Necessary and sufficient conditions for Lur'e system incremental stability

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Abstract

In this paper, we propose a necessary and sufficient condition for incremental stability of the interconnection of an LTI system with a static nonlinearity. The complexity of the condition is discussed.

1 Introduction

Incremental stability was recently proposed as a powerful tool for analyzing qualitative properties [11, 12, 13] and quantitative properties [10, 14, 16] of nonlinear systems. Zames [39] proposed a sufficiency test for incremental stability based on incremental conicity conditions. However, related Popov-type multiplier stability criteria (e.g., [41, 42]) do not in general assure *incremental* stability even when elements of the feedback system satisfy incremental conicity conditions [18].

The issue of precise tests for incremental stability that are both *necessary and sufficient* is, in the general case, a difficult problem though there is a result showing that the problem is equivalent to solution of certain Hamilton Jacobi Isaac inequalities [10, 27]. In this paper, we derive a simpler alternative necessary and sufficient condition for the special case of nonlinear systems of the Lur'e-type, which consist of a feedback interconnection of a Linear Time-Invariant (LTI) element and a memoryless single-input-single-output nonlinear element. Based on the strong connection between the incremental stability of a nonlinear operator and the exponential stability of its time varying linearizations, we prove that the incremental stability of an interconnection between an LTI system and a nonlinearity is equivalent to the exponential stability of a specific Linear Differential Inclusion (LDI).

Based on [25] and [21, 22, 23], we deduce a new necessary and sufficient condition allowing to prove that the interconnection between a specific C^1 nonlinearity and an LTI system is incrementally stable. Unfortunately, exact testing of the obtained condition is an NP hard problem.

At the end of the paper, the conservatism of the quadratic condition (introduced in [14]) in this specific context is emphasized. Furthermore, we point out that polyhedric type condition allows to obtain more interesting algorithms. By this way, using the approaches proposed by [24] , [4], [43] and [20], we are able to provide algorithms allowing to obtain a compromise between complexity and computation time.

2 Preliminary

2.1 Notation and definitions

The notations and terminology, here used, are classical in the input-output context [29, 30, 39, 40, 35, 8, 28]. The $\mathcal{L}_2$-norm of $f : [t_0, \infty) \mapsto \mathcal{R}^n$ is $\|f\|_2 = \sqrt{\int_{t_0}^{\infty} \|f(t)\|^2 dt}$. The *causal truncation* at $T \in [t_0, \infty)$, denoted by $P_T f$ gives $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *extended space*, $\mathcal{L}_2^e$ is composed with the functions whose causal truncations belong to $\mathcal{L}_2$. For convenience, $\|P_T u\|_2$ is denoted by $\|u\|_{2,T}$.

In the sequel, we consider systems exhibiting the differential representation:

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t), u(t)) \\ x(t_0) & = & x_0 \end{array} \right. \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^l$. f and h, defined from $\mathcal{R}^n \times \mathcal{R}^l$ into $\mathcal{R}^n$ and $\mathcal{R}^l$ respectively, are assumed to be C^1 and uniformly Lipschitz. The unique solution $x(t) = \phi(t, t_0, x_0, u)$ belongs to $\mathcal{L}_2^e$ for all $x_0 \in \mathcal{R}^n$ and for any $u \in \mathcal{L}_2^e$. It is assumed that there exists x_{0e} such that $f(x_{0e}, 0) = 0$ and $h(x_{0e}, 0) = 0$, *i.e.* the system initialized at x_{0e} is unbiased, $\Sigma(0) = 0$. The notion of incremental $\mathcal{L}_2$-gain can now be recalled.

Definition 2.1 *Σ is said to be a weakly finite gain stable system if there exists $\gamma \geq 0$, $\beta \geq 0$ such that $\|\Sigma(u)\|_2 \leq \gamma\|u\|_2 + \beta$ for all $u \in \mathcal{L}_2$. Σ is said to be finite gain stable when $\beta = 0$. The gain of Σ coincides with the minimum value of γ and is denoted by $\|\Sigma\|_i$.*

Definition 2.2 *Σ has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta\|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. The incremental gain of Σ coincides with the minimum value of η and is denoted by $\|\Sigma\|_\Delta$. Σ is said to be incrementally stable if it is stable, i.e. it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and has a finite incremental gain.*

