THE EQUIVALENCE OF THE ANTI-WINDUP CONTROL DESIGN AND THE EXPLICIT MODEL-BASED PARAMETRIC CONTROLLER

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Abstract

In this paper it is shown that under certain conditions the explicit model-based tracking optimal control law for constrained input linear dynamic systems results in the same control actions as a certain class of anti-windup control design techniques. This work clearly demonstrates how parametric programming transforms systematically the complex model-based advanced control technology to a simple anti-windup control scheme that features the significant benefits of broad industrial recognition and applicability. The equivalence of the two schemes is shown via theoretical arguments and a simple SISO plant example.

1 Introduction

Of special interest and common occurrence are systems with control input constraints in an otherwise linear system. All physical systems are subject to actuator saturation that is an ubiquitous nonlinearity in the control problem. For example, the valve in an exhaust automobile system can only operate between fully open and fully closed. A popular way of treating these control characteristics has been developed, grouped under the generic name of anti-windup techniques. Anti-windup synthesis is based on first designing a linear controller ignoring the control input nonlinearity and thereafter adding an anti-windup compensator to minimize the adverse effects arising from the input nonlinearity. Anti-windup schemes are usually derived independently in an ad-hoc fashion, however, recently a large number of unified anti-windup designs with formal stability and feasibility guarantees have been proposed [8, 4, 7] elevating this technique from an industrial based practical tool to a theoretical research field.

Another control design technique for dealing with a broad class of complex constrained and multivariable processes is model predictive control (MPC), [9]. MPC determines the optimal future control profile according to a prediction of the system behaviour over a receding time horizon. The control actions are computed by solving repetitively a constrained on-line optimal control problem over a receding horizon every time a state measurement or estimate becomes available. The capabilities of MPC are limited mainly by the significant on-line calculations that make it applicable mostly to slowly varying processes. This shortcoming is surpassed by employing a different type of model-based controllers the so-called parametric controllers [10, 3]. These controllers are based on recently proposed novel parametric programming algorithms, developed in our research group at Imperial College, and succeed in obtaining the explicit mapping of the optimal control actions in the space of the current states. Thus, a state feedback control law for the system is derived off-line, respecting the input constraints and avoiding the restrictive on-line computations.

Despite these developments, advanced model based control technology such as MPC or the explicit parametric controllers is not fully appreciated or exploited by many industrial practitioners that view these methods as over-complicating academic inventions. In contrast to these notions, Dona and Goodwin [5] showed that the implicit MPC scheme can be equivalent to the saturation control function. In this paper we go a step further elucidating how the explicit optimization-based parametric controller can represent exactly a wider variety of conventional anti-windup schemes for SISO and SIMO systems, by merely tuning appropriately two controller parameters. Thus, it is clearly indicated that the implementation of model-based control entails no additional complexity, while it (i) provides a systematic way for tuning the compensator performance and (ii) opens the avenue for further improvements in closed-loop system capabilities. In § 2 a brief overview of the theory behind the parametric controller and the anti-windup control structure is presented. In the following section is it shown how the two technologies provide an identical control action under certain circumstances. Then an example is used to demonstrate our findings and the conclusions from this work are finally drawn.

2 Preliminaries

Consider the input constrained plant:

$$x_{t+1} = Ax_t + Bv_t$$

$$y_t = Cx_t + Dv_t$$
$$v_{\min} \leq v_t \leq v_{\max} \tag{1}$$

For deriving the explicit model - based optimal control law for (1), the following receding horizon optimal control problem is formulated [3, 11]:

$$\hat{\phi}(x_t) = \min_{v^N \in V^N} \begin{bmatrix} x_{t+N|t} \\ \xi_{t+k|t} \end{bmatrix}^T P \begin{bmatrix} x_{t+N|t} \\ \xi_{t+k|t} \end{bmatrix}$$
$$+ \sum_{k=0}^{N-1} [y_{t+k|t}^T Q y_{t+k|t} + \xi_{t+k|t}^T Q_1 \xi_{t+k|t}$$
$$+ v_{t+k}^T R v_{t+k}]$$
s.t.
$$x_{t+k+1|t} = A x_{t+k|t} + B v_{t+k}$$
$$y_{t+k|t} = C x_{t+k|t} + D v_{t+k}$$
$$\xi_{t+k+1|t} = \xi_{t+k|t} + y_{t+k|t}$$
$$v_{\min} \leq v_{t+k} \leq v_{\max}$$
$$k = 0, 1, 2, ..N - 1$$
$$x_{t|t} = x_t, \ \xi_{t|t} = \xi_t \tag{2}$$

