ON UPPER BOUNDS FOR REAL PROPORTIONAL STABILISING CONTROLLERS

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Abstract

The explicit determination of the largest admissable constant controller is difficult. For linear systems with all poles in the domain of stability but at least one zero outside, an improved upper bound on real proportional gain controllers is given. An example shows the improvement and further possible margins.

1 Introduction

Given the transfer function of a plant $p(z)$ and a controller $c(z)$ stabilising it in closed loop. What conditions are imposed by the plant configuration on the size of the controller?

Designate by $S(z) = (1 + p(z) \cdot c(z))^{-1}$ the so-called sensitivity function. It is then well-known that asymptotic internal stability of the closed feedback loop is equivalent to the following three properties [5] :

i) $S(z)$ is without poles outside the domain of stability (left half-plane or unit disc).

ii) Zeros $(S(z)) \supset$ Poles $(p(z))$ (outside the domain of stability).

iii) Zeros $(S(z) - 1) \supset$ Zeros $(p(z))$ (outside the domain of stability).

In this presentation, we study the following question: How large may we take a stabilising proportional controller? We will exhibit a new upper bound for rational (or more general: meromorphic) systems with all poles in the left half-plane, but at least one zero in the right half-plane. The bound is related to the sensitivity of zeros. It does not depend on the number of zeros.

We will outline analytic limitations to proportional controllers in the next section, present the steps leading to the bound in the following section, give the proofs subsequently, and close the paper with a worked example.

2 Known limitation for proportional control

Consider the closed right half-plane and suppose the rational function $p(z)$ has no poles there.

If we consider only scalar, real controllers $c(z) = k$, we find that of conditions i), ii), iii) as above only i) means a limitation. For $S(z) = (1 + p(z) \cdot c(z))^{-1}$ to have no poles in the right half-plane, while $c(z) = k$, this means $\frac{-1}{p(z)} \neq k$. This yields the following.

Fact: *Every real controller*

$$c(z) = k, |k| < [\sup \{|p(z)| : \Re z \geq 0, \Im(p(z)) = 0\}]^{-1} \quad (1)$$

is stabilising.

The right hand side of (1) was denoted L_{opt} by Blondel and Bertilsson in [1] as this figure is the maximal size of the largest proportional gain controller such that all smaller ones are stabilising. This value L_{opt} might in practice often be replaced by the unconstrained figure $\underline{L} := [\sup \{|p(z)| : \Re z \geq 0\}]^{-1}$ as $\underline{L} \leq L_{opt}$ is easier to compute. To assess any such approximation of the optimum upper bounds may be used. Thus, Blondel and Bertilsson [1] proved the following upper bound on *complex* proportional controllers k.

Theorem 1 *Let $p(s) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$ be the transfer function of a system with no poles in the closed right-half plane. Suppose $p(s)$ has at least one zero s_0 in the open right half-plane. Let m designate the number of zeros of $p(s)$ in the open right half plane, q the multiplicity of the zero s_0 and $\Re(s_0)$ the real part of the zero s_0. Suppose $k \in \mathbb{C}$ to be a stabilising proportional controller. Then* [1]

$$|k| < \frac{9 \cdot (m+1)\, q!}{|2 \cdot \Re(s_0)|^q \cdot \left|p^{(q)}(s_0)\right|}. \quad (2)$$

The above bound holds a fortiori for real k. The proof in [1] relies on the study of functions omitting two values. As the number of zeros is supposed to be finite and known, a complex function omitting the values 1 and 0 is constructed. Using the sharp version of a special result by Landau and Carathéodory (see [3]), the above bound is established. The quantity on the right hand side of (2) is denoted by $\overline{U}$ in [1], and is such that no complex controller of larger absolute value is stabilising. Hence, we have

$$\underline{L} \leq L_{opt} \leq \overline{U}. \quad (3)$$

We might want to inquire which real proportional controllers are not stabilising the system. The general bound for complex

[1] Please note: The term $q!$ is missing in [1].

proportional gain may be too conservative for this important case. Hence, we consider the largest absolute value $\overline{R}$ such that all *real* proportional compensators smaller in absolute value are stabilising. This allows to lower the factor $9(m+1)$ in (2) to a constant 2. Thus, we establish a bound $\overline{R}$ for *real* gain controllers with

$$\underline{L} \leq L_{opt} \leq \overline{R} < 9\overline{R} \leq \overline{U}.$$

We achieve this result by studying the maximal possible set of image values. This value set must be bounded for the functions considered. Taking into consideration the image covered by $p(z)$ depending on the maximum real value, we may establish a new bound using 'standard' conformal mappings to the unit disc. As we will see, we can sharpen this further to obtain bounds near the order of magnitude of the optimal value L_{opt}.