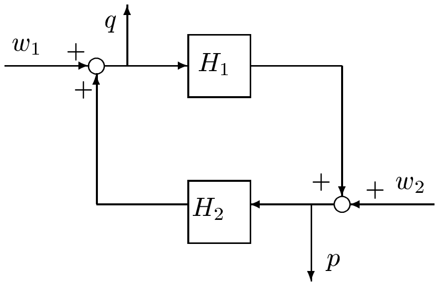

Figure 1: Internal stability

Definition 2.3 *The interconnection between two causal operators H_1 and H_2 is said to be* incrementally stable *if the associated operator $\Sigma : (w_1, w_2) \to (q, p)$*

$$\Sigma = \begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix}^{-1}.$$

exists and is incrementally stable.

2.2 Gâteaux derivative and mean value theorem in norm

In the functional analysis frame, there exist, at least, five notions of derivative (see [1]). The differences between these various notions are related to the fact that we work on spaces of infinity dimension. The Fréchet derivative, which is "similar" to the classical derivative on $\mathcal{R}$, cannot be used since it is not defined for many dynamical systems [34, 31]. We then use a weaker notion of the derivative, the Gâteaux one.

Definition 2.4 [1] *Given a causal operator Σ, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, let be $u_0 \in \mathcal{L}_2^e$ and let us assume that for any $T \in [t_0, \infty)$ and for any $h \in \mathcal{L}_2^e$, there exists a continuous linear operator $D\Sigma_G[u_0]$ from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ such that*

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_{2,T} = 0.$$

Then $D\Sigma_G[u_0]$ is the Gâteaux derivative (the linearization) of Σ at u_0.

When the system is defined by differential equations, Definition 2.4 reduces to the usual linearization definition.

Proposition 2.1 [15] *Let us consider Σ defined by (1) and let us assume that f and h are uniformly Lipschitz and C^1. Then, for any $u_r \in \mathcal{L}_2^e$, the system has a Gâteaux derivative that satisfies the following differential equations:*

$$\bar{y} = D\Sigma_G[u_r](\bar{u}) \left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) & = & C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) & = & 0 \end{array} \right. \quad (2)$$

with $A(t) = \frac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \frac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \frac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \frac{\partial h}{\partial u}(x_r(t), u_r(t))$ and where $x_r(t) = \phi(t, t_0, x_0, u_r)$ is the solution of system (1) with the input $u_r(t)$ and $x(t_0) = x_0$.

Definition 2.5 [32] *$D\Sigma_G$, a Gâteaux derivative of Σ defined by (2), is said to have a minimal state-space realization if the pair $[A(t), B(t)]$ is uniformly controllable and the pair $[A(t), C(t)]$ is uniformly observable.*

We now recall a powerful result in the context of incrementally bounded systems.

Proposition 2.2 [15] *Let us assume that Σ is Gâteaux differentiable on $\mathcal{L}_2^e$. If the state space representation of each derivative of Σ is minimal then there exists a finite constant η such that for any $T \geq t_0$ and for any u_1 and u_2 belonging to $\mathcal{L}_2^e$ one has:*

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta\|u_1 - u_2\|_{2,T}$$

if and only if all its linearizations are uniformly exponentially stable.

3 Main result

Let us consider that the nonlinear system, namely Σ, defined as the interconnection between an LTI system

$$M(s) = \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bq(t) \\ p(t) & = & Cx(t) \\ x(t_0) & = & x_0, \end{array} \right. \quad (3)$$

where $x(t), x_0 \in \mathcal{R}^n$, $p(t)$ and $q(t) \in \mathcal{R}^l$ and $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times l}$ and $C \in \mathcal{R}^{l \times n}$ and the following nonlinearity:

$$q(t) = \Phi(p(t) + w_2(t)) + w_1(t)$$

with

$$\Phi(p + w_2) \triangleq \operatorname{diag}(\varphi_1(p_1 + w_{2_1}), \cdots, \varphi_i(p_i + w_{2_i}), \cdots, \cdots, \varphi_l(p_l + w_{2_p})) \quad (4)$$

where $w_1(t)$ and $w_2(t) \in \mathcal{R}^l$ and where each nonlinearity is assumed uniformly Lipschitz continuous and C^1 on $\mathcal{R}$.