where $x \in \Re^n$ are the process states; $\xi \in \Re^m$ are an extra set of states representing the integrated outputs, artificially incorporated in the dynamics to ensure offset free asymptotic tracking of the reference signal; $y \in \Re^m$ are the outputs that we aim to control, i.e. to drive to their set-point; and $v \in V \subseteq \Re$ are the manipulated inputs; $Q, P, \succeq 0, Q_1, R \succ 0$ are weight matrices penalizing the input and output deviations; t is the time when a measurement is taken, k are the future time instants, $x_{t+k|t}$ is the prediction of x for time $t + k$ at time t and N is the prediction horizon. $v^N = [v_t^T, v_{t+1}^T, ..., v_{t+N-1}^T]^T$ denotes the sequence of the control vector over the receding horizon. We assume that the pair (A, B) is stabilizable and the pair (A, C) detectable. The terminal cost satisfies the algebraic Riccati or the Lyapunov equation for ensuring stability. By considering the current states $x_t \in X$ as parameters and eliminating the equalities in (2) by substituting $x_{t+k|t} = A^k x_t + \sum_{j=0}^{k-1} (A^j B v_{t+k-1-j})$ and $\xi_{t+k|t} = \xi_t + \sum_{i=0}^{k-1} [C A^i x_t + C \sum_{j=0}^{i-1} (A^j B v_{t+i-1-j}) + D v_{t+i}]$ for the states, problem (2) is recast as a multiparametric quadratic program (mp-QP):

$$\hat{\phi}(\bar{x}_t) = \min_{v^N} \frac{1}{2} (v^N)^T U v^N + (\bar{x}_t)^T S v^N$$
$$\text{s.t. } A_I v^N \leq b \tag{3}$$

where $\bar{x} = [x^T, \xi^T]^T$, $b = [(v_{\max}^N)^T, -(v_{\min}^N)^T]^T$, U, S are matrices that are a functions of the original problem components $(A, B, C, D, P, Q, Q_1, R)$ and $A_I = [I_v, -I_v]^T$; where I_v is the square identity matrix with dimensions $N \times N$. The explicit solution of that problem can be derived provided the following conditions hold: (i) the objective function in (3) is strictly convex, (ii) the active constraints are linearly independent and (iii) strict complementarity slackness holds, i.e. if

a constraint is active its lagrange multiplier is strictly positive. The explicit solution of (3) [6] consists of a set of affine control functions in terms of the states and a set of polyhedral regions where these functions are valid. This mapping of the manipulating inputs in the state space constitutes a control law for the system. The mathematical form of the tracking parametric controller is as follows:

$$\hat{v}_t(x_t, \xi_t) = \mathcal{A}_c \cdot x_t + b_c + \mathcal{D}_c \cdot \xi_t;$$
$$\mathcal{CR}_c(x_t, \xi_t) \leq 0 \equiv \{CR_c^1 \cdot x_t + cr_c^2 + CR_c^3 \cdot \xi_t \leq 0\}$$
$$\text{for } c = 1, \cdots, N_c \tag{4}$$

where N_c is the number of regions in the state space, $\mathcal{A}_c, \mathcal{D}_c, CR_c^1, CR_c^3$ and b_c, cr_c^2 are constant matrices and vectors respectively and the index c designates that each region admits a different control law. The scalar $\hat{v}_t$ is the first element of the optimal control sequence implemented to the plant, whereas similar expressions are derived for the rest of the control elements. The integral states are calculated as follows:

$$\xi_{t+1} = \xi_t + f_1(CR_c^1, cr_c^2, CR_c^3) \cdot y_t + f_2(CR_c^1, cr_c^2, CR_c^3)$$
$$c = 1, \cdots N_c \tag{5}$$

where the functions f_1, f_2 are computed a posteriori to the derivation of the parametric solution.