3 Functions with image restrictions

We build our analysis on the fact that the coefficients of a bounded function are bounded. One of the quantitative expressions for this is Gutzmer's lemma (see for ex. [4]), a consequence of the maximum principle. We denote the open unit disc by $\mathbb{D}$.

Lemma 1 *(Gutzmer)*

Suppose $g(z) : \mathbb{D} \to \mathbb{D}$ given as $g(z) = \sum_{v=0}^{\infty} g_v z^v$ is holomorphic. Then

$$\sum_{v=0}^{\infty} |g_v|^2 \leq 1.$$

We will derive the new bound as a consequence of the following observation, which is proved together with the theorem in the next section.

Proposition 1 *Suppose we have a meromorphic function $p(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$ with no poles in the closed unit disc $\overline{\mathbb{D}}$, but a zero z_0 at the origin of arbitrary multiplicity $q \geq 1$. Define $\delta := \sup\left\{|p(z)| : z \in \overline{\mathbb{D}}, \Im(p(z)) = 0\right\}$.*

Then

$$\frac{1}{2}|p_q| = \frac{1}{2}|\frac{p^{(q)}(0)}{q!}| \leq \delta.$$

Moreover, for a simple root

$$|\frac{p_1}{2\delta}|^2 + |\frac{p_2}{2\delta}|^2 \leq 1.$$

For the half-plane $\Re s > 0$ and an arbitrary zero there, the proposition translates via conformal mapping to the following.

Theorem 2 *Given a meromorphic transfer function $p(z)$ with no poles in the closed right half-plane. Define $\kappa := \sup\left\{|p(s)| : \Re s \geq 0, Im(p(s)) = 0\right\}$.*

Suppose that $s_0, \Re s_0 > 0$ is a zero of $p(s)$ with multiplicity q. Then

$$\frac{1}{2}\left|2 \cdot \Re\left(s_0\right)\right|^q \cdot \left|p^{(q)}\left(s_0\right)/q!\right| \leq \kappa. \tag{4}$$

With

$$f_1 := -2\Re s_0 |p'(s_0)|,$$
$$f_2 := 2\left(\frac{s_0+1}{\overline{s_0+1}}\right)^2 \left(p''(s_0)(\Re s_0)^2 - p'(s_0)\Re s_0 \frac{1}{s_0+2}\right),$$

for a simple root s_0 the bound κ is no less than the smallest positive root of

$$|\frac{f_1}{2\kappa}|^2 + |\frac{f_2}{2\kappa}|^2 = 1. \tag{5}$$

As it is clear from the above discussion of (1), the inverse of the maximum real value on the half-plane, i.e. $1/\kappa$, gives the supremum bound such that all smaller real proportional gains are stabilising controllers, i.e. L_{opt}. We infer our desired bound $\overline{R}$ for proportional controllers by taking the inverse of our function bound.

Corollary 1 *Given $p(z)$ meromorphic, with no poles in the closed right half-plane and at least one zero s_0 in the open half-plane. The largest real value $\overline{R}$ such that all real k with $|k| < \overline{R}$ are proportional stabilising $p(z)$ is bounded by*

$$\overline{R} < \frac{2}{\left|2 \cdot \Re\left(s_0\right)\right|^q \cdot \left|p^{(q)}\left(s_0\right)/q!\right|}, \tag{6}$$

where q denotes the multiplicity of the zero. Moreover, for a simple root an upper bound to $\overline{R}$ is the inverse of the smallest positive root of (5)*.*

Remark. Using the full force of Lemma 1, computing more coefficients we might improve this further. We give no explicit formulas here.

