DYNAMIC OUTPUT FEEDBACK SLIDING MODE CONTROL FOR NONLINEAR SYSTEMS WITH MISMATCHED UNCERTAINTY

Xing-Gang Yan, Sarah K. Spurgeon†, Christopher Edwards

Department of Engineering, University of Leicester
University Road, Leicester LE1 7RH, UK

†fax: +44 (0)116 252 2619
†e-mail: eon@le.ac.uk

Keywords: nonlinear system, robust control, sliding mode control, observer design, dynamic output feedback.

Abstract

In this paper, the robust stabilization problem for a class of nonlinear uncertain systems is studied using sliding mode techniques. Matched and mismatched uncertainties are both considered. By employing the sliding surface proposed by Zak and Hui [7], the stability of the sliding mode is shown first. Then, an asymptotic observer is established to estimate the system state variables based on a constrained Lyapunov equation, and a variable structure controller is proposed to stabilize the system by exploiting the estimated state and system output. The two major limitations of [7] are eliminated. Finally a numerical example is presented to show the approach in detail.

1 Introduction

Sliding mode ideas have been successfully applied to the control of uncertain dynamical systems [2]. A great deal of the work focuses on matched uncertainty and requires that all state variables are accessible. However, in many practical systems, state variables are not always accessible. This motivates the need for output feedback control or observer based schemes which estimate the system states. It is worth noting that the separation principle is no longer true for nonlinear systems. This means that different results may be obtained if a controller is designed for a system using estimated states and true states respectively. Therefore, the study of observer-based control is necessary for nonlinear systems.

There has been significant work which focuses on output feedback control. In the approach proposed by Zak and Hui [7] geometric conditions were presented for the existence of a sliding mode and an associated design algorithm was also derived. However, as pointed out by Kwan [4] and Shyu *et al* [5], there are two major assumptions which restrict the application of the corresponding results. In order to overcome these shortcomings in [4], a class of SISO system is studied using a specific 1^{st} order dynamic feedback controller. In this work, it is assumed that the uncertainties are matched: the same as the requirement of the work of Zak and Hui in [7]. Shyu *et al* [5] have proposed a dynamical output feedback approach which is applicable to MIMO systems with mismatched uncertainty based on the work of Kwan [4]. Unfortunately, in all the results above, it is required that the uncertainty is bounded by a function of the output or a first-order polynomial of the norm of the state variables.

In this paper, a class of nonlinear systems is considered involving both matched and mismatched uncertainties. By employing the sliding surface prescribed by Zak and Hui [7], the stability of the sliding mode is shown first. Then, an asymptotic observer is established to estimate the state variables based on a constrained Lyapunov equation. The observation error is shown to converge to zero exponentially. Further, a variable structure control is proposed using the estimated state and system output. The two major limitations in [7] are both eliminated, and the approach can be applied to a wider class of systems. In this paper, the known nonlinearity and the nonlinear disturbances are dealt with separately. This is in contrast with other work in which all nonlinearities are considered as disturbances. In addition, the uncertainty bounds considered here are the products of functions of the outputs and functions of the state variables, which not only have more general forms but also make it possible to use the function of the output completely in the observer and controller design. Therefore, conservatism is reduced and the robustness is enhanced.

2 System description and preliminaries

Consider the system

$$\dot{x} = Ax + B[u + \Delta g(x,t)] + \Delta f(x,t) + \Phi(x) \quad (1)$$

$$y = Cx \quad (2)$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $y \in \mathcal{R}^p$ $(p \geq m)$ are system state variables, inputs and outputs respectively; A, B, C are constant matrices of appropriate dimensions with B and C both being of full rank; $\Delta g(x,t)$ and $\Delta f(x,t)$ are the matched and the mismatched uncertainties respectively and the known nonlinear vector $\Phi(x)$ is sufficiently smooth with $\Phi(0) = 0$. Since $\Phi(x)$ is smooth, and $\Phi(0) = 0$, there exists a matrix $H(x) \in \mathcal{R}^{n\times n}$ such that $\Phi(x) = H(x)x$.

Assumption 1. The pair (A, B) is controllable, the pair (A, C) observable, and the nonlinear function $\Phi(x)$ is Lipschitz in its defined domain.