Up to now, we have presented the SIMO system we are considering and the MPC control problem together with its explicit solution. The aim of the paper is to compare MPC with classical PI control anti-windup schemes. As the structure of the parametric controller (4) implies a state-feedback control law, we need to consider state feedback anti-windup control algorithms translated in the discrete-time domain for a consistent comparison. The anti-windup schemes that we examine here are:

1. In the absence of control states a large number of SIMO anti-windup designs [13, 8] pertain to a saturating control signal:
$$\hat{v}_t = sat(K x_t) \tag{6}$$
   where K is the static state feedback gain.

2. When the actuator saturates, in the presence of the integral states, the simplest approach is to stop updating the integrator [2]:
$$\hat{v}_t = sat(v_t), \ v_t = K \cdot x_t + L \cdot \sum_{i=0}^{t-1} \delta_i \cdot y_i \cdot \Delta t$$
$$\delta_i = \begin{cases} 1 \text{ if } v_t = \hat{v}_t \\ 0 \text{ otherwise} \end{cases}, \ i = 0, \cdots, t - 1 \tag{7}$$
   where Δt is the sampling time.

3. Another anti-windup approach usually called *anti-reset windup* forms an error signal as the difference between the actuator and the controller output. This error is then

fed to the input of the integrator through the vector gain K_r [8, 1]:

$$\hat{v}_t = sat(v_t),$$

$$v_t = K \cdot x_t + \sum_{i=0}^{t-1}[L \cdot y_i - K_r \cdot (v_i - \hat{v}_i)]\Delta t \quad (8)$$

3 Theoretical Developments

In this section, we establish that the closed form (4) of the model-based parametric controller under particular considerations for the representation of the integral states is equivalent to the anti-windup algorithms (6)-(8). The saturation function is first defined as :

$$\hat{v}_t = \begin{cases} v_{\min} & \text{if } v_{\min} > K\,x_t \\ K\,x_t & \text{if } v_{\min} \le K\,x_t \le v_{\max} \\ v_{\max} & \text{if } v_{\max} < K\,x_t \end{cases} \quad (9)$$

The substript t is removed in this paragraph for simplicity. Hereof, we make the following assumption:

Assumption 3.1 *The control law (4) satisfies the following condition:*

$$v_{\min} < v_t(x_t, \xi_t) < v_{\max} \Rightarrow v_{\min}^N < v^N(x_t, \xi_t) < v_{\max}^N \quad (10)$$

Next, it is useful to define the positively invariant set for the unconstrained control law $v_t = K x_t$.

Definition 3.1 $\Omega \subseteq \Re^n$ *is a positively invariant set for the system:* $x_{t+1} = (A+BK)x_t$ *if for all $x_0 \in \Omega$ the system evolution satisfies $x_t \in \Omega$, $t > 0$*

Here, three cases are stated where assumption 3.1 holds:

1. When the receding MPC horizon is equal to $N = 1$ then $v^N = v_t$ and condition (10) is trivially satisfied.

2. If $\forall x_t \in X$ an admissible control input exists which will drive the system to a positively invariant set $\Omega \subseteq \Re^n$, as defined in definition 3.1, in one step, then v_{t+k}, $k > 1$ is always unconstrained, hence condition (10) is satisfied. In that particular case the asymptotic stability of the closed-loop system is guaranteed.

3. If the magnitude of the optimal open-loop control sequence is monotonically decreasing for all $x \in X$, then $\not\exists x_t \in X$ such that $|v_{t+k+1}| > |v_{t+k}|$, or equivalently $v_{t+k+1}^2 > v_{t+k}^2$ for $k = 0, \cdots, N-2$. In that case condition (10) is always satisfied.