Assumption 3.1 *Φ is a Lipschitz and C^1 operator such that there exists for each $i \in \{1, \cdots, l\}$, two finite constants $\alpha_i < \beta_i$ such that*

$$\alpha_i = \min_{\xi \in \mathcal{R}} \frac{\partial \varphi_i}{\partial \sigma}(\xi) \text{ and } \beta_i = \max_{\xi \in \mathcal{R}} \frac{\partial \varphi_i}{\partial \sigma}(\xi)$$

We now propose a necessary and sufficient condition such the interconnection between Φ and $M(s)$ is internally incrementally stable.

Theorem 3.1 *Let Φ be a Lipschitz and C^1 operator satisfying assumption 3.1 and let $M(s)$ be an LTI system with the minimal realization (3). Let us denote $\mathcal{A} \subset \mathcal{R}^{n \times n}$ the set of all time-varying and measurable matrices $A(t)$ that belong to the polytope of matrices defined as:*

$$\mathcal{A} = (A(t)|A(t) = A + B\,diag(k_1(t), \cdots, k_l(t))C \quad \exists k_i(t) \in [\alpha_i,\ \beta_i] \ \ \forall i \in \{1, \cdots, l\})$$

Then, the interconnection between Φ and $M(s)$ is internally incrementally stable if and only if all the solutions of the Linear Differential Inclusion associated to $\mathcal{A}$ go to zero as $t \to \infty$.

Proof: Following proposition 2.1, the linearization of Σ along a specific input $w_r = (w_{1r}, w_{2r})^T$ corresponds to the interconnection between this LTI system

$$\bar{M}(s) = \begin{cases} \dot{\bar{x}}(t) &= A\bar{x}(t) + B\bar{q}(t)\ ;\ \bar{x}(t_0) = 0 \\ \bar{p}(t) &= C\bar{x}(t) \end{cases}$$

and the following vector of time-varying gains

$$\bar{q}(t) = \mathrm{diag}(k_1(t), \cdots, k_l(t))(\bar{p}(t) + \bar{w}_2(t)) + \bar{w}_1(t) \quad (5)$$

with

$$k_i(t) = \frac{\partial \varphi_i}{\partial \sigma}(p_{ir}(t) + w_{2ir}(t))$$

and where $p_r(t)$ is associated to Σ for the input $w_r(t) = (w_{1r}(t), w_{2r}(t))^T$ and for the initial condition x_0. Finally, note that the realization of $D_G\Sigma[w_r]$ is bounded and minimal since the realization of $M(s)$ is minimal and all the $k_i(t)$ are bounded (see lemma 3 in [32]). So, following proposition 2.2, a necessary and sufficient condition for the incremental stability of Σ is the uniform exponential stability of $D_G\Sigma[w_r]$ defined as

$$\dot{z} = Az(t) + B\mathrm{diag}(k_1(t), \cdots, k_l(t))Cz(t) \quad (6)$$

where, by definition, each $k_i(t)$ belongs to $[\alpha_i, \beta_i]$ for any $t \in [t_0, \infty)$.

Sufficiency. From Lemma 2 p. 160 in [9], as all the solutions of the linear differential inclusion (LDI) associated to $\mathcal{A}$ go to zero as $t \to \infty$, the equilibrium point of the LDI is asymptotic stable. Moreover, an LDI is a homogeneous differential inclusion. It is then possible to conclude that the equilibrium point of the LDI system associated to $\mathcal{A}$ is uniformly exponentially stable (see remark on Theorem 4 in [9]). Proposition 2.2 allows to conclude the sufficiency part of the proof.

Necessity. The linear differential inclusion can be rewritten in the following equivalent form:

$$\dot{z} = Az(t) + B\mathrm{diag}(u_1(t), \cdots, u_l(t))Cz(t) \quad (7)$$

where each input $u_i(t)$ is a measurable signal such that $\alpha_i \leq u_i(t) \leq \beta_i$ for any $t \in [t_0, \infty)$.

