mathcal{S}_i, \mathcal{S}_i + \mathcal{R}_c^*)$ characterized in Theorem 4. ¥

Lemma 6 Consider that $\mathcal{S}_{(\mathcal{C},\mathcal{D})}^* \subset \mathcal{V}_{(\mathcal{B},\mathcal{E})}^*$ and let us define $\mathcal{R}_e^* = \mathcal{N}_c^* + \mathcal{R}_c^*$, and $\langle A \mid \mathcal{B} + \mathcal{N}_c^*\rangle = \langle A \mid \mathcal{B}\rangle + \mathcal{N}_c^* + A\mathcal{N}_c^* + A^2\mathcal{N}_c^* + \cdots + A^{n-1}\mathcal{N}_c^*$. Then: $\langle A \mid \mathcal{B} + \mathcal{N}_c^*\rangle = \langle A \mid \mathcal{B}\rangle + \mathcal{R}_e^*$

Proof: Consider that $\mathcal{S}_{(\mathcal{C},\mathcal{D})}^* \subset \mathcal{V}_{(\mathcal{B},\mathcal{E})}^*$. It has been proved in [2], Property A.5 that if $\mathcal{D} \subset \mathcal{V}_{(\mathcal{B},\mathcal{E})}^*$ then $\langle A \mid \mathcal{B} + \mathcal{D}\rangle = \langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^*$ (remember that $\mathcal{R}_c^* = \mathcal{R}_{(\mathcal{B}+\mathcal{D},\mathcal{E})}^*$). From [2], Corollary A.2, $\mathcal{R}_e^* = \mathcal{R}_{(\mathcal{B}+\mathcal{N}_c^*,\mathcal{E})}^*$. Then, considering $\mathcal{D} = \mathcal{N}_c^*$, it is clear that $\langle A \mid \mathcal{B} + \mathcal{N}_c^*\rangle = \langle A \mid \mathcal{B}\rangle + \mathcal{R}_e^*$¥

This lemma allows us to derive a structural characterization of the F.P.

Theorem 7 Assume that the system is $(\begin{bmatrix} C \\ E \end{bmatrix}, A)$ observable and that the DRMF problem is solvable. Let us define E_e such that ker$E_e = \mathcal{R}_c^*$ and D_e such that im $D_e = \mathcal{N}_c^*$. The FP of the DRMF problem are given by:

$$\left\{\mathcal{Z}(A,D,C) - \mathcal{Z}\left(A, D, \begin{bmatrix} C \\ E_e \end{bmatrix}\right)\right\} \tag{8}$$

$$\dot{\cup} \tag{9}$$

$$\left\{\sigma(\overline{\langle A \mid \mathcal{B} + \mathcal{N}_c^* \rangle})\dot{\cup}\left\{\mathcal{Z}(A,B,E) - \mathcal{Z}(A,[B \vdots D],E)\right\}\right\} \tag{10}$$

where $\sigma(\overline{\langle A \mid \mathcal{B} + \mathcal{N}_c^* \rangle})$ are the non controllable poles of the composed system $(A,[B \vdots D_e])$.

Proof: The proof is a consequence of Theorem 5 and Lemma 6. Consider $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subset \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$. From (5):

$$\begin{gathered}\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) = \\ \sigma_{fix}(\mathcal{N}_c^*)\dot{\cup}\sigma(F|\left(\mathcal{R}_c^* + \mathcal{N}_c^*\right)/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}) \\ \dot{\cup}\sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^* + \mathcal{N}_c^*}\right)\end{gathered}$$

As $\sigma\left(F_{\mathcal{N}_c^* + \mathcal{R}_c^*} \mid \frac{\mathcal{N}_c^* + \mathcal{R}_c^*}{\mathcal{R}_c^*}\right) = \sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}_c^*}{\mathcal{R}_c^* \cap \mathcal{N}_c^*}\right)$ then

$$\begin{gathered}\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) = \sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}^*}{\mathcal{R}_c^* \cap \mathcal{N}_c^*}\right) \\ \dot{\cup}\sigma(F|\mathcal{R}_c^*/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}) \\ \dot{\cup}\sigma\left(F_{\mathcal{S}_i + \mathcal{R}_c^*} \mid \frac{\mathcal{X}}{\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^* + \mathcal{N}_c^*}\right)\end{gathered}$$

From Lemma 6, $\langle A \mid \mathcal{B}\rangle + \mathcal{R}_c^* + \mathcal{N}_c^* = \langle A \mid \mathcal{B} + \mathcal{N}_c^*\rangle$. Then:

$$\begin{gathered}\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) = \sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}^*}{\mathcal{R}_c^* \cap \mathcal{N}_c^*}\right) \\ \dot{\cup}\sigma(F|\mathcal{R}_c^*/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})})\dot{\cup}\sigma(\overline{\langle A \mid \mathcal{B} + \mathcal{N}_c^*\rangle}\end{gathered}$$

As

$$\begin{gathered}\sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}^*}{\mathcal{R}_c^* \cap \mathcal{N}_c^*}\right) = \\ \sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}^*}{\mathcal{S}^*_{(\mathcal{C},\mathcal{D})}}\right) - \sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{R}_c^* \cap \mathcal{N}_c^*}{\mathcal{S}^*_{(\mathcal{C},\mathcal{D})}}\right)\end{gathered}$$