Thereafter, we state the following lemmas:

Lemma 3.1 *For given system (1), the control law given by the state feedback anti-windup design (6) is equivalent to the parametric controller scheme (4), (5) by appropriate manipulation of the functions f_1, f_2 provided assumption 3.1 holds.*

Proof: The absence of controller states implies the absence of the integral states in (2). Thus, we set in (5) $f_1 = f_2 = 0$ which results in $\xi_t = 0$, $\forall t \ge 0$. The resulting parametric controller is equivalent to (6) if and only if (i) the control function and (ii) the inequalities as defined in (9) are identical to the control structure of (4). The proof is thereby two-stage as follows:

i) Take the control function for a polyhedral region $\mathcal{CR}_{unc}(x_t) \le 0$ where no input saturation occurs in the first control element of sequence v^N. This implies, based on assumption 3.1 that all the elements in v^N are unconstrained and that the control function in this region is unique. Thus, the mp-QP theory and reformulation (3) yield: $v_{unc}^N = -U^{-1}S^T x_t$. But product $-U^{-1}S^T$ can be interpreted as:

$$-U^{-1}S^T = \begin{bmatrix} K_t \\ K_{t+1} \\ \vdots \\ K_{t+N-1} \end{bmatrix} \quad (11)$$

Since U, S are an explicit function of A, B, C, D, P, Q, Q_1, R, by the appropriately choosing the values of the elements of Q, R we can obtain: $K_t = K$, which indicates that both control schemes provide the same unique control action in the unconstrained region.

For a region $\mathcal{CR}_{const}(x_t) \le 0$ where one of the constraints $v_{\min} \le v_{t+k} \le v_{\max}$, $k = 0, \cdots, N$ is active, according to assumption 3.1 the constraint on the first control element is active. Thus, the pertinent control function in (4) if only some of the constraints $v_{t+k} \le v_{\max}$, $k = 0, \cdots, N$ are active is identical to [6]:

$$\tilde{\lambda} = -(\tilde{A}_I U^{-1}\tilde{A}_I^T)^{-1} \cdot (v_{\max} + \tilde{A}_I(U^{-1}S^T)x_t),$$

$$\check{\lambda} = 0 \quad (12)$$

$$v^N = -U^{-1}A_I^T\lambda - U^{-1}Sx_t, \quad \lambda = [\tilde{\lambda}^T, \ \check{\lambda}^T]^T \quad (13)$$

where λ is the vector of lagrange multipliers associated with each constraint and $\tilde{\lambda}, \tilde{A}_I$ denotes the vector (or matrix) consisting of only the active constraint rows of the vector (or matrix) λ, A_I. From now on we assume that only constraint $v_t \le v_{\max}$ is active; it follows that $\tilde{A}_I$ is just a single row matrix with the first element being unity and the rest zeros. So by substituting (12) into (13) on condition that only $v_t \le v_{\max}$ is active, we get:

$$v^N = \begin{bmatrix} -U^{-1}A_I^T \end{bmatrix} \begin{bmatrix} -(\tilde{A}_I U^{-1}\tilde{A}_I^T)^{-1}(v_{\max} + \tilde{A}_I U^{-1}S^T x_t) \\ 0 \end{bmatrix} - U^{-1}S^T x_t \quad (14)$$

In (14) 0 denotes a column vector of zeros of size $2N-1$. Now using the definition of A_I, (14) can be decomposed into:

$$v_t = (U^{-1})_{11} \cdot ((U^{-1})_{11})^{-1} \cdot (v_{\max} + (U^{-1}S^T)|_1 x_t)$$
$$- (U^{-1}S^T)|_1 x_t$$
$$= v_{\max}$$

$$v_{t+1} = (U^{-1})_{21} \cdot ((U^{-1})_{11})^{-1} \cdot (v_{\max} + (U^{-1}S^T)|_1 x_t)$$
$$- (U^{-1}S^T)|_2 x_t$$
$$\vdots$$
$$v_{t+1} = (U^{-1})_{N1} \cdot ((U^{-1})_{11})^{-1} \cdot (v_{\max} + (U^{-1}S^T)|_1 x_t)$$
$$- (U^{-1}S^T)|_N x_t \qquad (15)$$

where $(U^{-1})_{ij}$ denotes the element ij of matrix U^{-1} and $(U^{-1}S^T)|i$ denotes the ith row of matrix $U^{-1}S^T$. In the case that more than one constraint is active it is readily shown that (15) still holds for the first element of the control sequence.

ii) The state inequalities representing the boundaries of the unconstrained region $\mathcal{CR}_{unc}(x_t) \leq 0$ derive from the conditions:

$$\tilde{\lambda} > 0, \quad v_{\min}^N < v_{unc}^N < v_{\max}^N \qquad (16)$$

By definition of the unconstrained region:

$$\tilde{\lambda} \in \emptyset, \quad v_{\min}^N < -U^{-1}S^T x_t < v_{\max}^N \qquad (17)$$