Assumption 1 is a limitation on the nominal system from (1)–(2). In view of the observability of (A, C), there exists a constant gain L such that $A - LC$ is Hurwitz stable. Therefore, for any positive definite $Q \in \mathcal{R}^{n\times n}$, there exists a positive definite $P \in \mathcal{R}^{n\times n}$ satisfying the following Lyapunov equation

$$(A - LC)^{\tau} P + P(A - LC) = -Q. \qquad (3)$$

Assumption 2. There exist known continuous functions ξ_1, ξ_2, γ_1 and γ_2 such that the matched uncertainty $\Delta g(x, t)$ satisfies

$$\|\Delta g(x, t)\| \le \xi_1(y, t)\xi_2(x, t), \qquad (4)$$

and the mismatched uncertainty $\Delta f(x, t)$ is described by

$$\Delta f(x, t) = E\Delta\eta(x, t) \qquad (5)$$

with $\|\Delta\eta(x, t)\| \le \gamma_1(\|y\|)\gamma_2(x, t)$, where γ_1 is differentiable at origin with $\gamma_1(0) = 0$, ξ_2 and γ_2 are both Lipschitz about x uniformly for $t \in \mathcal{R}^+$, and E is a constant matrix.

From $\gamma_1(0) = 0$, and the differentiability of $\gamma_1(\cdot)$ at the origin, there exists a continuous function $\zeta(\cdot)$ satisfying

$$\gamma_1(\tau) \;\;=\;\; \zeta(\tau)\tau. \qquad (6)$$

In Zak and Hui [7], the sliding surface is defined by

$$\delta(x) = Sx = 0 \qquad (7)$$

and the matrix $S \in \mathcal{R}^{m\times n}$ is assumed to have the following decomposition

$$S = FC. \qquad (8)$$

The algorithm to choose F and S has been given by Zak and Hui in [7], where it is claimed that the desired, distinct, real negative eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$ can be assigned freely. However, only matched uncertainty is considered in [7] and the approach has the following two limiting assumptions as pointed out in [4, 5].

- The matched uncertainty is bounded by a function of the output y.
- There exists a matrix M such that $FCA = MC$ or $SA = MC$.

It is shown in [4] that the former condition is restrictive, and in [7] that the latter limitation is strong. In most cases it is impossible to achieve $FCA = MC$ for some M even for the SISO case. This limits the application of the approach greatly. Although the results given by Kwan [4] and Shyu *et al* [5] avoid these two limitations, it is required that the uncertain bounds have the form $k_1\|x\| + k_2$, and further limitations are necessary for the sliding mode dynamics.

Assumption 3. For a given set of negative real values $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$, there exist full rank matrices $W \in \mathcal{R}^{n\times(n-m)}$, and $W^g \in \mathcal{R}^{(n-m)\times n}$ such that $W^g W = I_{n-m}$, $W^g B = 0$ and $W^g AW = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$. Then, according to Zak and Hui [7] there exists a matrix $S \in \mathcal{R}^{m\times n}$ such that the sliding motion associated with (7) is governed by $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$.

Assumption 4. The matrix S in (7) has the decomposition (8).

Remark 1. Assumption 3 is mainly used to guarantee the existence of the sliding surface $Sx = 0$. Under Assumption 3, Zak and Hui [7] show that $SW = 0$, SB is nonsingular, and the nominal linear system is asymptotically stable when restricted to the sliding surface (7). Assumption 4 guarantees that the state switching surface can be replaced by an output switching surface. $\triangleleft$

3 Stability analysis of sliding mode

Consider system (1)-(2). By Assumption 4, the sliding surface (7) can be expressed as $Fy = 0$. Then, under Assumption 3, it is observed from [7] that the matrix $[\,W \quad B\,]$ is invertible, and

$$[\,W \quad B\,]^{-1} = \begin{bmatrix} W^g \\ B^g \end{bmatrix}$$

where W^g and B^g denote the generalized inverse of W and B respectively. Now, introduce the coordinate transformation $z = Tx$ with T defined by

$$T = \begin{bmatrix} W^g \\ B^g \end{bmatrix}. \qquad (9)$$

In the new coordinates z, system (1)-(2) is described by

$$\begin{aligned}
\dot{z}_1 &= Dz_1 + W^g ABz_2 \\
&\quad + W^g\left[\Delta f(T^{-1}z, t) + \Phi(T^{-1}z)\right], && (10)\\
\dot{z}_2 &= B^g AWz_1 + B^g ABz_2 + u + \Delta g(T^{-1}z, t) \\
&\quad + B^g\left[\Delta f(T^{-1}z, t) + \Phi(T^{-1}z)\right], && (11)\\
y &= CT^{-1}z, && (12)
\end{aligned}$$

where $z_1 \in \mathcal{R}^{n-m}$, $z_2 \in \mathcal{R}^m$, $z = \mathrm{col}(z_1, z_2)$ and the matrix $D \equiv: \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$. It should be noted that Assumption 3 and $B^g B = I_m$ are used in the above.

