A FRACTIONAL IDEAL APPROACH TO STABILIZATION PROBLEMS

A. Quadrat*

* INRIA Sophia Antipolis, CAFE project, 2004 route des lucioles, BP 93, 06902 Sophia Antipolis cedex, France.
Alban.Quadrat@sophia.inria.fr

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Abstract

The purpose of this paper is to show how the theory of fractional ideals is a powerful mathematical framework for the study of stabilization problems of linear SISO systems. In particular, in terms of fractional ideals, we give necessary and sufficient conditions for a plant to be internally/strongly/bistably stabilizable or to admit a (weakly) coprime factorization. Finally, we show how to extend the Youla-Kučera parametrization of all stabilizing controllers to every stabilizable plant which does not (necessarily) admit coprime factorizations.

1 Introduction

In the eighties, *the fractional representation approach to analysis and synthesis problems* has been developed by M. Vidyasagar, C. A. Desoer and co-authors in order to settle in a unique mathematical framework different stabilization problems (internal/strong/simultaneous stabilization, parametrization of all stabilizing controllers, graph topology, gap metric, margins of robustness, optimal controllers ...) [3, 10]. In the nineties, certain ideas of this approach have been at the core of the successful development of H_∞-control for linear finite dimensional systems. However, for certain classes of linear infinite dimensional systems and multidimensional systems, some questions on stabilization problems are still open [3, 4, 8, 10].

In this paper, we show that the introduction of *fractional ideals* [2] within the fractional representation approach to synthesis problems allows us to obtain general necessary and sufficient conditions for internal/strong/bistable stabilization or for the existence of (weakly) coprime factorizations. The main idea is to replace transfer functions by means of ideals [8], and then, to use the powerful theory of ideals to obtain certain information that could be difficult to find if we use different approaches. This approach gives simple and tractable characterizations of the structural properties of SISO systems. For instance, it is possible to clarify the relationship between internal stabilizability and the existence of coprime factorizations. In particular, we recover the fact that internal stabilizability does not generally imply the existence of coprime factorizations [8]. The Youla-Kučera parametrization of all stabilizing controllers was developed for plants that admit coprime factorizations. We extend the Youla-Kučera parametrization to every stabilizable plant which does not (necessarily) admit coprime factorizations. As the Youla-Kučera parametrization, this new parametrization is affine in the free parameters but generally has two free parameters. If p admits a coprime factorization, then we show that this parametrization is the Youla-Kučera one and if p does not admit a coprime factorization but p^2 does, then this parametrization only admits one free parameter. Finally, using the concept of *Picard group* [2], we characterize the rings A of SISO (proper) stable plants [3, 10] over which every stabilizable plant, defined by a transfer function $p = n/d$, $0 \neq d,\ n \in A$, admits a Youla-Kučera parametrization or a parametrization with one or two free parameters.

Notation: A will denote a commutative integral domain ($\forall\, a,\, b \in A,\ \ a\,b = b\,a,\ a\,b = 0,\ \ b \neq 0 \Rightarrow a = 0$) and $K = Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}$ the *quotient field* of A [2]. $\mathrm{U}(A) = \{a \in A \mid \exists\, b \in A:\ a\,b = 1\}$ will be the *group of invertible elements* of A, $M_q(A)$ the ring of $q \times q$ matrices with entries in A. Finally, if $a_1, \ldots, a_n \in K = Q(A)$, then $(a_1, \ldots, a_n)$ will denote the A-module $A\,a_1 + \ldots + A\,a_n = \{\sum_{i=1}^n \lambda_i\, a_i \mid \lambda_i \in A\}$ and $\triangleq$ will mean "by definition".

2 Fractional representation approach to synthesis problems

The main idea of the fractional representation approach to synthesis problems [3, 10] is to write the stabilization problems into general forms so that they can be applied to different classes of linear systems (e.g. ordinary differential equations, time-delay systems, multidimensional systems, infinite dimensional systems, discrete systems ...). More precisely, we first consider *a ring A of (proper) stable SISO plants* (e.g. $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$, $l_1(\mathbb{Z}_+)$) [3, 10]. Then, the class of systems is defined by the set of plants whose transfer functions are of the form $p = n/d$ with $0 \neq d,\ n \in A$. Hence, $K = Q(A)$ modelizes the set of stable/unstable or proper/improper plants which have transfer functions $p \in K = Q(A)$.

Example 1. For finite dimensional systems, we consider *the integral domain $A = RH_\infty$ of proper stable real rational transfer function* [10], i.e. we have:

$$A = \{n/d \mid n,\, d \in \mathbb{R}[s], \quad \deg n \leq \deg d, \qquad d(s^\star) = 0 \Rightarrow \mathrm{Re}\, s^\star < 0\}.$$

A transfer function which belongs to A corresponds to a proper and stable plant. Now, we have $K = Q(A) = \mathbb{R}(s)$, i.e. every element $p \in K$ can be written as $p = n/d$ with $n,\, d \in A$. For instance, the unstable plant $1/(s-1)$ can be written as $p = n/d$ with $n = 1/(s+1) \in A$ and $d = (s-1)/(s+1) \in A$.