4 Derivation of results

Proof of the Proposition:

The meromorphic function $p(z)$ has by assumption no poles in the disc $|z| \leq 1$. The function values taken on the unit disc hence lie in a bounded domain. Thus, the limes superior of all real values, δ, is finite. Therefore, the image of $r(z)$ is contained in the doubly slit plane $M := \mathbb{C} \setminus \{(-\infty, -\delta] \cup [\delta, \infty)\}$.

We map this maximal image domain M back to the unit circle $\mathbb{D}$ via

$$\phi(s) = \frac{1 - \sqrt{\frac{1-s/\delta}{1+s/\delta}}}{1 + \sqrt{\frac{1-s/\delta}{1+s/\delta}}}. \tag{7}$$

Obviously, $\phi(0) = 0$. We have $\phi(M) \subset \mathbb{D}$, especially

$$\phi(p(\mathbb{D})) \subset \mathbb{D}. \tag{8}$$

Let the power series expansion of $p(z)$ be given as: $p(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$. Consider the composite function $\phi(p(z))$, and compute the first coefficients of the square root term in (7)

$$\begin{aligned}
\sqrt{\frac{1-p(z)/\delta}{1+p(z)/\delta}} &= \sqrt{1-\frac{2}{\delta}\frac{p(z)}{1+\frac{p(z)}{\delta}}} \\
&= \sqrt{1-\frac{2}{\delta}\cdot\frac{(p_0+p_1\cdot z+p_2z^2+\ldots)}{1+1/\delta\,(p_0+p_1z+p_2z^2+\ldots)}} \\
&\overset{p(0)=0}{=} \sqrt{1-\frac{2}{\delta}\cdot\frac{(p_1\cdot z+p_2z^2+\ldots)}{1+(1/\delta)\,(p_1z+p_2z^2+\ldots)}} \\
&= \sqrt{1-\frac{2}{\delta}\left(p_1z+\left(p_2-\frac{p_1^2}{\delta}\right)z^2+\ldots\right)} \\
&= 1-\left(\frac{1}{2}\right)\left(\frac{2}{\delta}\right)\left(p_1\cdot z+\left(p_2-\frac{p_1^2}{\delta}\right)z^2+\ldots\right) \\
&\quad -\frac{1}{8}\left(2/\delta\right)^2\left(p_1\cdot z+\left(p_2-(p_1/\delta)^2\right)z^2+\ldots\right)^2 \\
&\quad +\ldots. \\
&= 1-\frac{p_1}{\delta}\cdot z+\left(\frac{1}{2}\frac{p_1^2}{\delta^2}-\frac{p_2}{\delta}\right)z^2+z^3\cdot(\ldots)\,.
\end{aligned}$$

Thus, if we consider our mapping $\phi(p(z))$ we have

$$\begin{aligned}
&\phi(p(z)) \\
&\quad= \frac{1-\left(1-\frac{p_1\cdot z}{\delta}+\left[\frac{1}{2}\frac{p_1^2}{\delta^2}-\frac{p_2}{\delta}\right]\cdot z^2+\ldots\right)}{1+\left(1-\frac{p_1}{\delta}\cdot z+\left[\frac{1}{2}\frac{p_1^2}{\delta^2}-\frac{p_2}{\delta}\right]\cdot z^2\ldots\right)} \\
&\quad= \frac{\frac{p_1}{\delta}\cdot z-\left[\frac{1}{2}\frac{p_1^2}{\delta^2}-\frac{p_2}{\delta}\right]z^2+\ldots}{2-\frac{p_1}{\delta}\cdot z+\ldots} \\
&\quad= \frac{p_1}{2\delta}\cdot z+\left(\frac{p_2}{2\delta}\right)\cdot z^2+\ldots\,.
\end{aligned}$$

Using now Gutzmer's coefficient bound for the first coefficient $\phi(p(z))$, we find esp.$\left|\frac{1}{2\cdot\delta}p_1\right|^2 \leq 1$, hence $\left|\frac{p_1}{2}\right| \leq \delta$. We observe, that the analogue result holds true for p_q in place of p_1, if $0 = p_0 = p_1 = \ldots = p_{q-1}$. The first claim of the proposition follows as $p_q = p^{(q)}(0)/q!$.