Consider the sliding surface (7). In the new coordinate system z:

$$SWz_1 + SBz_2 = 0.$$

Then, from Remark 1, the sliding surface is given by

$$z_2 = 0$$

due to $SW=0$ and the nonsingularity of SB. Therefore, the sliding mode may be prescribed by

$$\dot{z}_1 \quad = \quad Dz_1 + W^g\Big[\Delta f(Wz_1,t)+\Phi(Wz_1)\Big] \qquad (13)$$

Theorem 1 *Consider system (1)–(2). For a given set of negative numbers $\{\lambda_1,\lambda_2,\cdots,\lambda_{n-m}\}$, under Assumptions 1-4, the sliding mode (13) of system (1)–(2) is asymptotically stable if there exists a neighborhood Ω of the origin such that in $\Omega\setminus\{0\}$*

$$\|W^g\|\Big(\|E\|\,\|CW\|\zeta(\|CWz_1\|)\gamma_2(Wz_1,t) + \|H(Wz_1)W\|\Big) < \mu \qquad (14)$$

where $\mu \equiv: min\{|\lambda_1|,|\lambda_2|,\cdots,|\lambda_{n-m}|\}$, W and W^g are determined by Assumption 3, and ζ and γ_2 are defined by (6) and Assumption 2 respectively.

Proof: For the sliding mode (13), consider the Lyapunov function candidate $V=z_1^\tau z_1$. Then, the time derivative of V along the trajectories of the dynamic system (13) is given by

$$\begin{aligned}\dot{V}\mid_{(13)} &= 2z_1^\tau Dz_1 + 2z_1^\tau W^g\left[\Delta f(Wz_1,t)+\Phi(Wz_1)\right]\\ &\le -2\mu\|z_1\|^2 + 2\|z_1\|\,\|W^g\|\Big[\|E\|\gamma_1\left(\|CWz_1\|\right)\\ &\quad \cdot\gamma_2(Wz_1,t)+\|H(Wz_1)Wz_1\|\Big]\\ &\le -2\mu\|z_1\|^2 + 2\|W^g\|\Big[\zeta\left(\|CWz_1\|\right)\gamma_2(Wz_1,t)\\ &\quad \cdot\|E\|\,\|CW\|+\|H(Wz_1)W\|\Big]\|z_1\|^2\\ &= -2\Big[\mu-\|W^g\|\Big(\|CW\|\,\|E\|\zeta\left(\|CWz_1\|\right)\\ &\quad \cdot\gamma_2(Wz_1,t)+\|H(Wz_1)W\|\Big)\Big]\|z_1\|^2.\end{aligned}$$

Combining with condition (14), $\dot{V}\mid_{(13)}$ is negative definite. Hence, the conclusion follows. #

Remark 2. From (13), the matched uncertainty does not affect the stability of the sliding mode. However, the mismatched uncertainty affects the dynamics of the sliding mode and is closely connected with stability. It is therefore necessary to impose some constraints on the mismatched part. The limitation (14) on the bound of the mismatched component is used to guarantee the stability of the sliding mode. ◁

Remark 3. In most cases, Theorem 1 is local due to (14). However, from the proof above, it is observed that the result is global if condition (14) is satisfied globally. Further, the conclusion in Theorem 1 may be global if the bounds of all mismatched nonlinearities degenerate to the case of the previous work (see e.g. [4, 5]). ◁

4 Variable structure controller design

In this section, an asymptotic observer is developed to estimate the state variables of system (1)–(2). Based on this estimate and the system output, a variable structure control is developed.

4.1 Asymptotic observer design

Assumption 5. There exist constant matrices Γ and Υ with appropriate dimension such that the solution of the Lyapunov equation (3) satisfies the following constraints

i). $B^\tau P = \Gamma C$; ii) $E^\tau P = \Upsilon C$,

where E is defined as in (5).