There are different classes of "stable" infinite dimensional systems depending on what criterium of stability we are dealing with. A first example is the integral domain $A = H_\infty(\mathbb{C}_+)$ of the holomorphic functions f in $\mathbb{C}_+ \triangleq \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$ which are bounded w.r.t. the norm $\| f \|_\infty \triangleq \sup_{s \in \mathbb{C}_+} |f(s)|$. $H_\infty(\mathbb{C}_+)$ is a *Banach algebra* [3, 10]. A transfer function $p \in A$ is stable in the sense that the linear operator

$$\begin{array}{rcl} T_p : H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ u & \longmapsto & p\, u, \end{array}$$

is *bounded*, where $H_2(\mathbb{C}_+)$ denotes the Hilbert space of holomorphic functions in $\mathbb{C}_+$ which are bounded w.r.t. to the norm $\| u \|_2 \triangleq \sup_{x \in \mathbb{R}_+} (\int_{-\infty}^{+\infty} |u(x + i\, y)|^2\, dy)^{1/2}$ [3, 7]. For instance, $p = e^{-s}/(s-1) \notin A$ because p has the unstable pole $1 \in \mathbb{C}_+$. However, we have $p = n/d \in K = Q(A)$, with $n = e^{-s}/(s+1) \in A$ and $d = (s-1)/(s+1) \in A$.

Another class of "stable" infinite dimensional systems is the Wiener algebra [3] defined by:

$$\begin{array}{l} \mathcal{A} = \{\ f(t) + \sum_{i=0}^{\infty} a_i\, \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+), \\ \qquad (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots\}. \end{array}$$

The two operations of $A = \mathcal{A}$ are $+$ and the convolution $\star$ and the Dirac distribution δ is the unit of A. A is an integral domain [3]. Endowed with the topology defined by the norm

$$\begin{array}{rl} \| g \|_{\mathcal{A}} & = \| f \|_{L_1(\mathbb{R}_+)} + \| (a_i)_{i \geq 0} \|_{l_1(\mathbb{Z}_+)} \\ & \triangleq \int_0^{+\infty} |f(t)|\, dt + \sum_{i=0}^{+\infty} |a_i|, \end{array}$$

$\mathcal{A}$ becomes a Banach algebra [3, 10]. The same properties hold for $\hat{\mathcal{A}} = \{\hat{f} \mid f \in \mathcal{A}\}$, where $\hat{\ }$ is the Laplace transform, and with the norm $\| \hat{f} \|_{\hat{\mathcal{A}}} \triangleq \| f \|_{\mathcal{A}}$. A transfer function $p \in A$ is stable in the sense that, for any $1 \leq p \leq +\infty$, the linear operator $T_p : L_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+)$, defined by $T_p(u) = p \star u$, is bounded [3, 7].

Definition 1.

- [3, 10] A transfer function $p \in K = Q(A)$ is *internally stabilizable* if there exists *a stabilizing controller* of p, namely $c \in K = Q(A)$, such that:

$$\begin{array}{rl} H(p, c) & = \left(\begin{array}{cc} 1 & c \\ p & 1 \end{array}\right)^{-1} \\ & = \left(\begin{array}{cc} \frac{1}{1 - p\, c} & -\frac{c}{1 - p\, c} \\ -\frac{p}{1 - p\, c} & \frac{1}{1 - p\, c} \end{array}\right) \in M_2(A). \end{array} \quad (1)$$

  Equivalently, this means that $1/(1 - p\, c),\ p/(1 - p\, c)$ and $c/(1 - p\, c)$ are A-stable.

- [10] A transfer function $p \in K = Q(A)$ is *strongly stabilizable* (resp. *bistably stabilizable*) if there exists a stable (resp. stable with a stable inverse) stabilizing controller c of p, i.e. $c \in A$ (resp. $c \in \mathrm{U}(A)$).

- [4, 5, 9] A transfer function $p \in K = Q(A)$ admits a *weakly coprime factorization* if there exist $0 \neq d,\ n \in A$ such that $p = n/d$ and:

$$\forall\, k \in K :\ k\, n,\ k\, d \in A \Rightarrow k \in A.$$

- [3, 10] A transfer function $p \in K = Q(A)$ admits a *coprime factorization* if there exist $0 \neq d,\ n \in A$ and $x,\, y \in A$ such that $p = n/d$ and:

$$d\, x - n\, y = 1.$$

For other structural properties and stabilization problems, we refer to [3, 7, 10] and the references therein. For a lack of space, in this paper, we shall only study the previous ones.

3 Theory of fractional ideals

Let us recall that an ideal I of A, defined by $a_1, \ldots, a_n$ of A, is the A-submodule of A defined by $I = \sum_{i=1}^{n} A\, a_i$. The theory of fractional ideals is an extention of the well-known theory of ideals of a ring A. The main idea of this theory is to develop a mathematical framework which can treat at the same time the elements of A and the elements of $K = Q(A)$. The motivation of the introduction of this theory within the fractional representation approach to systems seems clear: we shall have the possibility to study stable and unstable/improper plants in the same mathematical framework.