Consider the special case of a simple zero. Using Gutzmer's bound again, we find

$$\left|\frac{p_1}{2\delta}\right|^2 + \left|\frac{p_2}{2\delta}\right|^2 \leq 1$$

□

Proof of the Theorem:

We proceed as in [1], to compose a suitable mapping of the unit disc to the right-half plane which transfers the origin to the zero s_0:

For the canonical mapping of the unit disc to the right half-plane $H^+ := \{z : \Re z > 0\}$,

$$\sigma : \mathbb{D} \to H^+,\ \omega \longmapsto \frac{1+\omega}{1-\omega}$$

define z_0 by

$$\sigma(z_0) = s_0 \ \Leftrightarrow\ \frac{1+z_0}{1-z_0} = s_0 \Leftrightarrow z_0 = \frac{s_0-1}{s_0+1} = 1-\frac{2}{s_0+1}.$$

Thus, given z_0 we define

$$\mu : \mathbb{D} \to \mathbb{D}\ ,\ \lambda \longmapsto \frac{\lambda - z_0}{\lambda\cdot\overline{z_0}-1},$$

and the composite function $\sigma\left(\mu\left(\cdot\right)\right)$ maps the unit disc to the right half plane. As $\mu\left(0\right) = z_0$ and $\sigma\left(z_0\right) = s_0$, the origin is mapped to s_0.

As by assumption, s_0 is a q-fold zero of $p(z)$, there is a q-fold zero of $f(z) := p\left(\sigma\left(\mu\left(z\right)\right)\right) = \sum f_\nu z^\nu$ at the origin. Using our proposition, we find that the real maximum δ of f on the unit circle which is equal the real maximum κ of p on the half-plane is lower bounded by

$$\left|\frac{f_q}{2}\right|^2 \leq \kappa.$$

For a simple zero we obtain

$$\left|\frac{f_1}{2\kappa}\right|^2 + \left|\frac{f_2}{2\kappa}\right|^2 \leq 1. \tag{9}$$

This leaves us the task to compute $f_1,\ f_2$ for simple zeros in terms of our original function $p(z)$. (The case of a multiple zero is completely analogue).

We find for

$$f\left(z\right) = p\left(\sigma\left(\mu\left(z\right)\right)\right)$$

that

$$f'(z) = p'\left(\sigma\left(\mu\left(z\right)\right)\right)\cdot\sigma'\left(\mu\left(z\right)\right)\cdot\mu'\left(z\right) \tag{10}$$

and

$$\begin{aligned}
&f''\left(z\right) = \\
&\quad p''\left(\sigma\left(\mu\left(z\right)\right)\right)\cdot\left(\sigma'\left(\mu\left(z\right)\right)\right)\cdot\mu'\left(z\right)^2 + \\
&\quad \ldots + p'\left(\sigma''\left(z\right)\right)\left[\sigma''\left(\mu\left(z\right)\right)\left(\mu'\left(z\right)\right)^2+\sigma'\left(\mu\left(z\right)\right)\mu''\left(z\right)\right]
\end{aligned} \tag{11}$$

From the definition, we have

$$
\begin{aligned}
\sigma(z) &= \frac{1+z}{1-z} = 1 + \frac{2z}{(1-z)}, \\
\sigma'(z) &= \frac{2(1-z)+2z}{(1-z)^2} = \frac{2}{(1-z)^2} \\
\sigma''(z) &= \frac{-2(-2)(1-z)}{(1-z)^4} = \frac{4}{(1-z)^3} = \frac{2\sigma'(z)}{(1-z)} \\
\mu(z) &= \frac{\lambda - z_0}{\lambda \cdot \overline{z_0} - 1}, \\
\mu'(z) &= \frac{(\lambda \cdot \overline{z_0} - 1) - \overline{z_0} \cdot (\lambda - z_0)}{(\lambda \cdot \overline{z_0} - 1)^2} \\
&= \frac{\lambda(\overline{z_0} - \overline{z_0}) + z_0 \cdot \overline{z_0} - 1}{(\lambda \cdot \overline{z_0} - 1)^2} = \frac{z_0 \cdot \overline{z_0} - 1}{(\lambda \cdot \overline{z_0} - 1)^2} \\
\mu''(z) &= -2\overline{z_0}\,(\lambda \cdot \overline{z_0} - 1) \cdot (z_0 \cdot \overline{z_0} - 1) \Big/ (\lambda\overline{z_0} - 1)^4 \\
\mu'(0) &= z_0 \cdot \overline{z_0} - 1, \\
\mu''(0) &= 2 \cdot \overline{z_0} \cdot (z_0 \cdot \overline{z_0} - 1) = 2 \cdot \overline{z_0} \cdot (\mu'(0))
\end{aligned}
$$