Remark 4. It should be noted that if there exists a matrix L such that $(A-LC,B,C)$ is passive, then Assumption 5 i) is satisfied with $\Gamma=I$. Similarly, if $(A-LC,E,C)$ is passive, then, Assumption 5 ii) is also satisfied. Similar conditions are adopted by many other authors such as Cheng [1] and Yan *et al.* [6]. ◁

Construct the following dynamical system associated with the system (1)–(2):

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x}+L(y-C\hat{x})+B\Big[u+\Pi_1(\hat{x},y,t)(y-C\hat{x})\Big]\\ &\quad +\Pi_2(\hat{x},y,t)(y-C\hat{x})+\Phi(\hat{x}) \qquad (15)\end{aligned}$$

where L is determined by (3), and Π_1 and Π_2 are given by

$$\Pi_1(\hat{x},y,t) \equiv: \begin{cases}\frac{\Gamma}{\|\Gamma(y-C\hat{x})\|}\xi_1(y,t)\xi_2(\hat{x},t), & \Gamma(y-C\hat{x})\neq 0\\ 0, & \Gamma(y-C\hat{x})=0\end{cases} \qquad (16)$$

$$\Pi_2(\hat{x},y,t) \equiv: \begin{cases}\frac{E\Upsilon}{\|\Upsilon(y-C\hat{x})\|}\gamma_1(\|y\|)\gamma_2(\hat{x},t), & \Upsilon(y-C\hat{x})\neq 0\\ 0, & \Upsilon(y-C\hat{x})=0\end{cases} \qquad (17)$$

The following conclusion can be drawn:

Theorem 2 *Under Assumptions 1, 2 and 5, the dynamical system (15) is an asymptotic observer for system (1)–(2), that is $\lim_{t\to\infty}\|x(t)-\hat{x}(t)\|=0$, if*

$$\xi_1(y,t)\mathcal{L}_{\xi_2}\|\Gamma C\|+\gamma_1(\|y\|)\mathcal{L}_{\gamma_2}\|\Upsilon C\|+\mathcal{L}_\Phi\|P\| < \frac{1}{2}\underline{\lambda}(Q) \qquad (18)$$

where $\mathcal{L}$ denotes the corresponding function's Lipschitz constant. Further, there exists a nonnegative constant α_1 and a positive constant α_2 such that

$$\|x-\hat{x}\| \le \alpha_1\exp\{-\alpha_2 t\} \qquad (19)$$

if

$$\underline{\lambda}(Q)-2\sup_{y,t}\Big\{\mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\| + \mathcal{L}_{\gamma_2}\gamma_1(\|y\|)\|\Upsilon C\| + \mathcal{L}_\Phi\|P\|\Big\} > 0. \qquad (20)$$

Proof: Let $e=x-\hat{x}$. It is observed from (1) and (15) that the state error dynamical equation is described by

$$\begin{aligned}\dot{e} &= (A-LC)e+B\left[-\Pi_1(\hat{x},y,t)Ce+\Delta g(x,t)\right]\\ &\quad -\Pi_2(\hat{x},y,t)Ce+\Delta f(x,t)+\Phi(x)-\Phi(\hat{x}). \qquad (21)\end{aligned}$$

For system (21), consider a Lyapunov function candidate as $V = e^\tau Pe$. Then, the time derivative of V along the trajectories of system (21) is given as

$$\begin{aligned}\dot{V} &= -e^\tau Qe + 2e^\tau PB\big[-\Pi_1(\hat{x},y,t)Ce + \Delta g(x,t)\big] \\ &\quad +2e^\tau P\big(-\Pi_2(\hat{x},y,t)Ce + \Delta f(x,t)\big) \\ &\quad +2e^\tau P\big(\Phi(x) - \Phi(\hat{x})\big). \end{aligned} \tag{22}$$

From Assumption 2, Assumption 5, and (16), it follows that

i). If $\Gamma(y - C\hat{x}) = 0$, then, it is easy to see that

$$\begin{aligned} & e^\tau PB\Big[-\Pi_1(\hat{x},y,t)Ce + \Delta g(x,t)\Big] \\ &= (\Gamma Ce)^\tau\Big[-\Pi_1(\hat{x},y,t)Ce + \Delta g(x,t)\Big] \\ &= 0 \end{aligned}$$

ii). If $\Gamma(y - C\hat{x}) \neq 0$, then,

$$\begin{aligned} & e^\tau PB\big[-\Pi_1(\hat{x},y,t)Ce + \Delta g(x,t)\big] \\ &\leq -\|\Gamma Ce\|\xi_1(y,t)\xi_2(\hat{x},t) + \|\Gamma Ce\|\xi_1(y,t)\xi_2(x,t) \\ &= \|\Gamma Ce\|\Big(\xi_2(x,t) - \xi_2(\hat{x},t)\Big)\xi_1(y,t) \\ &\leq \mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\|\,\|e\|^2. \end{aligned}$$