Definition 2. We have the following definitions [2]:

- A *fractional ideal* J of A is an A-submodule of $K = Q(A)$ such that there exists $0 \neq a \in A$ satisfying:

$$(a)\, J \triangleq \{a\, b \in K \mid b \in J\} \subseteq A.$$

- A fractional ideal J of A such that $J \subseteq A$ is called an *integral ideal of* A.

- A fractional ideal J of A is *principal* if there exists $k \in K$ such that $J = (k) \triangleq A\, k$.

- A non-zero fractional ideal J is *invertible* if there exists a non-zero fractional ideal I of A such that we have:

$$I\, J = A.$$

Example 2. Let $p \in K$ be the transfer function of the system:

$$y = p\, u \Leftrightarrow (1 :\ -p) \left(\begin{array}{c} y \\ u \end{array}\right) = 0.$$

Module theory approach to linear system tells us that the structural properties only depend on the whole system, i.e. input and output together (without separation), and thus, on the A-module $J = A\,(1) + A\,(-p) = A + A\, p$ [4, 5]. In fact, $J = (1,\, p) \triangleq A + A\, p$ is a fractional ideal of A. Indeed, there exist $0 \neq d,\ n \in A$ and such that $p = n/d$. Thus, we have $J = (1,\, p) = (1/d)\,(d,\, n)$, where $I = (d,\, n)$ is an integral ideal of A, i.e. $I \subseteq A$. Therefore, we have $(d)\, J = (d,\, n) \subseteq A$.

We shall denote by $\mathcal{F}(A)$ *the set of non-zero fractional ideals of* A. Let $I,\, J \in \mathcal{F}(A)$, then the following ideals

$$\left\{\begin{array}{l} I \cap J \triangleq \{a \in I,\, a \in J\}, \\ I + J \triangleq \{a + b \mid a \in I,\, b \in J\}, \\ I\, J \triangleq \{\sum_{i=1}^{n} a_i\, b_i \mid a_i \in I,\, b_i \in J, n \in \mathbb{Z}_+\}, \\ I : J \triangleq \{k \in K = Q(A) \mid (k)\, J \subseteq I\}, \end{array}\right.$$

are also fractional ideals of A, i.e. belong to $\mathcal{F}(A)$. Hence, $\mathcal{F}(A)$ is stable under respectively intersections, sums, products and residuals. Moreover, we have the following relations.

Lemma 1. *[2] If $I,\, J,\, L$ are elements of $\mathcal{F}(A)$, then we have the following equalities:*

$$\left\{\begin{array}{l} I\,(J + L) = I\,J + I\,L, \\ I : (J + L) = (I : J) \cap (I : L), \\ I : (J\,L) = (I : J) : L = (I : L) : J. \end{array}\right.$$

Proposition 1.
- [2] If J is an invertible fractional ideal of A, then there exists a unique ideal $I \in \mathcal{F}(A)$ such that $I\,J = A$. This fractional ideal is denoted by J^{-1} and we have $J^{-1} = A : J = \{k \in K = Q(A) \mid (k)\, J \subseteq A\}$.
- [2] If J is an invertible fractional ideal of A, then we have $(J^{-1})^{-1} = J$. Hence, an invertible fractional ideal J of A is *divisorial*, namely J satisfies $J = A : (A : J)$.

The next theorem characterizes the structural properties of plants and the stabilization problems defined in Definition 1 in terms of the properties of the fractional ideal $J = (1,\, p)$.

Theorem 1. *Let A be an integral domain of (proper) stable SISO plants and $K = Q(A)$ its quotient field. Let $p \in K$ be a transfer function of a plant and $J = (1,\, p) \triangleq A + A\,p$ the fractional ideals of A defined by 1 and p. Then, we have:*

1. *p is stable, i.e. $p \in A$, iff $J = A$, or equivalently, iff $A : J = A$.*
2. *p admits a weakly coprime factorization iff $A : J$ is a non-zero principal integral ideal of A, i.e. there exists $0 \neq d \in A$ such that $A : J = (d)$. Then, $p = n/d$, where $n = d\,p \in A$, is a weakly coprime factorization of p.*
3. *p is internally stabilizable iff the fractional ideal J is invertible, or equivalently, there exist $a,\, b \in A$ such that:*
$$\left\{\begin{array}{l} a - b\,p = 1, \\ a\,p \in A. \end{array}\right. \quad (2)$$
If $a \neq 0$, then $c = b/a$ is a stabilizing controller of p and $J^{-1} = (a,\, b)$. $c \in K$ internally stabilizes $p \in K$ iff:
$$(1,\, p)\,(1,\, c) = (1 - p\,c). \quad (3)$$
4. *p admits a coprime factorization iff J is a non-zero principal fractional ideal of A, i.e. there exists $0 \neq k \in K$ such that $J = (k)$. Moreover, we have $d = 1/k \in A$, $n = d\,p \in A$ and $p = n/d$ is a coprime factorization of p.*
5. *p is strongly stabilizable iff there exists $c \in A$ such that $J = (1 - p\,c)$.*
6. *p is bistably stabilizable iff there exists $c \in \mathrm{U}(A)$ such that $J = (1 - p\,c)$.*

