LOCAL STABILITY ANALYSIS OF PIECEWISE AFFINE SYSTEMS

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Abstract

This paper is concerned with the local stability of a piecewise affine system. In terms of piecewise quadratic Lyapunov functions, we derive a new matrix inequality condition that explicitly characterizes an inner approximation of the domain of attraction for the piecewise affine system. We also show the effectiveness of the present method by two numerical examples.

1 Introduction

Hybrid systems are composed of both continuous dynamics governed by physical laws and discrete-event dynamics driven by logic and rules [14]. In particular, there are many works on piecewise affine (PWA) systems [2,5,6,9,11] as a typical model of the hybrid systems. Johansson and Rantzer [6,11] considered the global stability and the optimal control of the PWA system by means of linear matrix inequalities (LMIs). Hassibi and Boyd [2] proposed a design method of a piecewise state feedback controller. In general nonlinear control systems, the global stability can not be guaranteed. For example, it is impossible to globally stabilize an exponentially unstable linear system with input saturation [12].

In such a case, the above global stability conditions are no longer applicable, and hence we need to consider the local stability of the PWA system. Mignone *et al.* [9] considered the local stability of a discrete-time PWA system. Johansson [7] proposed a local stability condition of a saturating system based on the results in [6,11]. Although Johansson's condition is quite successful in the analysis of saturating systems, it is not applicable to a more general class of PWA systems. This is because the condition makes use of a quadratic Lyapunov function obtained from the circle criterion [4, 8, 13] in order to construct a less conservative piecewise quadratic Lyapunov function.

In this paper, we consider the local stability of a PWA system. We will characterize an inner approximation of the DA of a PWA system in terms of a level set of a piecewise quadratic Lyapunov function. By using this level set, we derive a new local stability condition for a PWA system which explicitly provides an inner approximation of the DA. Our result is a local version of the global stability condition by Johansson and Rantzer [6, 11]. We also demonstrate the effectiveness of the present method by two numerical examples.

We use the following notations in this paper. For two vectors u and v, $u > (\geq) v$ implies $u_j > (\geq) v_j,\ \forall j$ where u_j and v_j are the j-th elements of u and v, respectively. We define $\mathrm{He}(M) = M + M^{\mathrm{T}}$. The ellipsoid associated with a positive symmetric matrix $M \in \mathbb{R}^{n\times n}$ is defined by $\mathcal{E}(M) = \left\{x \in \mathbb{R}^n \middle|\, x^{\mathrm{T}} M x \leq 1\right\}$. For a symmetric matrix

$$\hat{N} = \begin{bmatrix} N_1 & N_2 \\ N_2^{\mathrm{T}} & N_3 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad N_1 \in \mathbb{R}^{n\times n},$$

we define a set

$$\hat{\mathcal{E}}(\hat{N}) = \left\{ x \in \mathbb{R}^n \,\middle|\, \hat{x}^{\mathrm{T}} \hat{N} \hat{x} \leq 1,\ \hat{x} = \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix}^{\mathrm{T}} \right\}.$$

The set $\hat{\mathcal{E}}(\hat{N})$ becomes an ellipsoid if and only if the conditions $N_1 = N_1^{\mathrm{T}} > 0$ and $N_3 - N_2^{\mathrm{T}} N_1^{-1} N_2 < 1$ hold [1]. The interior of a region $\mathcal{R}$ is represented by $\mathrm{int}(\mathcal{R})$.

2 Piecewise Affine (PWA) System

We consider a piecewise affine (PWA) system

$$\dot{x} = A_i x + a_i, \quad x \in \mathcal{X}_i, \quad i \in \mathcal{I}. \tag{1}$$