Therefore,

$$e^\tau PB\big[-\Pi_1(\hat{x},y,t)Ce + \Delta g(x,t)\big] \leq \mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\|\,\|e\|^2 \tag{23}$$

Then, by the same reasoning as above, it is observed from (5), (17) and Assumption 5 that

$$e^\tau P\big(-\Pi_2(\hat{x},y,t)Ce + \Delta f(x,t)\big) \leq \gamma_1(\|y\|)\mathcal{L}_{\gamma_2}\|\Upsilon C\|\,\|e\|^2 \tag{24}$$

From Assumption 1, $\Phi(x)$ is Lipschitz, and thus

$$e^\tau P\big(\Phi(x) - \Phi(\hat{x})\big) \quad\leq\quad \mathcal{L}_\Phi\|P\|\,\|e\|^2. \tag{25}$$

Now, substituting (23)–(25) into (22), it follows that

$$\begin{aligned}\dot{V} &\leq -e^\tau Qe + 2\Big[\mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\| + \mathcal{L}_{\gamma_2}\gamma_1(\|y\|)\|\Upsilon C\| \\ &\quad +\mathcal{L}_\Phi\|P\|\Big]\|e\|^2 \\ &\leq -2\Big(\frac{1}{2}\underline{\lambda}(Q) - \mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\| \\ &\quad -\mathcal{L}_{\gamma_2}\gamma_1(\|y\|)\|\Upsilon C\| - \mathcal{L}_\Phi\|P\|\Big)\|e\|^2. \end{aligned} \tag{26}$$

Using (18), the RHS of (26) is negative and hence the error dynamics (21) are asymptotically stable, and thus (15) is an asymptotic observer of system (1)–(2).

Define

$$\begin{aligned}\kappa = &\ \underline{\lambda}(Q) - 2\sup_{y,t}\big\{\mathcal{L}_{\xi_2}\xi_1(y,t)\|\Gamma C\| \\ &+\mathcal{L}_{\gamma_2}\gamma_1(\|y\|)\|\Upsilon C\| + \mathcal{L}_\Phi\|P\|\big\}.\end{aligned}$$

If $\kappa > 0$, then it follows from (26) that

$$\dot{V} \leq -\kappa\|e\|^2 \leq -\frac{\kappa}{\overline{\lambda}(P)}e^\tau Pe = -\frac{\kappa}{\overline{\lambda}(P)}V$$

Consequently, $V(t) \leq V(0) + \int_0^t -\frac{\kappa}{\overline{\lambda}(P)}V(t)dt$. Therefore, from the Gronwall-Bellman Inequality (See [3] PP. 68-69)

$$\|V\| \leq V(0)\exp\left\{-\frac{\kappa}{\overline{\lambda}(P)}t\right\}$$

Combined with the fact $\|e\| \leq \sqrt{\frac{\|V\|}{\underline{\lambda}(P)}}$, it follows that

$$\|e\| \leq \sqrt{\frac{V(0)}{\underline{\lambda}(P)}}\exp\{-\frac{\kappa}{2\overline{\lambda}(P)}\}$$

Hence, the conclusion follows by choosing $\alpha_1 = \sqrt{\frac{V(0)}{\underline{\lambda}(P)}}$ and $\alpha_2 = \frac{\kappa}{2\overline{\lambda}(P)}$. #

It should be noted that the observer (15) may be written as

$$\dot{\hat{x}} = A\hat{x} + \Theta(\hat{x},y,t)(y - C\hat{x}) + Bu + \Phi(\hat{x}),$$

where $\Theta(\hat{x},y,t) \equiv: L + B\Pi_1(\hat{x},y,t) + \Pi_2(\hat{x},y,t)$ is called the observer gain. The time-varying parts Π_1 and Π_2 are introduced here mainly to reject the effect of uncertainties $\Delta g(x,t)$ and $\Delta f(x,t)$ respectively. Theorem 2 shows that under some conditions, the observer error converges to zero exponentially.