Proof. 1. If p is stable, then $J = (1,\, p) = (1) = A$, and thus, $A : J = A : A = A$. Conversely, if we have $A : J = A$, then $1 \in A : J = \{k \in K \mid k,\ k\,p \in A\} = \{d \in A \mid d\,p \in A\}$, and thus, $1\,p = p \in A$, i.e. p is stable.

2. Let p admit a weakly coprime factorization $p = n/d$, $0 \neq d,\, n \in A$. Then, by definition of a weakly coprime factorization, we have $A : (d,\, n) = \{k \in K \mid k\,d,\ k\,n \in A\} = A$. Thus, $J = (1,\, p) = (1,\, n/d) = (1/d)\,(d,\, n)$ which implies that $A : J = (A : (d,\, n)) : (1/d) = A : (1/d) = (d)$, by the third equality of Lemma 1, and thus, $A : J$ is a principal integral ideal of A.

Conversely, if $A : J$ is a non-zero principal integral ideal of A, then there exists $0 \neq d \in A$ such that $A : J = (d)$. But, we have $A : J = \{d \in A \mid d\,p \in A\}$, and thus, $n \triangleq d\,p \in A$. Let us prove that $p = n/d$ is a weakly coprime factorization of p. We have

$$A : (d,\, n) = A : ((d)\, J) = (A : J) : (d) = (d) : (d) = A,$$

and thus, $A : (d,\, n) = \{k \in K \mid k\,d,\ k\,n \in A\} = A$, which shows that $p = n/d$ is a weakly coprime factorization of p.

3. Let us suppose that p is internally stabilizable. Then, there exists a stabilizing controller $c \in K$ such that we have (1). Let us denote $a = 1/(1 - p\,c) \in A$, $b = c/(1 - p\,c) \in A$ and $I = (a,\, b)$ the integral ideal of A defined by a and b. We have $a\,p = p/(1 - p\,c) \in A$, $b\,p = (p\,c)/(1 - p\,c) = -1 + a \in A$

$$\Rightarrow 1 = a - b\,p \in I\,J = (a,\, b,\, a\,p,\, b\,p) \subseteq A \Rightarrow I\,J = A,$$

which shows that $J^{-1} = I = (a,\, b)$. Moreover, $c = b/a$ is a stabilizing controller of p.

Conversely, if J is an invertible ideal of A, then we have

$$(A : J)\, J = \{u - v\,p \mid u,\, v \in A : J\} = A,$$

where $J^{-1} = A : J = \{d \in A \mid d\,p \in A\}$. Thus, there exist $a,\, b \in J^{-1}$, i.e. $a,\, b,\, a\,p,\, b\,p \in A$, such that $a - b\,p = 1$. If $a \neq 0$, then let us define $c = b/a$. Then, we have:

$$\left\{\begin{array}{l} 1/(1 - p\,c) = a \in A, \\ p/(1 - p\,c) = a\,p \in A, \\ c/(1 - p\,c) = b \in A, \end{array}\right.$$
$$\Rightarrow H(p,\, c) = \left(\begin{array}{cc} a & -b \\ -a\,p & a \end{array}\right) \in M_2(A).$$

Morever, we have $(a,\, b) \subseteq J^{-1}$. Using $a - b\,p = 1$ and the fact that $u \in J^{-1}$ satisfies $u,\, u\,p \in A$, we obtain $u = (u)\,a - (u\,p)\,b$, i.e. $u \in (a,\, b) \Rightarrow J^{-1} \subseteq (a,\, b) \Rightarrow J^{-1} = (a,\, b)$.

Let us notice that $a\,p \in A$ and $a - b\,p = 1$ implies that we have $b\,p = a - 1 \in A$, and thus, the condition $b\,p \in A$ is redundant in (2). Finally, using the fact that $c = b/a$, we have

$$A = J\,J^{-1} = (1,\, p)\,(a,\, b) = (a)\,(1,\, p)\,(1,\, c)$$

and thus, $(1,\, p)\,(1,\, c) = (1/a) = (1 - p\,c)$.

4. Let us suppose that p admits a coprime factorization. Then, there exist $d,\, n,\, x,\, y \in A$ such that $p = n/d$ and $d\,x - n\,y = 1$. Then, we have $1 = d\,x - n\,y \in (d,\, n) \subseteq A \Rightarrow (d,\, n) = A$. Thus, we have $J = (1,\, p) = (1/d)\,(d,\, n) = (1/d)$, i.e. J is a principal fractional ideal of A.