The necessity is proved using the following fact: for the same initial condition, the solutions of system (7) are the solutions of system (6), *i.e.* for any measurable input $u(t)$ such that $\alpha_i \leq u_i(t) \leq \beta_i$ there at least exists an input w_{2r}, such that for any $i \in \{1, \cdots, l\}$, one has:

$$u_i(t) = \frac{\partial \varphi_i}{\partial \sigma}(p_{ir}(t) + w_{2ir}(t)) \text{ almost everywhere}$$

This fact can be proved in several steps. The first step is to prove that it is always possible to choose the input of Φ. To this purpose, let us assume that the input of Φ is $\nu(t)$. Let us consider the output of the open-loop system associated to the connection between $M(s)$ and Φ, *i.e.* $q_\nu(t) = M(\Phi(\nu(t))$. Now, if we consider the closed-loop system and if we define w_{2r} as

$$w_{2r}(t) = \nu(t) - q_\nu(t)$$

then by definition the input of Φ is $\nu(t)$.
Let us now prove that for any measurable $u(t)$, there exists $w_{2r}(t) \in \mathcal{L}_2^e$ such that for any $i \in \{1, \cdots, l\}$, one has

$$u_i(t) = \frac{\partial \varphi_i}{\partial \sigma}(p_{ir}(t) + w_{2ir}(t))$$

To this purpose, let us recall that a measurable function is a step function limit[1] (see *e.g.* [6]) then for any $u(t)$, there exist step functions, ϕ_n such that

$$\lim_{n \to \infty} \phi_n(t) = u(t) \text{ almost everywhere.}$$

Moreover, since $\frac{\partial \varphi_i}{\partial \sigma}$ is a continuous function, there exists for any ϕ_n, at least a step function ψ_n, such that

$$\frac{\partial \varphi_i}{\partial \sigma}(\psi_n) = \phi_n$$

We then deduce that there exists an input belonging to $\mathcal{L}_2^e$ defined by

$$w_{2ir}(t) = \lim_{n \to \infty} \psi_n(t) - p_{ir}(t)$$

and such that

$$u(t) = \frac{\partial \varphi_i}{\partial \sigma}(p_{ir}(t) + w_{2ir}(t)) \text{ almost everywhere}$$

Indeed, $w_{2ir}(t)$ is the sum of two functions belonging to $\mathcal{L}_2^e$ since

[1] A function ϕ, defined on a closed set $[a, b]$ of $\mathcal{R}$ into $\mathcal{R}$, is called a *step function* if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ of the interval such that in every subinterval $I_k = (t_{k-1}, t_k)$ the function ϕ is constant, *i.e.*, $\phi(t) = a_k$ for $t \in I_k$ for $k = 1, 2, \ldots, n$.

1. the closed-loop is assumed well-defined that ensures that $p_{ir}(t) \in \mathcal{L}_2^e$;

2. there exists a finite constant $K > 0$ such that $|\psi_n(t)| \leq K$ where

$$K = \max(|\arg\min_{\xi \in \mathcal{R}} \frac{\partial \varphi_i}{\partial \sigma}(\xi)|, |\arg\max_{\xi \in \mathcal{R}} \frac{\partial \varphi_i}{\partial \sigma}(\xi)|).$$

We thus deduce that $f(t)$ defined by $f(t) = \lim_{n \to \infty} \psi_n(t)$, is square integrable on any finite support since $\|f(t)\|^2$ is a bounded and measurable function on any finite interval of time. This last claim allows to conclude the proof. □

Remarks

(i) Theorem 3.1 conditions are closely related to conditions ensuring absolute stability. Remember that the absolute stability problem focus on the stability of the interconnection between an LTI system and a time-varying and memoryless sector nonlinearities, see [25, 21, 23]. In the absolute stability case, necessity corresponds to a robustness result: if the condition is not satisfied, then there at least exists one particular time-varying and memoryless sector nonlinearity such that the interconnected system is not stable. From this remark, we conclude that if we apply this result for proving stability of the closed-loop system with a given (sector) nonlinearity, the obtained condition is only sufficient and it is generally conservative.

In the incremental case, this conclusion is no longer true. Indeed, necessary and sufficient condition of theorem 3.1 ensures incremental stability of the closed loop system for all C^1 nonlinearities such that for any $i \in \{1, \cdots, l\}$, there exist two finite constants ξ_1 and ξ_2 such that

$$\frac{\partial \varphi_i}{\partial x}(\xi_1) = \alpha_i \text{ and } \frac{\partial \varphi_i}{\partial x}(\xi_2) = \beta_i$$

The condition is thus necessary and sufficient for each nonlinearity of the prescribed sector (and such that the min and max values of the sector are reached).