According to assumption 3.1 if any of the future control elements $\{v_{t+1}, \cdots, v_{t+N-1}\}$ takes a value such that any of the inequalities in (17) becomes active it implies that the inequality enforced on the first control element becomes active too. Hence, the inequality:

$$v_{\min} < K_t x_t < v_{\max} \qquad (18)$$

is non-redundant in region $\mathcal{CR}_{unc}(x_t)$. But since for constructing the control law (4) we have assumed strict complementarity slackness and that the active constraints are linearly independent it follows that region $\mathcal{CR}_{unc}(x_t) \leq 0$ is fully defined by the state inequality (18). Accordingly, the state boundaries for the constrained region $\mathcal{CR}_{const}(x_t) \leq 0$ where $v_t \leq v_{\max}$ is active derive from the same conditions (16) and are reformulated using (12) as:

$$-(\tilde{A}_I U^{-1} \tilde{A}_I^{\;T})^{-1} \cdot (v_{\max} + \tilde{A}_I (U^{-1}S^T) x_t) \geq 0,$$
$$v_{\min} < \breve{v}_{unc_{t+k}} < v_{\max}, \quad k = 1, \cdots N-1 \qquad (19)$$

The first inequality in (19) yields:

$$v_{\max} \leq -\tilde{A}_I (U^{-1}S^T) x_t \;\Rightarrow\; v_{\max} \leq -(U^{-1}S^T)|_1 x_t$$
$$\Rightarrow v_{\max} \leq K_t x_t \qquad (20)$$

(20) represents the common border between region $\mathcal{CR}_{const}(x_t) \leq 0$ and $\mathcal{CR}_{unc}(x_t) \leq 0$. This implies that the control law (4) can be translated to:

$$\text{If } v_{\max} \;\leq\; K_t x_t \;\Rightarrow\; v_t = v_{\max}$$
$$\text{If } v_{\max} \;>\; K_t x_t \;\Rightarrow\; v_t = K_t x_t \qquad (21)$$
$$\text{and similarly for the inequality } v_{\min} \leq v_t.$$

Letting $K = K_t$ (21) and (6) are equivalent. The same result can be shown if more than one inequalities are active.
$\square$

Lemma 3.2 *Given system (1), the control law given by the state feedback anti-windup design (7) is equivalent to the parametric controller scheme (4), (5) by appropriate manipulation of the functions f_1, f_2 provided assumption 3.1 holds.*

Proof:

(i) As a tuning parameter in (5) we choose $f_2 = 0$:

$$f_1 = \begin{cases} 1 \text{ if } \mathcal{CR}_{unc}(x_t) \leq 0, \\ 0 \text{ if } \mathcal{CR}_{const}(x_t) \leq 0 \end{cases} \qquad (22)$$

Similarly to equation (11), in the unconstrained region we have:

$$v_{unc}^N = -U^{-1} S^T [x_t^T, \; \xi_t^T] \qquad (23)$$

and we define:

$$-U^{-1} S^T = \begin{bmatrix} K_t & L_t \\ K_{t+1} & L_{t+1} \\ \vdots & \\ K_{t+N-1} & L_{t+N-1} \end{bmatrix} \qquad (24)$$

By choosing appropriate values for Q, P, R, Q_1 we can obtain $K = K_t, L = L_t$. Thus the control law in the unconstrained region from the definition of ξ in (5) becomes:

$$v_{t_{unc}} = K \cdot x_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t, \qquad (25)$$

which is the same as (7).

In region $\mathcal{CR}_{const}(x_t, \xi_t) \leq 0$ where $v_t \leq v_{\max}$ is active the control action is similar to (15):

$$v_t = (U^{-1})_{11} \cdot (\tilde{A}_I U^{-1} \tilde{A}_I^{\;T})^{-1} \cdot (v_{\max} + \tilde{A}_I U^{-1} S^T [x_t^T, \; \xi_t^T]^T)$$
$$- (U^{-1}S^T)|_1 [x_t^T, \; \xi_t^T]^T \qquad (26)$$

which eventually yields:

$$v_t = v_{\max} \qquad (27)$$

The same result can readily be obtained if more than one constraints are active.