Conversely, if J is a non-zero principal fractional ideal of A, then there exists $0 \neq k \in K$ such that $J = (k)$. Thus, there exist $x,\, y,\, d,\, n \in A$ such that

$$\left\{ \begin{array}{l} k = x - y\,p, \\ 1 = d\,k, \\ p = n\,k, \end{array} \right. \Rightarrow d = 1/k,$$

and thus, $p = n/d$ and $1/d = x - y\,(n/d) \Rightarrow d\,x - n\,y = 1$.

5 (resp. 6). Let us suppose that p is strongly stabilizable (resp. bistably stabilizable). Then, there exists a stabilizing controller of the form $c \in A$ (resp. $c \in \mathrm{U}(A)$), and thus, $(1,\, c) = A$. Using (3), we obtain that $J = (1,\, p) = (1 - p\,c)$.

Conversely, let us suppose that we have $J = (1 - p\,c)$ where $c \in A$ (resp. $c \in \mathrm{U}(A)$). Then, $1 \in J \Rightarrow J \neq 0$, and thus, we have $1 - p\,c \neq 0$. Moreover, there exist $d,\, n \in A$ such that

$$\left\{ \begin{array}{l} 1 = d\,(1 - p\,c), \\ p = n\,(1 - p\,c), \end{array} \right. \Rightarrow \left\{ \begin{array}{l} p = n/d, \\ p = (d-1)/(d\,c), \\ p = n/(1 + n\,c), \end{array} \right.$$

and $1 - p\,c = d\,(1 - p\,c) - (n\,c)\,(1 - p\,c) = (d - n\,c)\,(1 - p\,c) \Rightarrow d - n\,c = 1$. The controller $c \in A$ (resp. $c \in \mathrm{U}(A)$) is a stabilizing controller of p, i.e. p is strongly (resp. bistably) stabilizable, because we have:

$$\begin{array}{rl} H(p,\, c) & = \left(\begin{array}{cc} 1 & c \\ n/d & 1 \end{array} \right)^{-1} = \frac{1}{(d - n\,c)} \left(\begin{array}{cc} d & -d\,c \\ -n & d \end{array} \right) \\ & = \left(\begin{array}{cc} d & -d\,c \\ -n & d \end{array} \right) \in M_2(A). \end{array}$$

□

Remark 1. Firstly, from 2 and 4 of Theorem 1, a coprime factorization is a weakly coprime factorization (this explains the name). Indeed, if p admits a coprime factorization, then, by 4 of Theorem 1, $J = (1/d)$, for a certain $0 \neq d \in A$, and J is invertible. Thus, we have $A : J = J^{-1} = (d)$ and $A : J \subseteq A$. Hence, $A : J$ is a principal integral ideal of A, i.e. by 2 of Theorem 1, p admits a weakly coprime factorization.

Secondly, let us notice that if p admits a coprime factorization, then p is internally stabilizable. Indeed, if p admits a coprime factorization, then, by 4 of Theorem 1, there exists $0 \neq k \in K$ such that $J = (k)$. Thus, J is invertible with inverse $J^{-1} = (1/k)$, and thus, by 3 of Theorem 1, p is internally stabilizable.

A strongly (resp. bistably) stabilizable plant always admits a coprime factorization: there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $J = (1 - p\,c)$ is a principal fractional ideal of A. Thus, by 4 of Theorem 1, p admits a coprime factorization.

Corollary 1. *Let $p \in K = Q(A)$ a transfer function. We have the following equivalences:*

1. *p admits a coprime factorization,*
2. *p admits a weakly coprime factorization and $J = (1,\, p)$ of A is divisorial ideal of A.*

Proof. $1 \Rightarrow 2$. By the first point of Remark 1, p admits a weakly coprime factorization. Moreover, we have $J = (k)$ for a certain $0 \neq k \in K$. Thus, we have $A : (A : J) = A : J^{-1} = (J^{-1})^{-1} = J$, by the second point of Proposition 1, and thus, J is divisorial.

$2 \Rightarrow 1$. By 2 of Theorem 1, there exists $0 \neq d \in A$ such that $A : J = (d)$. Using the fact that J is divisorial, we obtain that $J = A : (A : J) = A : (d) = (1/d)$, and thus, J is a principal fractional ideal of A, i.e. p admits a coprime factorization by 4 of Theorem 1. □

Proposition 2. *Let $p = n/d \in K = Q(A),\ 0 \neq d,\, n \in A$, be any fractional representation of p and $J = (1,\, p)$ the fractional ideal of A defined by 1 and p. Then, we have:*

1. *$(d) \cap (n) = (n)\,(A : J)$.*
2. *A is a* greatest common divisor domain *(GCDD), namely every couple of elements of A admits a greatest common divisor, iff every transfer function $p \in K$ admits a weakly coprime factorization.*
3. *If A is a GCDD, then p admits a coprime factorization iff J is a divisorial fractional ideal, i.e. $J = A : (A : J)$.*

We let the reader proves this theorem by himself. We refer to [5] for the proof and examples.