(ii) All the qualitative properties of the quadratically incrementally stable systems presented in [14] are also true for the systems considered in the main result. Actually, the system steady state is unique, its behavior for any periodic (resp. constant) input is asymptotically periodic (resp. constant). Finally, all the unperturbed trajectories of the system are globally exponentially stable (see also [17]).

(iii) Theorem 3.1 remains valid on $\mathcal{L}_\infty$.

(iv) Following the Y.S. Pyatnitskii's proof in [25], in the case of a non incrementally stable interconnection, the corresponding nonlinearity (step function) inputs are such that the derivative of each nonlinearity only takes the two extreme values *i.e.* $\frac{\partial \varphi_i}{\partial \sigma}(\xi) = \alpha_i$ or $\frac{\partial \varphi_i}{\partial \sigma}(\xi) = \beta_i$. We then conclude that theorem 3.1 condition holds if we just consider the space of piecewise continuous functions equipped with the $\mathcal{L}_2$ or $\mathcal{L}_\infty$ norms.

4 Computation aspects: necessary and sufficient conditions

For testing incremental stability, a sufficient condition was previously proposed in [14]. Testing this condition is computationally attractive (convex optimization over Linear Matrix Inequality constraints). In the previous section, we propose a necessary and sufficient condition for ensuring incremental stability. From this condition, we can suspect that the quadratic condition of [14] is overly conservative. In the sequel, this fact is emphasized by an example. In this section, we discuss on testing theorem 3.1 condition. This theorem allows to recast testing incremental stability as testing exponential stability of the associated Linear Differential Inclusion. In the general case, this problem is difficult.

Due to the theoretical importance of the absolute stability problem, or more recently, the interest of the robustness conditions against real time-varying parameters, many approaches were proposed for this problem.

More generally, this problem is closely related to various classical results of the stability theory[2], see e.g.:

- Absolute stability problem (see *e.g.*[25, 23, 20, 19, 43]);
- Time-varying stability radius (see *e.g.* [7] and see also [26])
- Joint spectral radius of a set of matrices (see *e.g.* [4, 2, 33])
- Invariant sets and associated Lyapunov functions (see *e.g.* [3])

Based on these different results, we discuss different (sufficient) conditions for testing the exponential stability of an LDI system. Their interest is due to the (NP hard) complexity of the necessary and sufficient condition.

4.1 Necessary and sufficient condition: specific Lyapunov function

Let us adapt Theorem 2 in [21] to our problem. To this problem, let us introduce $F(x)$, a compact set in $\mathcal{R}^n$, defined as

$$F(x) = \{y | y = Ax + B\text{diag}(\lambda_1, \cdots, \lambda_l)Cx, \quad \alpha_i \leq \lambda_i \leq \beta_i, i \in \{1, \cdots, l\}\} \tag{8}$$

Proposition 4.1 *Let Φ be a Lipschitz and C^1 operator satisfying assumption 3.1 and let $M(s)$ be an LTI system defined by (3). The interconnection between Φ and $M(s)$ is internally incrementally stable if and only if there exist an*

[2]The number of references on these different results is so important that it is not possible to propose an exhaustive list of references. We just point out some of them.

integer $m \geq n$, a strictly positive constant δ and a positive definite function V defined by

$$V(x) = \max_{1 \leq i \leq m} x^T P_i, x \qquad (9)$$

where P_i for any $i \in \{1, \ldots, m\}$ is a positive definite matrix and such that

$$\max_{y \in F(x)} D^+ V(x, y) \leq -\delta \|x\|_2^2 \qquad (10)$$

for any $x \in \mathcal{R}^n$ and where

$$D^+ V(x, y) = \lim_{\lambda \downarrow 0} \frac{V(x + \lambda y) - V(x)}{\lambda}$$

is the directional derivative of $V(x)$ at x in the direction $y \in F(x)$.

The main interest of this proposition is that the usual quadratic condition is recovered with $m = 1$ [14]. Unfortunately, conservatism of the condition for $m = 1$ can be proved by a simple counterexample.