Here,

$$\mathcal{X}_i = \{x \in \mathbb{R}^n |\, E_i x \geq e_i\}, \quad i \in \mathcal{I}_1, \tag{2}$$

$$\mathcal{X}_i = \{x \in \mathbb{R}^n |\, E_i x \geq 0\} \setminus \mathrm{int}\left(\bigcup_{j\in\mathcal{I}_1} \mathcal{X}_j \right), \quad i \in \mathcal{I}_0 \tag{3}$$

are polyhedral cells on the state space. Note that each of the cells is identified by the index i. $\mathcal{I}$ denotes the index set. Let $\mathcal{I}_0$ be the index set for the cells that contain the origin, and define $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0$. All coefficient matrices $A_i \in \mathbb{R}^{n\times n}$, $a_i \in \mathbb{R}^n$, $E_i \in \mathbb{R}^{q_i\times n}$, $e_i \in \mathbb{R}^{q_i}$ $(i \in \mathcal{I})$ are known. The state vector $x : [0,\infty) \to \mathbb{R}^n$ is a continuous piecewise $\mathcal{C}^1$ function on the time interval $[0,\infty)$, and we assume that the state trajectory $x(t)$ does not exhibit singular phenomena such as jumps of the state, sliding modes, dead-locks, live-locks, and so on.

On the boundary between two different cells $\mathcal{X}_i$ and $\mathcal{X}_j$, i.e., $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$ $(i, j \in \mathcal{I},\ i \neq j)$,

$$F_i x + f_i = F_j x + f_j, \quad x \in \mathcal{X}_i \cap \mathcal{X}_j \tag{4}$$

holds, where $F_i \in \mathbb{R}^{r\times n}$, $f_i \in \mathbb{R}^r$ are given matrices. The affine terms satisfy $a_i,\ f_i = 0,\ \forall i \in \mathcal{I}_0$.

We can express the PWA system (1) as

$$\begin{aligned} &\dot{x} = A_i x, \quad x \in \mathcal{X}_i, \quad i \in \mathcal{I}_0, \\ &\dot{\hat{x}} = \hat{A}_i \hat{x}, \quad x \in \mathcal{X}_i = \left\{ x \in \mathbb{R}^n \,\middle|\, \hat{E}_i \hat{x} \geq 0 \right\}, \quad i \in \mathcal{I}_1, \end{aligned}$$

where

$$\hat{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}, \quad \hat{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \hat{E}_i = \begin{bmatrix} E_i & -e_i \end{bmatrix}.$$

Then, the boundary condition (4) is equivalent to

$$\hat{F}_i \hat{x} = \hat{F}_j \hat{x}, \quad \hat{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}, \quad x \in \mathcal{X}_i \cap \mathcal{X}_j.$$

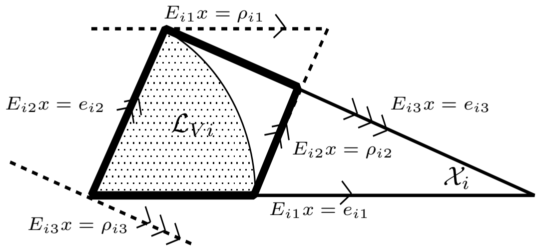

Figure 1: Outer polyhedral bounding of $\mathcal{L}_{Vi}$ $(i \in \mathcal{I}_{V1})$.

3 Local Stability Analysis

Definition 1 The domain of attraction (DA) $\mathcal{D}$ is the set of the initial states for which the state trajectories converge to the origin as time goes to infinity, namely,

$$\mathcal{D} = \left\{ x_0 \in \mathbb{R}^n \,\middle|\, \lim_{t\to\infty} x(t) = 0,\ x(0) = x_0 \right\}.$$

The DA of the PWA system is too complicated to compute exactly. Instead, we wish to find its inner approximation in terms of a level set of a piecewise quadratic Lyapunov function.

We define a piecewise quadratic function

$$V(x) = \begin{cases} x^{\mathrm{T}} P_i x,\ P_i = F_i^{\mathrm{T}} T F_i, & x \in \mathcal{X}_i,\ i \in \mathcal{I}_0, \\ \hat{x}^{\mathrm{T}} \hat{P}_i \hat{x},\ \hat{P}_i = \hat{F}_i^{\mathrm{T}} T \hat{F}_i, & x \in \mathcal{X}_i,\ i \in \mathcal{I}_1. \end{cases} \tag{5}$$

for a symmetric matrix $T \in \mathbb{R}^{r\times r}$. We choose T so that

(i) $\mathcal{E}(P_i)$ and $\hat{\mathcal{E}}(\hat{P}_i)$ are ellipsoidal regions, as described in Section 1,

(ii) the level set

$$\mathcal{L}_V = \{x \in \mathbb{R}^n |\ V(x) \le 1\} \tag{6}$$

is a connected set.