Example 3. Let us consider the following wave equation [3]:

$$\left\{ \begin{array}{l} \frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ \frac{\partial z}{\partial x}(0,t) = 0, \\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1,t), \end{array} \right. \Rightarrow \hat{y}(s) = \frac{(e^s + e^{-s})}{(e^s - e^{-s})}\,\hat{u}(s).$$

The transfer function

$$p = (e^s + e^{-s})/(e^s - e^{-s}) = (1 + e^{-2\,s})/(1 - e^{-2\,s})$$

belongs to the field of fractions of $A = H_\infty(\mathbb{C}_+)$ (or $A = \hat{\mathcal{A}}$) because $1 + e^{-s},\, 1 - e^{-s} \in A$. Let us consider the fractional ideal $J = (1,\, p)$ of A. It is known that A is a GCDD [9] (see also [4]). We have $A : J = \{d \in A \,|\, d\,p \in A\} = (1 - e^{-2\,s})$, and thus, by 2 of Theorem 1, $p = (1 + e^{-2\,s})/(1 - e^{-2\,s})$ is a weakly coprime factorization of p. Moreover, we have $J\,J^{-1} = (1 + e^{-2\,s},\, 1 - e^{-2\,s}) = A$ because $1 = (1 + e^{-2\,s} + 1 - e^{-2\,s})/2$. Thus, p is internally stabilizable and, using the fact that $a - b\,p = 1$ with $a = -b = (1 - e^{-2\,s})/2 \in (A : J)$, we obtain that $c = b/a = -1$ is a stabilizing controller of

p . Hence, $A : J = J^{-1} = (1 - e^{-2s})$, and thus, $J = (J^{-1})^{-1} = (1/(1 - e^{-2s}))$ is a principal fractional ideal of A, and thus, by 4 of Theorem 1, p admits the coprime fatorization $p = (1+e^{-2s})/(1-e^{-2s})$, $\frac{1}{2}(1+e^{-2s}) - \frac{1}{2}(1-e^{-2s}) = 1$. Finally, we have $1 - p\,c = 1 + p = 2/(1 - e^{-2s})$, and thus, $J = (1/(1-e^{-2s})) = (1-p\,c)$. Therefore, by 5 of Theorem 1, we find that p is bistably stabilizable ($-1 \in A$ is a stabilizing controller of p).

4 Parametrization of all stabilizing controllers

In Remark 1, we saw that the existence of a coprime factorization for a transfer function implies that the plant is internally stabilizable. The converse is generally not true because the fact that J is an invertible fractional ideal of A does not imply that it is principal (see 3 and 4 of Theorem 1). The Youla-Kučera parametrization of all stabilizing controllers of a plant was developed under the condition that the plant p admits a coprime factorization [3, 10]. Therefore, we may wonder if there exists a parametrization of all stabilizing controllers for a stabilizable plant which does not necessarily admit a coprime factorization. We are going to show that such a parametrization exists and it generalizes the Youla-Kučera one.

Lemma 2. *Let $J = (1,\, p)$ be an invertible fractional ideal of A. Then, we have:*

1. *$J^2 = (1,\, p^2)$ and J^2 is also invertible whose inverse $J^{-2} = (J^{-1})^2 = (a^2,\, b^2)$.*
2. *$J^{-2} = (J^2)^{-1} = \{a \in A \,|\, a\,p^2 \in A\} = (r_1,\, r_2)$, where $r_1,\, r_2 \in A$ are such that $r_1 - r_2\,p^2 = 1$ and $r_1\,p^2 \in A$.*
3. *p^2 admits a coprime factorization iff J^2 is a non-zero principal fractional ideal of A, or equivalently, iff J^{-2} is a non-zero principal fractional ideal of A.*
4. *If p admits a coprime factorization, then p^2 also admits a coprime factorization.*

Proof. 1. We have $J^2 = (1,\, p,\, p^2)$, and thus, $(1,\, p^2) \subseteq J^2$. Using the fact that J is invertible, from 3 of Theorem 1, there exist $a,\, b \in A$ such that $a - b\,p = 1$ and $a\,p \in A$. Multiplying this last equality by p, we obtain $p = (a\,p)\,1 - (b\,p)\,p$, where $a\,p,\, b\,p \in A$, and thus, $p \in (1,\, p^2)$, which proves that we have $J^2 = (1,\, p^2)$. Now, let us notice that we have

$$(J\,J^{-1})\,(J\,J^{-1}) = J^2\,(J^{-1})^2 = A \Rightarrow (J^2)^{-1} = (J^{-1})^2,$$

and thus, $J^{-2} \triangleq (J^2)^{-1} = (J^{-1})^2 = (a^2,\, b^2,\, ab) \supseteq (a^2,\, b^2)$. From $a - b\,p = 1$, we obtain $ab = (b)\,a^2 - (a\,p)\,b^2 \in (a^2,\, b^2)$, because $b,\, a\,p \in A$, and thus, we obtain $J^{-2} = (a^2,\, b^2)$.