This gives the first coefficient of f by (10) as

$$
\begin{aligned}
f_1 = f'(0) &= p'(s_0) \cdot \sigma'(z_0) \cdot \mu'(0) \\
&= p'(s_0) \cdot \frac{2}{(1-z_0)} \cdot (z_0 \cdot \overline{z_0} - 1)
\end{aligned}
$$

With $z_0 = \sigma(s_0) = \frac{s_0 - 1}{s_0 + 1} = 1 - \frac{2}{s_0+1}$ we find

$$
\begin{aligned}
f_1 &= f'(0) \\
&= -4p'(s_0)\,\frac{1}{2}\,(s_0+1)^2\,\frac{Re\,(s_0)}{(s_0+1) \cdot (\overline{s_0+1})} \qquad (12) \\
&= -2\,p'(s_0) \cdot Re\,(s_0) \cdot \frac{s_0+1}{\overline{s_0+1}} \qquad (13)
\end{aligned}
$$

With s_0 inside the right half-plane the absolute value of this coefficient is:

$$
|f_1| = |p'(s_0)| \cdot 2 \cdot Re\,(s_0)
$$

Computing the derivatives from (10), this translates (just as in [1]) to the situation of a multiple zero as:

$$
|f_q| = \left| p^{(q)}(s_0)\,/q! \right| \cdot |2 \cdot \Re(s_0)|^q
$$

Suppose now finally the case of a simple zero. The computation of f_2 from (11) is slightly more tedious. We have

$$
\begin{aligned}
& p''(\sigma(\mu(z))) \cdot (\sigma'(\mu(z)) \cdot \mu'(z))^2_{|z=s_0} \\
&= p''(s_0) \cdot \left(-2 \cdot \Re(s_0) \cdot \frac{s_0+1}{\overline{s_0+1}}\right)^2.
\end{aligned}
$$

The term $\left[\sigma''(\mu(z)) \cdot (\mu'(z))^2 + \sigma'(\mu(z)) \cdot \mu''(z)\right]$ evaluated at zero gives:

$$
\begin{aligned}
& \sigma''(\mu(0)) \cdot (\mu'(0))^2 + \sigma'(\mu(0)) \cdot \mu''(0) \\
&= \sigma'(\mu(0)) \cdot (\mu'(0)) \cdot \frac{2 \cdot \mu'(0)}{1 - \mu(0)} + \sigma'(\mu(0)) \cdot \mu'(0) \cdot 2 \cdot \overline{z_0} \\
&= \sigma'(\mu(0)) \cdot (\mu'(0)) \left[\frac{2 \cdot \mu'(0)}{1 - \mu(0)} + 2\overline{z_0}\right].
\end{aligned}
$$

The term $\left[\frac{2 \cdot \mu'(0)}{1-\mu(0)} + 2\overline{z_0}\right]$ where $\mu'(0) = z_0 \cdot \overline{z_0} - 1$, $\mu(0) = z_0$ gives

$$
\begin{aligned}
& 2 \cdot \frac{(z_0 \cdot \overline{z_0} - 1)}{1 - z_0} + 2 \cdot \overline{z_0}\frac{(1 - z_0)}{1 - z_0} \\
&= \frac{-2 + 2\overline{z_0}}{1 - z_0} = -2 \cdot \frac{(\overline{z_0} - 1)}{z_0 + 1} = -2 \cdot \left(\frac{-2\,/\overline{s_0+1}}{2 + \frac{2}{s_0+1}}\right) \\
&= 4 \cdot \frac{(s_0+1)/(\overline{s_0+1})}{2 \cdot s_0 + 2 + 2} = 2 \cdot \frac{s_0+1}{\overline{s_0+1}} \cdot \frac{1}{s_0+2}.
\end{aligned}
$$