We see from (4) that $V(x)$ is continuous on the boundaries $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$ $(i, j \in \mathcal{I},\ i \neq j)$. Our purpose is to derive a sufficient matrix inequality condition for $\mathcal{L}_V \subseteq \mathcal{D}$.

Because the level set $\mathcal{L}_V$ is a local region containing the origin, $\mathcal{L}_V$ may not intersect all cells $\mathcal{X}_i$ $(i \in \mathcal{I})$. Let $\mathcal{I}_V$ denote the index set of the cells that intersect $\mathcal{L}_V$. $\mathcal{I}_{V0}$ and $\mathcal{I}_{V1}$ are index sets defined by

$$\begin{aligned} &\mathcal{I}_V = \mathcal{I}_{V0} \cup \mathcal{I}_{V1} \subseteq \mathcal{I}, \\ &\mathcal{I}_{V0} = \{i \in \mathcal{I}_0 |\ \mathcal{L}_V \cap \mathcal{X}_i \neq \emptyset,\ \mathcal{L}_V \cap \mathcal{X}_i \neq \{0\}\}, \\ &\mathcal{I}_{V1} = \{i \in \mathcal{I}_1 |\ \mathcal{L}_V \cap \mathcal{X}_i \neq \emptyset\}. \end{aligned}$$

Then, we can express the level set $\mathcal{L}_V$ as

$$\mathcal{L}_V = \bigcup_{i\in\mathcal{I}_V} \mathcal{L}_{Vi} \subseteq \mathcal{D}, \quad \mathcal{L}_{Vi} = \begin{cases} \mathcal{E}(P_i) \cap \mathcal{X}_i, & i \in \mathcal{I}_{V0}, \\ \hat{\mathcal{E}}(\hat{P}_i) \cap \mathcal{X}_i, & i \in \mathcal{I}_{V1}. \end{cases}$$

Since each $\mathcal{L}_{Vi}$ is a bounded region, there exist vectors $\rho_i \in \mathbb{R}^{q_i}$ $(i \in \mathcal{I}_V)$ such that

$$\begin{aligned} 0 \le E_i x \le \rho_i, &\quad \forall x \in \mathcal{L}_{Vi}, \quad i \in \mathcal{I}_{V0}, \\ e_i \le E_i x \le \rho_i, &\quad \forall x \in \mathcal{L}_{Vi}, \quad i \in \mathcal{I}_{V1}. \end{aligned}$$

This means that we can confine $\mathcal{L}_{Vi}$ into a polyhedron which is smaller than $\mathcal{X}_i$ (see Figure 1).

Theorem 1 *For the PWA system (1), let T be a given constant symmetric matrix such that the level set $\mathcal{L}_V$ defined by (6) is a connected set. Suppose that there exist symmetric matrices U_i, Y_i, V_{ij}, with non-negative entries, a matrix Z_i with non-negative entries, non-negative scalars α_i, ρ_{ij} $(j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_V)$ and positive scalars δ_1, δ_3 satisfying*

$$P_i - E_i^{\mathrm{T}} U_i E_i \ge \delta_1 I, \quad P_i = F_i^{\mathrm{T}} T F_i \quad (i \in \mathcal{I}_{V0}), \tag{7}$$

$$\hat{P}_i - \hat{E}_i^{\mathrm{T}} U_i \hat{E}_i \ge \begin{bmatrix} \delta_1 I & 0 \\ 0 & 0 \end{bmatrix}, \ \hat{P}_i = \hat{F}_i^{\mathrm{T}} T \hat{F}_i \quad (i \in \mathcal{I}_{V1}), \tag{8}$$