From [2], Corollary A.2, $\mathcal{R}_c^* \cap \mathcal{N}_c^* = \mathcal{N}^*_{(\mathcal{C} \cap \mathcal{R}_c^*, \mathcal{E})}$, and $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} = \mathcal{S}^*_{(\mathcal{C} \cap \mathcal{R}_c^*, \mathcal{E})}$. Then

$$\begin{gathered}\sigma\left(G_{\mathcal{R}_c^* \cap \mathcal{N}_c^*} \mid \frac{\mathcal{N}^*}{\mathcal{R}_c^* \cap \mathcal{N}_c^*}\right) = \\ \sigma\left(G \mid \frac{\mathcal{N}^*_{(\mathcal{C},\mathcal{D})}}{\mathcal{S}^*_{(\mathcal{C},\mathcal{D})}}\right) - \sigma\left(G \mid \frac{\mathcal{N}^*_{(\mathcal{C} \cap \mathcal{R}_c^*, \mathcal{E})}}{\mathcal{S}^*_{(\mathcal{C} \cap \mathcal{R}_c^*, \mathcal{E})}}\right) \\ = \mathcal{Z}(A,D,C) - \mathcal{Z}\left(A, D, \begin{bmatrix} C \\ E_e \end{bmatrix}\right)\end{gathered}$$

with E_e such that ker$E_e = \mathcal{R}_c^*$. On the other hand, as $\mathcal{D} \subset \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$, $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} = \mathcal{V}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}$ (see [1]) and

$$\begin{gathered}\sigma\left(F_{\mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}} \mid \mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}\right) = \\ \sigma\left(F_{\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}} \mid \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}\right) - \\ \sigma\left(F_{\mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}} \mid \mathcal{V}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}/\mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}\right) = \\ \mathcal{Z}(A,B,E) - \mathcal{Z}(A,[B \vdots D],E)\end{gathered}$$

Finally:

$$\sigma_{fix}(\mathcal{N}_c^*, \mathcal{R}_c^* + \mathcal{N}_c^*) = \tag{11}$$

$$\left\{\mathcal{Z}(A,D,C) - \mathcal{Z}\left(A, D, \begin{bmatrix} C \\ E_e \end{bmatrix}\right)\right\}\dot{\cup} \tag{12}$$

$$\left\{\sigma(\overline{\langle A \mid \mathcal{B} + \mathcal{N}_c^* \rangle})\dot{\cup}\left\{\mathcal{Z}(A,B,E) - \mathcal{Z}(A,[B \vdots D],E)\right\}\right\} \tag{13}$$

By a dual procedure, we can easily obtain a similar result under the assumption $(A,[BD])$ controllable and without any observability consideration. In this case, DRMF FP $= \sigma_{fix}(\mathcal{N}_c^* \cap \mathcal{R}_c^*, \mathcal{R}_c^*)$:

Theorem 8 Consider that the system is $(A, \begin{bmatrix} B & D \end{bmatrix})$ controllable and assume that the DRMF problem is solvable. Then:

- DRMF FP $= \sigma_{fix}(\mathcal{N}_c^* \cap \mathcal{R}_c^*, \mathcal{R}_c^*)$.
- $(\mathcal{N}_c^* \cap \mathcal{R}_c^*, \mathcal{R}_c^*)$ is a (C,A,B)-pair solutions which leads to an optimal solution (to the DRMF problem) in the sense of maximal pole placement, i.e., from which all the poles can be freely placed (modulo symmetry/real axis) in the closed loop system, except the DRMF FP.

Theorem 9 Consider that the system is $(A, \begin{bmatrix} B & D \end{bmatrix})$ controllable and assume that the DRMF problem is solvable. Let us define E_e such that ker$E_e = \mathcal{R}_c^*$ and D_e such that im $D_e = \mathcal{N}_c^*$. The FP of the DRMF problem are given by:

$$\left\{\mathcal{Z}(A,D,C) - \mathcal{Z}\left(A, D, \begin{bmatrix} C \\ E \end{bmatrix}\right)\right\}\dot{\cup} \tag{14}$$

$$\sigma(\overline{\langle \mathcal{C} \cap \mathcal{R}_c^* \mid A \rangle})\dot{\cup}\left\{\mathcal{Z}(A,B,E) - \mathcal{Z}(A,[B \vdots D_e],E)\right\} \tag{15}$$

where $\sigma(\overline{\langle \mathcal{C} \cap \mathcal{R}_c^* \mid A \rangle})$ are the unobservable poles of the composed system $\left(\begin{bmatrix} C \\ E_e \end{bmatrix}, A\right)$.

5 Conclusions

We have presented here a new characterization of the fixed poles present in any solution of the DRMF problem under the hypothesis of $(A, \begin{bmatrix} C \\ E \end{bmatrix})$ observability and the dual case, the hypothesis of $(A, \begin{bmatrix} B & D \end{bmatrix})$ controllability. These fixed poles are imposed by the solution of the corresponding problems and are independent on the way the solution is obtained. Natural conclusions about the solvability of the corresponding disturbance rejection problem with internal stability can easily be stated: the necessary and sufficient condition is that the DRMF problem be solvable and the FP be stable.

Our next objective will be to consider the completely general case without neither controllability nor observability assumptions. This is important since in practice, the DROF problem may have to be considered on a part of a large and complex system with no guarantee that either controllability or observability holds for this part. The solution of the general case will also be helpful in tackling the H_2 Optimal Control problem without controllability or observability assumptions. Indeed, in [10], a non exact Disturbance Rejection problem has been considered using an H_2 Optimal Control approach. The aim is to minimize the H_2 norm of the transfer from the disturbance to the controlled output, while applying an internally stabilizing controller. There, H_2 Optimal FP also occur. However, in the case of dynamic measurement feedback, i.e. for the DRMF problem, the FP characterizations given in [10] are always related to particular classes of compensators. It has been proved in [5] that under controllability and observability assumptions, there exists an unique set of H_2 Optimal FP by dynamic measurement feedback, whatever be the way used to find the compensator. Our objective is to extend this result without using controllability or observability assumptions (just using stabilizability and detectability).

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