The second coefficient is finally computed as

$$
\begin{aligned}
f_2 &= f''(0)/2 \\
&= 2\left(\frac{s_0+1}{\overline{s_0+1}}\right)^2 \left(p''(s_0)(\Re s_0)^2 - \frac{p'(s_0)\Re s_0}{s_0+2}\right) \qquad (14)
\end{aligned}
$$

Hence, the lower bound for simple zeros is obtained from (9) with (13) and (14). □

Remark. The Corollary may be strengthened by further coefficient computations. We may have given more formulas to state a further improved estimate for multiple zeros as well. The way to proceed is strictly as above.

5 Example

In [1] the following transfer function was considered

$$
p(s) = \frac{(s-2)(s+1)}{2s^3 + s^2 + 3s + 1}. \qquad (15)
$$

There is no pole in the closed right half plane and a single zero s_0 with $Re\, s_0 > 0$, thus $s_0 = 2$, $m = 1$. The derivative of $p(z)$ is

$$
p'(z) = \frac{-2 \cdot s^4 + 4 \cdot s^3 + 16 \cdot s^2 + 6s + 5}{(2 \cdot s^3 + s^2 + 3 \cdot s + 1)^2}
$$

hence $p'(s_0) = p'(2) = 1/9$.

Blondel and Bertilsson derived the upper bound for the modulus of a *complex* gain as (2), namely

$$
\frac{9 \cdot (m+1)\,q!}{|2Re\,(s_0)|^q\,|p^q(s_0)|},
$$

which evaluates to

$$\frac{9 \cdot 2}{4 \cdot 1/9} = 40.5.$$

From the first bound in Theorem 2 we find the upper bound $\overline{R}$ to be smaller than

$$\frac{2}{f_1} = \frac{2}{|2 \cdot Re\,(s_0)| \cdot |p'\,(s_0)|} = \frac{9}{2} = 4.5.$$

Willing to use more information from the transfer function we calculate first $p''\,(s_0) = \frac{-44}{243}$, and then

$$\begin{aligned} f_2 = f''\,(0)\,/2 &= 2\left(\frac{3}{3}\right)^2 \left(p''(2)2^2 - p'(2)\frac{2}{4}\right) \\ &= 2(\frac{-44}{243} \cdot 4 - \frac{1}{9} \cdot \frac{2}{4}) = \frac{-758}{486}. \end{aligned}$$

This gives by the second estimate of our Theorem 2 a lower bound of the maximal real value κ via

$$|\frac{f_1}{2\kappa}|^2 + |\frac{f_2}{2\kappa}|^2 \leq 1 \quad \text{as} \quad \sqrt{146557}/243 \leq \kappa.$$

Taking the inverse gives the second controller bound of Corollary 1. Hence we find that the upper bound $\overline{R}$ to real proportional gain is no greater than

$$0.635.$$

What is the actual largest admissable value for a constant real controller k? Computing the root loci of the family

$$2s^3 + s^2 + 3s + 1 + k \cdot (s^2 - s - 2),$$

we find that all positive, real values k, $0 \leq k < 0.5$ are admissable, while $k = 0.5$ yields an unstable function. The negative proportional controllers may not be chosen smaller than -0.1625. Thus, we must compare the new upper bounds 4.5 and 0.635 to 0.1625 which shows that further improvement of the latter bound is limited to a factor smaller than 4.

From the numerical example above, we infer that the first part of Corollary 1 does not hold true when the bound is multiplied by a constant smaller than $1/28$. The second part of Corollary 1 does not hold true when the bound is multiplied by a factor smaller than $1/4$.

6 Conclusion

Using complex analysis, we have given general upper bounds for real proportional controllers. The new bound improves the known one by a factor of at least 9, and is in contrast not dependent on the number of roots. A systematic way to improve the new bound has been outlined. The general bound may not be improved by any constant factor smaller $1/27$, while the first of the improved bounds may not be lowered by a factor smaller than $1/4$.

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