(ii) The state inequalities of the unconstrained region are:

$$v_{\min}^N < v_{unc}^N < v_{\max}^N \qquad (28)$$

from which the inequality:

$$v_{\min} < v_{t_{unc}} < v_{\max} \qquad (29)$$

is always non-redundant and yields from (25):

$$v_{\min} < K x_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t < v_{\max} \qquad (30)$$

The constrained region boundaries derive from:

$$-(\tilde{A}_I U^{-1} \tilde{A}_I^{\ T})^{-1} \cdot (v_{\max} + \tilde{A}_I (U^{-1} S^T)[x_t^T, \quad \xi_t^T]^T) \geq 0,$$
$$v_{\min} < v_{t+k_{unc}} < v_{\max}, \quad k = 1, \cdots N-1 \tag{31}$$

resulting in:

$$v_{\max} \leq K \cdot x_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t \tag{32}$$

which represents exactly the common border between regions $\mathcal{CR}_{const} \leq 0$ and $\mathcal{CR}_{unc} \leq 0$. Thus, we have proved that the control law (4) and (5), when using (22) as the condition for parameters f_1, f_2 can be reformulated as:

$$v_t = \begin{cases} v_{\max} \text{ if } v_{\max} \leq Kx_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t \\ Kx_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t \\ \quad \text{if } v_{\max} > Kx_t + L \cdot \sum_{i=0}^{t-1} f_1 \cdot y_i \cdot \Delta t \end{cases}$$
$$\text{and similarly for the inequality } v_{\min} \leq v_t \tag{33}$$

Letting $K = K_t$ and $L = L_t$ (33) and (7) are equivalent.

$\square$

Lemma 3.3 *Given system (1), the control law given by the state feedback anti-windup design (8) is equivalent to the parametric controller scheme (4), (5) by appropriate manipulation of the functions f_1, f_2 provided assumption 3.1 holds.*

Proof: The equivalence of the parametric controller (4), (5) to the anti-windup controller of (8) follows with a similar proof by choosing:

$$f_1 = 1$$
$$f_2 = \begin{cases} -\{\mathcal{D}_{unc}\}^{-1} \cdot K_r \cdot (v_{unc} - v_{\max}) \\ \quad \text{if } \mathcal{CR}_{const}^{\max}(x_t) \leq 0 \\ -\{\mathcal{D}_{unc}\}^{-1} \cdot K_r \cdot (v_{unc} - v_{\min}) \\ \quad \text{if } \mathcal{CR}_{const}^{\min}(x_t) \leq 0 \\ 0 \text{ if } \mathcal{CR}_{unc}(x_t) \leq 0 \end{cases} \tag{34}$$

where $\mathcal{CR}_{const}^{\max / \min}(x_t) \leq 0$ is the region where the inequalities $v \leq v_{\max}$ and $v \geq v_{\min}$ are active respectively; $\{\mathcal{D}_{unc}\}^{-1}$ is a column vector of size m whose elements are the inverse of the corresponding elements of $\mathcal{D}_{unc}$.
$\square$

4 Remarks

For SIMO systems that we study here, assumption 3.1 holds usually for small horizons N, while it is always the case for $N = 1$.

The equivalence between the anti-windup and the explicit MPC controllers in the MIMO case can be shown by posing stronger

assumptions on the MPC solution that renders the MPC scheme unable to account for the interactions between the manipulated variables and thus deteriorates the performance. Therefore, this case is not considered here.

Note that if there are output and state constraints present, or if the assumption 3.1 does not hold then the two schemes are generally not providing identical control actions. However, in that case the explicit model-based optimal controller will exhibit superior performance to the anti-windup scheme for the following reasons:
i) Assumption 3.1 is restrictive for the parametric controller as it removes its predictive benefits. ii) It has been shown [11] that in the presence of output constraints the parametric controller provides anti-windup action in terms of output rather than control saturation. This feature is extremely beneficial in the presence of hard output restrictions and it is certainly not captured by the traditional anti-windup design.