Counterexample Let us consider that the LTI system is the second order system:

$$M(s) = \frac{1}{s^2 + 2s + 1}$$

and let us consider that $\varphi(x)$ belongs to the incremental sector $[0, \beta]$ *i.e.* $0 \leq \frac{\partial \varphi_i}{\partial \sigma}(\xi) \leq \beta$. From the result presented in [36], a quadratic ($m = 1$) solution exists if (and only if)

$$\beta < (\min_{\omega \in [0, \infty]} Re(M(j\omega)))^{-1} = 8.065.$$

Furthermore, Brockett proved (see p. 222 in [5]) that the linearization of the interconnected system is exponentially stable for $\beta \leq 11.6$. We thus conclude that, in this example, the quadratic condition is only a sufficient condition.

As a consequence, quadratic based conditions (such as small gain theorem and passive theorem) are conservative. Let us note that another possibility exists for decreasing the conservatism of the small gain theorem: the use of multipliers [41, 42, 8, 18]. Unfortunately, as it is proved in [18], only constant multipliers can be used for incremental memoryless nonlinearities. Following this result, this approach does not allow to decrease the conservatism of the quadratic based condition.
In order to reduce conservatism, it is thus necessary to apply proposition 4.1 with $m > 1$. Unfortunately, as it was pointed out in [21] (see also [37]), if $m > 1$ then the reformulation of Proposition 4.1 condition as a standard optimization problem[3] is an open problem. For this purpose, a usual approach is based on the use of the so-called S-procedure. Unfortunately when quadratic constraints are non integral, the S-procedure is only lossless[4] in the case of two quadratic constraints [38]. As a consequence, to our best knowledge, testing Proposition 4.1 condition cannot be equivalently transformed in a standard (efficient) optimization problem.

To overcome this problem, the authors in [21] propose the use of a piecewise linear Lyapunov function of the polyhedral vector norm instead of the use of the piecewise linear Lyapunov function of the quadratic vector norm.

The main interest of the second approach is that the constraint (10) reduces in the polyhedral case to a set of linear inequalities. With these constraints, the S procedure is replaced by the Minkowski-Farkas lemma which is lossless for any value of m.

Proposition 4.2 *Let Φ be a Lipschitz and C^1 operator satisfying assumption 3.1 and let $M(s)$ be an LTI system defined by (3). The interconnection between Φ and $M(s)$ is internally incrementally stable if and only if there exist for some integer $m \geq n$ a full column rank matrix $H \in \mathcal{R}^{m \times n}$ and $m \times m$ matrices Γ_i, $i = 1, \ldots, 2^l$, satisfying*[5]

$$\mu_\infty(\Gamma_i) < 0 \textit{ for all } i = 1, \ldots, 2^l;$$

such that the matrix relations

$$H A_i = \Gamma_i H \textit{ for all } i = 1, \ldots, 2^l;$$

are satisfied where A_i are the vertices of a convex polyhedron of $n \times n$ matrices in the n^2 dimensional space which embedded all the matrices $A(t)$ that belong to the polytope of matrices defined as:

$$\{\forall i \in \{1, \cdots, l\}, \exists k_i(t) \in [\alpha_i, \beta_i], \\ A(t) = A + B \mathrm{diag}(k_1(t), \cdots, k_l(t)) C\}$$

Even if the previous problem is difficult to compute, it is a first step in reducing the quadratic approach conservatism. Actually, if the value of m is set, the previous condition is then only sufficient. In this particular case, an algorithm is proposed by [24]. Other classes of algorithms exist, see especially [4], [43] or [20]. Finally, in many cases, the necessary and sufficient conditions are reduced to the verification of conditions on particular sets of matrices, (*e.g.* matrix polytope edges or vertices [19]).

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[2] A. Bhaya and F. Das Chagas Mota. Equivalence of stability concepts for discrete time-varying systems. *Int. J. Robust and Nonlinear Control*,4:725–740, 1994.

[3] For instance convex optimization such linear, quadratic or LMI optimization.

[4] Lossless means that the obtained condition by applying S-procedure is also necessary.

[5] For a matrix $A \in \mathcal{R}^{n \times n}$, $\mu_\infty(A) \triangleq \max_{1 \leq i \leq n} \left\{ a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}| \right\}$.

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