4.2 Variable structure control design

A control must be designed so that the system is driven to the sliding surface and forced to remain there. Consider the following output feedback variable structure controller

$$\begin{aligned} u = &-(SB)^{-1}\Big[\frac{Fy}{\|Fy\|}k(y,t) + S(A\hat{x} + \Phi(\hat{x})) + \frac{Fy}{\|Fy\|} \\ &\Big(\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) + \|SE\|\gamma_1(\|y\|)\gamma_2(\hat{x},t)\Big)\Big] \end{aligned} \tag{27}$$

where $\hat{x}$ is given by (15), and the control gain $k(y,t)$ is to be developed to satisfy the reachability condition

$$\sigma^\tau(x)\dot{\sigma}(x) < -\beta\|\sigma(x)\| \tag{28}$$

with β a positive constant. Then:

Theorem 3 *Suppose that (20) is satisfied. Under Assumptions 1-5, system (1)–(2), driven by control (27), converges to the sliding surface (7) and remains on it if the control gain $k(y,t)$ is chosen such that*

$$\begin{aligned} k(y,t) > &\ \alpha_1\exp\{-\alpha_2 t\}\big[\|SA\| + \|S\|\mathcal{L}_\Phi + \xi_1(y,t)\mathcal{L}_{\xi_2} \\ &\cdot\|SB\| + \gamma_1(\|y\|)\mathcal{L}_{\gamma_2}\|SE\|\big] + \beta \end{aligned} \tag{29}$$

where β is chosen as a positive constant, α_1 and α_2 are given as in Theorem 2, S satisfies (8), and ξ_1, ξ_2, γ_1 and γ_2 are defined by Assumption 2.

Proof: From (1)–(2) and (7), it is observed that the sliding dynamics can be written as

$$\dot{\sigma}(x) = S\Big(Ax + B[u + \Delta g(x,t)] + \Delta f(x,t) + \Phi(x)\Big). \quad (30)$$

Then, by (27) and (30), it follows that

$$\begin{aligned} &\sigma^\tau(x)\dot{\sigma}(x) \\ &= -(Sx)^\tau \frac{Fy}{\|Fy\|}k(y,t) + \big[(Sx)^\tau SB\Delta g(x,t) \\ &\quad -(Sx)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t)\big] + \\ &\quad \big[(Sx)^\tau S\Delta f(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SE\|\gamma_1(\|y\|) \\ &\quad \cdot\gamma_2(\hat{x},t)\big] + \big[(Sx)^\tau SAx - (Sx)^\tau SA\hat{x}\big] \\ &\quad +\big[(Sx)^\tau S\Phi(x) - (Sx)^\tau S\Phi(\hat{x})\big]. \end{aligned} \quad (31)$$

By Assumption 2, $S = FC$ and (19), it follows that

$$\begin{aligned} &(Sx)^\tau SB\Delta g(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) \\ &\leq \|Fy\|\,\|SB\|\xi_1(y,t)\xi_2(x,t) \\ &\quad -(Fy)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) \\ &= \|Fy\|\,\|SB\|\xi_1(y,t)\left(\xi_2(x,t) - \xi_2(\hat{x},t)\right) \\ &\leq \alpha_1\|Fy\|\,\|SB\|\xi_1(y,t)\mathcal{L}_{\xi_2}\exp\{-\alpha_2 t\}, \end{aligned} \quad (32)$$

and

$$\begin{aligned} &(Sx)^\tau S\Delta f(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SE\|\gamma_1(\|y\|)\gamma_2(\hat{x},t) \\ &= (Fy)^\tau SE\Delta\eta(x,t) - \\ &\quad (Fy)^\tau \frac{Fy}{\|Fy\|}\|SE\|\gamma_1(\|y\|)\gamma_2(\hat{x},t) \\ &\leq \|Fy\|\,\|SE\|\gamma_1(\|y\|)\gamma_2(x,t) \\ &\quad -\|Fy\|\,\|SE\|\gamma_1(\|y\|)\gamma_2(\hat{x},t) \\ &\leq \alpha_1\|Fy\|\,\|SE\|\gamma_1(\|y\|)\mathcal{L}_{\gamma_2}\exp\{-\alpha_2 t\}. \end{aligned} \quad (33)$$

By similar reasoning, it follows that

$$\begin{aligned} &(Sx)^\tau SAx - (Sx)^\tau SA\hat{x} \\ &\quad = (Fy)^\tau SA(x - \hat{x}) \\ &\quad \leq \alpha_1\|Fy\|\,\|SA\|\exp\{-\alpha_2 t\}. \end{aligned} \quad (34)$$

and

$$\begin{aligned} &(Sx)^\tau S\Phi(x) - (Sx)^\tau S\Phi(\hat{x}) \\ &\quad \leq \alpha_1\|Fy\|\,\|S\|\mathcal{L}_\Phi\exp\{-\alpha_2 t\}. \end{aligned} \quad (35)$$