$$\mathrm{He}\,(P_i A_i) + E_i^{\mathrm{T}} Y_i E_i \le -\delta_3 I \quad (i \in \mathcal{I}_{V0}), \tag{9}$$

$$\mathrm{He}\left(\hat{P}_i \hat{A}_i + \hat{E}_i^{\mathrm{T}} Z_i \hat{H}_i\right) + \alpha_i \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \hat{P}_i \right) \le \begin{bmatrix} -\delta_3 I & 0 \\ 0 & 0 \end{bmatrix} \quad (i \in \mathcal{I}_{V1}), \tag{10}$$

$$\rho_{ij}^2 P_i - E_i^{\mathrm{T}} V_{ij} E_i \ge E_{ij}^{\mathrm{T}} E_{ij}, \quad \rho_{ij} \ge 0 \quad (j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_{V0}), \tag{11}$$

$$\rho_{ij}^2 \hat{P}_i - \hat{E}_i^{\mathrm{T}} V_{ij} \hat{E}_i \ge \begin{bmatrix} E_{ij}^{\mathrm{T}} E_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad \rho_{ij} \ge e_{ij} \quad (j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_{V1}), \tag{12}$$

$$\hat{H}_i = \begin{bmatrix} -E_i & \rho_i \end{bmatrix}, \quad \rho_i = \begin{bmatrix} \rho_{i1} & \ldots & \rho_{iq_i} \end{bmatrix}^{\mathrm{T}} \quad (i \in \mathcal{I}_{V1}),$$

where E_{ij} and e_{ij} denote the j-th row of E_i and the j-th element of e_i, respectively. Then, the PWA system (1) is locally exponentially stable, and $\mathcal{L}_V \subseteq \mathcal{D}$ holds.

Proof From (11), we obtain

$$\begin{aligned} \rho_{ij}^2 x^{\mathrm{T}} P_i x \ge \rho_{ij}^2 x^{\mathrm{T}} P_i x - x^{\mathrm{T}} E_i^{\mathrm{T}} V_{ij} E_i x \ge |E_{ij} x|^2, \\ \forall x \in \mathcal{L}_{Vi} \quad (j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_{V0}). \end{aligned}$$

Since $x^{\mathrm{T}} P_i x \le 1$ and $E_i x \ge 0$ hold for all $x \in \mathcal{L}_{Vi}$ $(i \in \mathcal{I}_{V0})$, $0 \le E_{ij} x \le \rho_{ij}$ $(j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_{V0})$, namely, $0 \le E_i x \le \rho_i$ $(i \in \mathcal{I}_{V0})$ is satisfied. Moreover, we see from (12) that

$$\begin{aligned} \rho_{ij}^2 \hat{x}^{\mathrm{T}} \hat{P}_i \hat{x} \ge \rho_{ij}^2 \hat{x}^{\mathrm{T}} \hat{P}_i \hat{x} - \hat{x}^{\mathrm{T}} \hat{E}_i^{\mathrm{T}} V_{ij} \hat{E}_i \hat{x} \ge |E_{ij} x|^2, \\ \forall x \in \mathcal{L}_{Vi} \quad (j = 1, 2, \ldots, q_i;\ i \in \mathcal{I}_{V1}). \end{aligned}$$

It follows from $\hat{x}^{\mathrm{T}} \hat{P}_i \hat{x} \le 1$ and $E_i x \ge e_i$ for all $x \in \mathcal{L}_{Vi}$ $(i \in \mathcal{I}_{V1})$ that

$$e_i \le E_i x \le \rho_i \ \Leftrightarrow\ \hat{E}_i \hat{x} \ge 0,\ \hat{H}_i \hat{x} \ge 0 \quad (i \in \mathcal{I}_{V1}). \tag{13}$$