2. $J^{-2} = (1,\, p^2)^{-1} = \{a \in A \,|\, a\,p^2 \in A\}$ and, using the fact that $J^2\,J^{-2} = A$, there exist $r_1,\, r_2 \in J^{-2}$, i.e. $r_i \in A$, $r_i\,p^2 \in A$, such that $r_1 - r_2\,p^2 = 1$. In particular, we have $(r_1,\, r_2) \subseteq J^{-2}$. Finally, for all $r \in J^{-2}$, we have $r = (r)\,r_1 - (r\,p^2)\,r_2$, with $r,\, r\,p^2 \in A$, which shows that $r \in (r_1,\, r_2)$, and thus, $J^{-2} = (r_1,\, r_2)$.

3. From 1, we have $J^2 = (1,\, p^2)$. Thus, by 4 of Theorem 1, p^2 admits a coprime factorization iff J^2 is a non-zero principal fractional ideal of A. Moreover, $J^2 = (k)$, with $0 \neq k \in K$, is equivalent to $J^{-2} = (J^2)^{-1} = (1/k)$, which proves 3.

4. If p admits a coprime factorization, then, by 4 of Theorem 1, there exists $0 \neq k \in K$ such that $J = (k)$. Then, $J^2 = (1,\, p^2) = (k^2)$ is also a principal ideal of A and the result follows from 4 of Theorem 1. □

Theorem 2. *Let A be an integral domain of (proper) stable SISO systems, $K = Q(A)$, $p \in K$ and $J = (1,\, p) \triangleq A + A\,p$ the fractional ideal of A defined by 1 and p. Let us suppose that p is internally stabilizable. Then, all stabilizing controllers of p have the form*

$$c(q_1, q_2) = \frac{b + r_1\,q_1 + r_2\,q_2}{a + r_1\,p\,q_1 + r_2\,p\,q_2}, \qquad (4)$$

where $c = b/a$ is a stabilizing controller of p (see (2)), $\{r_1,\, r_2\}$ is a set of generators of J^{-2}, i.e. $J^{-2} = (r_1,\, r_2)$, and $q_1,\, q_2$ are free parameters of A such that $a + r_1\,q_1\,p + r_2\,q_2\,p \neq 0$.

In particular, we can choose $r_1 = a^2$ and $r_2 = b^2$ or any two elements $r_1,\, r_2 \in A$ satisfying $r_1 - r_2\,p^2 = 1$ and $r_1\,p^2 \in A$.

1. *(4) has only one free parameter, i.e. $q_2 = 0$, iff p^2 admits a coprime factorization.*
2. *If p^2 admits a coprime factorization $p^2 = s/r$, where $s,\, 0 \neq r \in A$, then (4) becomes:*

$$c(q) = \frac{b + r\,q}{a + r\,p\,q}, \quad \forall\, q \in A: \; a + r\,p\,q \neq 0. \qquad (5)$$

3. *If p admits a coprime parametrization $p = n/d$, where $d\,x - n\,y = 1$ for certain $x,\, y \in A$, then (4) becomes*

$$c(q) = \frac{y + d\,q}{x + n\,q}, \quad \forall\, q \in A: \; x + q\,n \neq 0, \qquad (6)$$

i.e. (4) is the Youla-Kučera parametrization of all stabilizing controllers of p [10].

Proof. Let $c_i = b_i/a_i$, $i = 1,\, 2$, be two stabilizing controllers of p. Then, by (2), we have:

$$\left\{\begin{array}{l} 0 \neq a_i,\, b_i \in A, \\ a_i - b_i\,p = 1, \\ a_i\,p \in A, \end{array}\right. \quad \Rightarrow \left\{\begin{array}{l} (b_2 - b_1) \in A, \\ (b_2 - b_1)\,p = (a_2 - a_1) \in A, \\ (b_2 - b_1)\,p^2 = (a_2 - a_1)\,p \in A, \end{array}\right.$$

which proves that $b_2 - b_1 \in (A : (1,\, p,\, p^2)) = J^{-2}$. Then, by 1 of Lemma 2, there exist $r_1,\, r_2 \in A$ such that $J^{-2} = (r_1,\, r_2)$. Thus, there exist $q_1,\, q_2 \in A$ such that $b_2 - b_1 = r_1\,q_1 + r_2\,q_2$ and we have $a_2 - a_1 = (b_2 - b_1)\,p = r_1\,p\,q_1 + r_2\,p\,q_2$. Thus, if $a + r_1\,p\,q_1 + r_2\,p\,q_2 \neq 0$, then we have:

$$c_2 = \frac{b_2}{a_2} = \frac{b_1 + r_1\,q_1 + r_2\,q_2}{a_1 + r_1\,p\,q_1 + r_2\,p\,q_2}. \qquad (7)$$

Hence, if c_1 and c_2 are two stabilizing controllers of p, then there exist $q_1,\, q_2 \in A$ such that we have (4). Let us show that any controller defined by (4) is a stabilizing controller of p. We have

$$(a+r_1\,p\,q_1+r_2\,p\,q_2)-(b+r_1\,q_1+r_2\,q_2)\,p=a-b\,p=1,$$
$$(a+r_1\,p\,q_1+r_2\,p\,q_2)\,p=a\,p+(r_1\,p^2)\,q_1+(r_2\,p^2)\,q_2\in A,$$

because $a\,p$, $r_1\,p^2$ and $r_2\,p^2$ belong to A. Hence, (4) is the family of all stabilizing controllers of p if $a+r_1\,p\,q_1+r_2\,p\,q_2\neq 0$.