Substituting (32)–(35) into (31), it is obtained that

$$\begin{aligned} &\sigma^\tau(x)\dot{\sigma}(x) \\ &\leq -\|Fy\|\Big\{k(y,t) - \alpha_1\exp\{-\alpha_2 t\}\left[\|SA\| + \|S\|\mathcal{L}_\Phi\right. \\ &\quad \left. +\xi_1(y,t)\mathcal{L}_{\xi_2}\|SB\| + \gamma_1(\|y\|)\mathcal{L}_{\gamma_2}\|SE\|\right]\Big\}. \end{aligned} \quad (36)$$

Then, it is observed by (29) that

$$\sigma^\tau(x)\dot{\sigma}(x) < -\beta\|\sigma(x)\|$$

if $\sigma(x) \neq 0$. Hence the result follows. #

Remark 5. From (27), the bounds of the matched and mismatched uncertainties are both used in the analysis and design. The uncertain nonlinearity $\Delta f(x,t)$ and the known nonlinearity $\Phi(x)$ are dealt with separately throughout the paper. Therefore, conservatism is reduced as seen from the proof of Theorems 2 and 3. The system considered in this paper includes the ones discussed in previous work [7, 4, 5] as special cases. $\triangleleft$

5 Simulation Example

Consider the nonlinear uncertain system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\Big(u + \Delta g(x,t)\Big) \\ &\quad +\Delta f(x,t) + \frac{1}{8}\begin{bmatrix} x_2 \\ (x_3)^2 \\ \sin x_1 \end{bmatrix}, \end{aligned} \quad (37)$$

$$y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \quad (38)$$

where $x = \mathrm{col}(x_1, x_2, x_3)$, $y = col(y_1, y_2)$ and $u \in \mathcal{R}$ are the state variable, output and input respectively. The matched uncertainty

$$|\Delta g(x,t)| \leq \frac{1}{6}\|x\|(\sin y_1)^2\exp\{-t\}$$

and the unmatched uncertainty

$$\Delta f(x,t) \equiv: \frac{1}{10}\begin{bmatrix} 2(\Delta\eta_1 + \Delta\eta_2) \\ 3(\Delta\eta_1 + \Delta\eta_2) \\ 0 \end{bmatrix}$$

where $\Delta\eta = \mathrm{col}(\Delta\eta_1, \Delta\eta_2)$, and $\|\Delta\eta\| \leq \frac{1}{4}\|y\|\|x\|\cos^2 t$. Let

$$L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad H(x) = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & x_3 \\ \phi(x_1) & 0 & 0 \end{bmatrix},$$

where $\phi(x_1) \equiv: \begin{cases} \frac{\sin x_1}{x_1}, & x_1 \neq 0 \\ 1, & x_1 = 0 \end{cases}$.

It is easy to check that $A - LC$ is Hurwitz stable. For

$$Q = \begin{bmatrix} 11 & -0.5 & -1 \\ -0.5 & 12 & 2.5 \\ -1 & 2.5 & 12 \end{bmatrix},$$

the unique solution of Lyapunov equation (3) is

$$P = \begin{bmatrix} 3 & 0 & 0.5 \\ 0 & 2 & 0.5 \\ 0.5 & 0.5 & 3 \end{bmatrix}.$$

Choose the sliding motion poles as $\{-2, -3\}$, and let

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad W^g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B^g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is easy to verify that Assumption 3 is satisfied, and the sliding surface matrix S can be chosen as $S = [0 \quad 0 \quad 1]$ which implies $F = [0 \quad 1]$. Now, let $\xi_1(y,t) = \frac{1}{6}\sin^2 y_1$, $\xi_2(x,t) = \|x\|\exp\{-t\}$, $\gamma_1(\tau) = \frac{1}{4}\tau$, $\gamma_2(x,t) = \|x\|\cos^2 t$. Then, $\zeta(\tau) = \frac{1}{4}$. By direct computation, Assumptions 1-4 are all satisfied, and (14) is satisfied globally. By Theorem 1 and Remark 4, the sliding mode is asymptotically stable globally. Further, let

$$\Gamma = [0.5 \quad 3], E = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.3 & 0 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}, \Upsilon = \begin{bmatrix} 0.6 & 0.25 \\ 0.05 & 0.3 \\ 0.7 & 0.85 \end{bmatrix}$$