In the remainder of this proof, we shall show the local exponential stability of the PWA system (1). We adopt $V(x)$ in (5) as a candidate of Lyapunov function. The matrix inequalities (7) and (8) yield

$$\begin{aligned} V(x) = x^{\mathrm{T}} P_i x \ge x^{\mathrm{T}} E_i^{\mathrm{T}} U_i E_i x + \delta_1 \|x\|^2 \ge \delta_1 \|x\|^2, \\ \forall x \in \mathcal{L}_{Vi} \quad (i \in \mathcal{I}_{V0}) \\ V(x) = \hat{x}^{\mathrm{T}} \hat{P}_i \hat{x} \ge \hat{x}^{\mathrm{T}} \hat{E}_i^{\mathrm{T}} U_i \hat{E}_i \hat{x} + \delta_1 \|x\|^2 \ge \delta_1 \|x\|^2, \\ \forall x \in \mathcal{L}_{Vi} \quad (i \in \mathcal{I}_{V1}). \end{aligned}$$

Hence, $V(x) \ge \delta_1 \|x\|^2,\ \forall x \in \mathcal{L}_V$ is satisfied. From the absence of affine terms in the Lyapunov candidate in an open neighborhood of the origin, we conclude that there exists a $\delta_2 > 0$ satisfying $V(x) \leq \delta_2\|x\|^2$, $\forall x \in \mathcal{L}_V$ [6]. In order to complete the proof, we need only to show that the time differentiation of $V(x(t))$ along the state trajectory $x(t) \in \mathcal{L}_V$ satisfies $\dot{V}(x(t)) \leq -\delta_3\|x(t)\|^2$ for almost all $t \geq 0$. If $x(t) \in \mathrm{int}(\mathcal{L}_{Vi})$ $(i \in \mathcal{I}_{V0})$, the inequality

$$\begin{aligned}\dot{V}(x(t)) &= \mathrm{He}\left(x^{\mathrm{T}}(t)P_iA_ix(t)\right)\\ &\leq -x^{\mathrm{T}}(t)E_i^{\mathrm{T}}Y_iE_ix(t) - \delta_3\|x(t)\|^2 \leq -\delta_3\|x(t)\|^2.\end{aligned}$$

immediately follows from (9). Similarly, if $x(t) \in \mathrm{int}(\mathcal{L}_{Vi})$ $(i \in \mathcal{I}_{V1})$, then we obtain from (10) and (13)

$$\begin{aligned}\dot{V}(x(t)) &= \mathrm{He}\left(\hat{x}^{\mathrm{T}}(t)\hat{P}_i\hat{A}_i\hat{x}(t)\right)\\ &\leq -\mathrm{He}\left(\hat{x}^{\mathrm{T}}(t)\hat{E}_i^{\mathrm{T}}Z_i\hat{H}_i\hat{x}(t)\right)\\ &\qquad - \alpha_i\left(1 - \hat{x}^{\mathrm{T}}(t)\hat{P}_i\hat{x}(t)\right) - \delta_3\|x(t)\|^2\\ &\leq -\delta_3\|x(t)\|^2.\end{aligned}$$

Hence, $\dot{V}(x(t)) \leq -\delta_3\|x(t)\|^2$ holds for almost all $t \geq 0$.

Consequently, we have shown that the PWA system (1) is locally exponentially stable, and $\mathcal{L}_V \subseteq \mathcal{D}$ holds. □

Remark 1 We regard the symmetric matrix T as an unknown variable in order to maximize the size of the level set $\mathcal{L}_V$ by a numerical calculation of the matrix inequalities (7)–(12). As for the objective function in the calculation, it is not easy to characterize its volume with a numerically tractable function. However, noting that $\mathcal{L}_V$ consists of several ellipsoids, we attempt to maximize the size of $\mathcal{L}_V$ by solving the optimization problem

$$\textbf{(OP)} \quad \inf_{T,U_i,Y_i,V_{ij},Z_i,\alpha_i,\rho_{ij},\delta_1,\delta_3} \sum_{i\in\mathcal{I}_{V0}} \mathrm{tr}(P_i) + \sum_{i\in\mathcal{I}_{V1}} \mathrm{tr}(\hat{P}_i)$$
$$\text{subject to (7)–(12).}$$

The objective function of **(OP)** is motivated by the fact that the minimization of $\mathrm{tr}(P_i)$ is often used in maximizing the size of $\mathcal{E}(P_i)$. If we fix the variables α_i and ρ_{ij}, **(OP)** is reduced to an optimization problem with LMI constraints.