1. It directly follows from 3 of Lemma 2 and the fact that $J^{-2}=(r_1,\,r_2)$.

2. If p^2 admits a coprime factorization, then, by 3 of Lemma 2, J^{-2} is a principal ideal of A, and thus, there exists $r\in A$ such that $J^{-2}=(r)$. Then, (5) follows directly from (4).

3. If $p=n/d$ is a coprime factorization and $d\,x-n\,y=1$, where $x,\,y\in A$, then we have $(d\,x)-(d\,y)\,p=1$, and thus, $a=d\,x$, $b=d\,y\in A$ and $a\,p=x\,n\in A$. From 4 of Theorem 1, we have $J=(1/d)$, and thus, $J^{-2}=(d^2)$. Using 3 of Lemma 2, then (5) becomes:

$$c(q)=\frac{d\,y+d^2\,q}{d\,x+d^2\,p\,q}=\frac{y+d\,q}{x+n\,q},\quad \forall\, q\in A:\; x+n\,q\neq 0.$$

□

Definition 3. [2, 5] Let us denote by $\mathcal{P}(A)$ the *group of non-zero principal fractional ideals* of A and $\mathcal{I}(A)$ the *group of non-zero invertible fractional ideals* of A. Then, the *Picard group* of A is defined by $\mathcal{C}(A)\triangleq\mathcal{I}(A)/\mathcal{P}(A)$.

Proposition 3. *1. If $\mathcal{C}(A)\cong\mathbb{Z}/2\,\mathbb{Z}$, then every stabilizable plant $p\in K=Q(A)$ has a parametrization of all its stabilizing controllers of the form (5).*

2. If $\mathcal{C}(A)\cong 1$, then every stabilizable plant $p\in K=Q(A)$ has a Youla-Kučera parametrization of its stabilizing controllers (e.g. $A=H_\infty(\mathbb{C}_+)$, RH_∞, Bézout domains).

Proof. 1. If $\mathcal{C}(A)\cong\mathbb{Z}/2\,\mathbb{Z}$, then every invertible fractional ideal J of A is such that J^2 is principal. Therefore, the result directly follows from 3 of Lemma 2 and 2 of Theorem 4.

2. If $\mathcal{C}(A)\cong 1$, then every invertible fractional ideal of A is principal. Then, the result directly follows from 4 of Theorem 1 and 3 of Theorem 4. □

Example 4. Let us consider the example defined in [1]:

$$p=(1+i\,\sqrt{5})/2\in K=Q(A),\quad A=\mathbb{Z}[i\,\sqrt{5}].$$

Let $J=(1,\,p)$ be the fractional ideal of A. Then, we have $A:J=(2,\,1-i\,\sqrt{5})$. By 2 of Theorem 1, we know that p does not admit weakly coprime factorizations, and thus, coprime factorizations [1]. We have $(-2)-(-(1-i\,\sqrt{5}))\,p=1$ and $2\,p\in A$, i.e. p is internally stabilized by $c=(1-i\,\sqrt{5})/2$. Let us find all the stabilizing controllers of p. The integral ideal $J^{-2}=(2,\,1-i\,\sqrt{5})^2=(2)$ is a principal ideal of A, and thus, $p^2=(-2+i\,\sqrt{5})/2$ admits a coprime factorization. By (5), we find that all the stabilizing controllers of p are given by:

$$c(q)=\frac{-1+i\,\sqrt{5}+2\,q}{-2+2\,((1+i\,\sqrt{5})/2))\,q}=\frac{1-i\,\sqrt{5}-2\,q}{2-(1+i\,\sqrt{5})\,q},\; q\in A.$$

$A=\mathbb{Z}[i\,\sqrt{5}]$ is a *Dedekind domain* [2, 4], i.e. an integral domain such that every non-zero fractional ideal of A is invertible, and $\mathcal{C}(A)\cong\mathbb{Z}/2\,\mathbb{Z}$ [5]. Thus, every SISO plant is internally stabilizable and has a parametrization of all the stabilizing controllers of the form (5) or (6).

5 Conclusion

The introduction of the theory of fractional ideals within the fractional representation approach to linear systems gives new insights to stabilization problems (internal stabilizability, parametrization of all stabilizing controllers ...). For more results and details, we refer the reader to [5]. The extension of these results to MIMO systems is obtained in [6] and a behavioural interpretation to the operator-theoretic approach is developed in [7]. Robust stabilization can also be recovered using the theory of fractional ideals. See forthcoming publications.

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