It is observed that Assumption 5 is satisfied. Computing directly, it follows that in the domain

$$\begin{aligned} \Omega' = \ & \{(x_1, x_2, x_3) \mid |x_1 + x_2| < 10.5, |x_3| \le \tfrac{1}{2}\} \\ & \cup \{(x_1, x_2, x_3) \mid |x_1 + x_2| < 3.5, \tfrac{1}{2} < |x_3| < 2.65\}, \end{aligned}$$

condition (20) is satisfied, and thus the observer for system (37)–(38) is

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & -1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ -y_2 + \hat{x}_3 \\ y_1 + 2y_2 - \hat{x}_1 - \hat{x}_2 - 2\hat{x}_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Big(u + \Pi_1(\hat{x}, y, t) \begin{bmatrix} y_1 - \hat{x}_1 - \hat{x}_2 \\ y_2 - \hat{x}_3 \end{bmatrix} \Big) \\ &\quad + \Pi_2(\hat{x}, y, t) \begin{bmatrix} y_1 - \hat{x}_1 - \hat{x}_2 \\ y_2 - \hat{x}_3 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} \hat{x}_2 \\ (\hat{x}_3)^2 \\ \sin\hat{x}_1 \end{bmatrix} \qquad (39) \end{aligned}$$

where $\hat{x} = \mathrm{col}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and Π_1 and Π_2 are directly obtained from (16) and (17) respectively.

Now, construct the control as

$$\begin{aligned} u &= -\Bigg[\frac{y_2}{|y_2|} k(y,t) - \hat{x}_2 + \frac{1}{8}\sin\hat{x}_1 + \frac{y_2}{|y_2|} \Big(\frac{1}{6}\|\hat{x}\|(\sin y_1)^2 \\ &\quad \cdot \exp\{-t\} + 0.0559\|y\|\,\|\hat{x}\|\cos^2 t \Big) \Bigg], \qquad (40) \end{aligned}$$

where $\hat{x}$ is given by (39), and $k(y,t)$ is chosen as

$$\begin{aligned} k(y,t) &= V(0)\exp\{-0.2098t\} \Big[1.6625 + \frac{1}{6}(\sin y_1)^2 \\ &\quad + 0.0559\|y\| \Big] + \beta. \qquad (41) \end{aligned}$$

Then, for the initial states $x(0) = (-3.6, 2.5, 1.0)$ and $\hat{x}(0) = (-3.2, 1.4, 0.8)$ and $\beta = 0.5$, the evolution of the system (37)–(38) is presented in Fig 1. The simulation shows that convergence is attained as proved in the theorems above.

6 Conclusion

Sliding mode control for a class of nonlinear systems is considered. An asymptotic observer is proposed with exponential observation error convergence based on the solution to a constrained Lyapunov equation. Using the estimated state and system output, a dynamic variable structure control is developed. The results are less conservative because the uncertain bounds are fully used in the controller and observer design, and the known nonlinearity and the uncertain nonlinearity are processed separately.

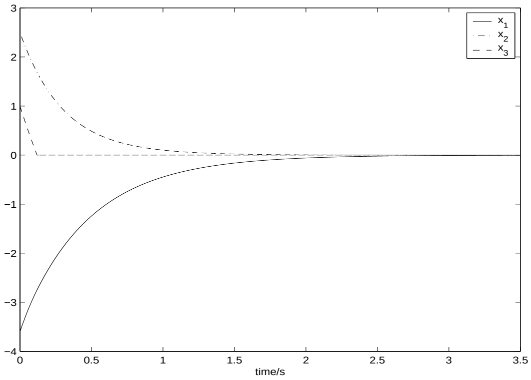

Fig.1. Evolution of state variables of system (37) under control (39)-(40).

References

[1] C. F. Cheng, W. J. Wang, and Y. P. Lin. Robust observer synthesis for nonlinear large-scale systems. *Int. J. Systems Sci.*, 25(6):1053–1066, 1994.

[2] C. Edwards and S. K. Spurgeon. *Sliding mode control: theory and application.* London: Taylor and Francis Ltd, 1998.

[3] H. K. Khalil. *Nonlinear Systems.* New York: Macmillan Publishing Company, 1992.

[4] C. M. Kwan. On variable structure output feedback controllers. *IEEE Trans. on Automat. Control*, 41(11):1691–1693, 1996.

[5] K. K. Shyu, Y. W. Tsai, and C. K. Lai. A dynamic output feedback controllers for mismatched uncertain variable structure systems. *Automatica*, 37(5):775–779, 2001.

[6] X. G. Yan, J. J. Wang, X. Y. Lü, and S. Y Zhang. Decentralized output feedback robust stabilization for a class of nonlinear interconnected systems with similarity. *IEEE Trans. on Automat. Control*, 43(2):294–299, 1998.

[7] S. H. Zak and S. Hui. On variable structure output feedback controllers for uncertain dynamic systems. *IEEE Trans. on Automat. Control*, 38(10):1509–1512, 1993.