In solving **(OP)**, it is difficult to specify the matrix inequalities to be solved, because the index set $\mathcal{I}_V$ depends on the variable T. To overcome this difficulty, we solve the optimization problem **(OP)** based on the following iterative algorithm.

Step 1: We choose $i_0 \in \mathcal{I}_0$ so that the matrix inequalities (7)–(12) are feasible, and set $\mathcal{I}_V = \mathcal{I}_{V0} := \{i_0\}$.

Step 2: Add an index $j \in \mathcal{I}$ which corresponds to a neighboring cell to $\mathcal{X}_i$ $(i \in \mathcal{I}_V)$ to the set $\mathcal{I}_V$.

Step 3: We solve (7)–(12). If (7)–(12) is infeasible, then we remove the index j from the set $\mathcal{I}_V$ and go to Step 2. If (7)–(12) is feasible, go to Step 2.

Step 4: If there is no index to be added to $\mathcal{I}_V$, then stop.

Note that we have to check in each step whether the level set $\mathcal{L}_V$ is connected or not.

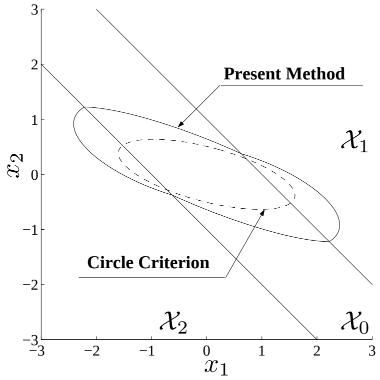

Figure 2: Inner approximations of the DA of the saturating system by **(OP)** and the local circle criterion [4, 8, 13].

4 Numerical Examples

4.1 Saturating System

First, we consider a saturating system given by

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = \sigma(y) = \mathrm{sgn}(y)\min\{1, |y|\},$$

where

$$A = \begin{bmatrix}0.5 & 6\\ 0.1 & 2.1\end{bmatrix}, \quad B = \begin{bmatrix}-1\\ -0.1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1\end{bmatrix}.$$

Corresponding to the saturation function σ, let $\mathcal{X}_0$, $\mathcal{X}_1$ and $\mathcal{X}_2$ be the unsaturated (linear) region ($|Cx| \leq 1$), the upper saturated region ($Cx \geq 1$) and the lower saturated region ($Cx \leq -1$), respectively. As shown below, this system can be represented as a PWA system (see also [7, 10]).

$$\begin{aligned}&\dot{x} = A_0x, \quad x \in \mathcal{X}_0,\\ &\dot{x} = A_ix + a_i, \quad x \in \mathcal{X}_i, \quad i = 1, 2,\\ &A_0 = A + BC, \quad A_1 = A_2 = A, \quad a_1 = B, \quad a_2 = -B.\end{aligned}$$

Obviously, the following equalities are satisfied on the saturation boundaries $Cx = \pm 1$.

$$\begin{aligned}&F_0x = \hat{F}_1\hat{x}, \quad \forall x \in \mathcal{X}_0 \cap \mathcal{X}_1 = \left\{x \in \mathbb{R}^2 \mid Cx = 1\right\},\\ &F_0x = \hat{F}_2\hat{x}, \quad \forall x \in \mathcal{X}_0 \cap \mathcal{X}_2 = \left\{x \in \mathbb{R}^2 \mid Cx = -1\right\},\\ &F_0 = \begin{bmatrix}I\\ 0\\ 0\end{bmatrix}, \quad \hat{F}_1 = \begin{bmatrix}I & 0\\ C & -1\\ 0 & 0\end{bmatrix}, \quad \hat{F}_2 = \begin{bmatrix}I & 0\\ 0 & 0\\ -C & -1\end{bmatrix}. \quad (14)\end{aligned}$$

The choice of F_0, $\hat{F}_1$ and $\hat{F}_2$ in (14) stems from the references [7, 10], though another choice is possible.

Johansson's condition [7] did not give a meaningful result for this example, because the resulting approximation of DA was not a connected set.

We have obtained the optimum $\mathrm{tr}(P_0) + \mathrm{tr}(\hat{P}_1) + \mathrm{tr}(\hat{P}_2) = 6.832$ at $(\alpha_i, \rho_{ij}) = (0.75, 1.71)$, $\forall i, j$ by the algorithm **(OP)** in the previous section.

Figure 2 illustrates the inner approximations of the DA obtained by **(OP)** (solid curve) and the local circle criterion (dashed curve) [4, 8, 13]. The two solid lines represent the saturation boundaries $Cx = \pm 1$.

From this figure, we can conclude that the present local stability condition (Theorem 1) gives a better approximation of the DA for the saturating system than the local circle criterion [4, 8, 13].

4.2 Local Flower System

We consider a local flower system described by

$$\dot{x} = A_i x, \quad x \in \mathcal{X}_i \quad (i = 1, 2, 3, 4), \tag{15}$$

$$\dot{x} = a_i, \qquad x \in \mathcal{X}_i \quad (i = 5, 6, 7, 8), \tag{16}$$

where

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix},\ A_2 = A_4 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},$$

$$a_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad a_7 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad a_8 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and the polyhedral cells $\mathcal{X}_i$ $(i = 1, 2, \ldots, 8)$ are given by (2), (3) with

$$E_1 = -E_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$E_5 = -E_7 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_6 = -E_8 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$e_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \quad (i = 5, 6, 7, 8).$$

The local flower system (15), (16) is obtained from a slight modification on the global flower system considered in [3,6,11]. Evidently, the DA of this system is limited in the cells $\mathcal{X}_i$ $(i = 1, 2, 3, 4)$, because the state trajectories starting from $\mathcal{X}_i$ $(i = 5, 6, 7, 8)$ diverge to infinity. Note that the previous analysis methods [2, 6, 7, 9, 11] can not be applied to this system.

The matrices and vectors in the boundary condition (4) are

$$F_i = \begin{bmatrix} I \\ E_i \\ 0 \end{bmatrix}, \quad f_i = 0 \qquad (i = 1, 2, 3, 4),$$

$$F_i = \begin{bmatrix} I \\ E_i \end{bmatrix}, \quad f_i = \begin{bmatrix} 0 \\ -e_i \end{bmatrix} \quad (i = 5, 6, 7, 8).$$

By solving **(OP)** with $\rho_{ij} = 1,\ \forall i, j$ fixed, we have obtained the optimum $\sum_{i=1}^{4} \mathrm{tr}(P_i) = 27.164$.

The resulting inner approximation of the DA is shown by solid curve in Figure 3. In this figure, the solid lines represent the boundaries between the polyhedral cells $\mathcal{X}_i$ $(i = 1, 2, \ldots, 8)$. The dashed curves denote the several state trajectories starting from some initial states (asterisks). We have obtained these trajectories by using PWLTOOL [3].

From the above numerical results, we can conclude that the present method is applicable to more general PWA systems as well as saturating systems.

5 Conclusion

In this paper, we have derived a local stability condition which explicitly characterizes an inner approximation of the DA of the PWA system in terms of a level set of a piecewise quadratic Lyapunov function. We have also applied the present condition to analyses of a saturating system and a local flower system. The numerical results have revealed the effectiveness of the present local stability condition.

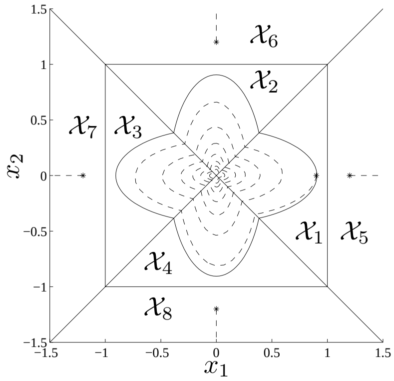

Figure 3: Inner approximation of the DA of the local flower system.

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