5 Illustrative Example

A simple 1-state SISO example from Scokaert and Mayne [12] is presented here. The problem is concerned with first deriving the explicit tracking predictive control law for the plant via parametric optimization and then recover from its structure the respective anti-windup schemes. The plant given by:

$$\begin{aligned} x_{t+k|t} &= x_t + v_t, \quad y_t = x_t \\ v_t &\in [-2, 2]; \end{aligned} \tag{35}$$

Three parametric controllers were derived with the common tuning penalties $R = 1, Q = 1, N = 2$ and P the solution of the Riccati equation and the sampling time of $\Delta t = 1$. The first controller does not feature integral action $Q_1 = 0$ and has the explicit representation shown in Table 1.

The second controller is a tracking parametric controller

CR01	
Control law	Critical Region
$v_t = -0.618034 \cdot x_t$	$-3.23607 \leq x_t \leq +3.23607$

CR02	
Control law	Critical Region
$v_t = +2$	$3.23607 \leq -1 \cdot x_t \leq +8$

CR03	
Control law	Critical Region
$v_t = -2$	$3.23607 \leq +1 \cdot x_t \leq +8$

Table 1: Critical Regions and Control laws for controller 1

that incorporates an integral state with a weighting penalty $Q_1 = 0.1$. The terminal cost P satisfies the Riccati eqaution and has the value: $P = \begin{bmatrix} 6.4721 & 1.4472 \\ 1.4472 & 1.0944 \end{bmatrix}$. Its explicit representation has the same regions as control functions that are shown in Figure 1. In region CR01 the control law is unconstrained with the function: $v_t = -0.7562 \cdot x_t - 0.1382 \cdot \xi_t$, while in the other two it is constrained to $v_t = 2$ and $v_t = -2$. The controller design is completed by assigning $f_1 = 1$ in CR01 and $f_1 = 0$ in CR02 and CR03 as defined in equation (22).

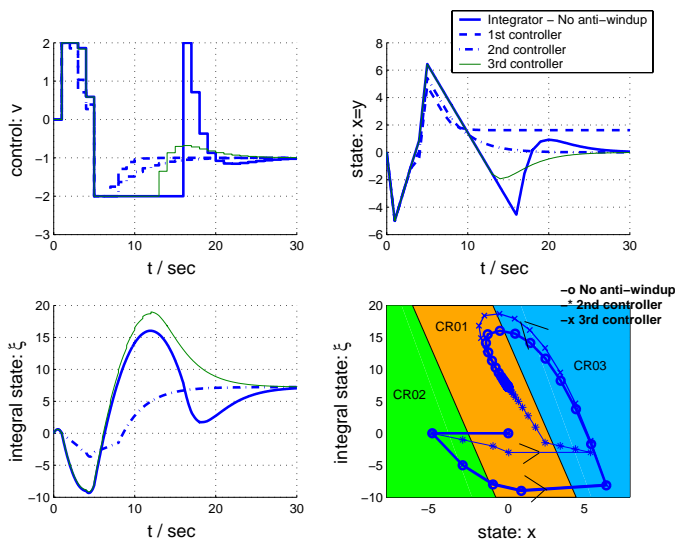The third controller has the same structure as the second

Figure 1: State - space partition and control and state profiles of the illustrative example

controller with the difference that it implements the integral action as described in (34), with $K_r = 0.5$ resulting in an $f_2 = 3.6179$.

The three parametric controllers in this example are respectively equivalent to (i) a P-controller with gain $K = -0.618034$ that saturates at ± 2; (ii) a PI-controller with gain $K = -0.7562$, time constant $\tau_I = 5.47$ (or L=-0.1382 as defined in (7)) and its integral state not being updated when $|v_t| > 2$; and (iii) a PI controller with the same tuning parameters but featuring a saturation penalty with weighting factor $K_r = 0.5$ as defined in (8). The response of those three controllers to a number of stepwise non-vanishing disturbances is shown in Figure 1.

6 Conclusions

In this paper a novel framework has been presented that demonstrates the equivalence of 3 typical anti-windup schemes and the optimization based explicit parametric controller. Our findings indicate clearly that from the simple explicit structure of the parametric controller we can postulate and retrieve in a systematic manner traditional control schemes that usually require a lot of effort for their development, tuning and implementation. The vast design capabilities and degrees of freedom of our controller can be utilized for improving further the process performance compared to the anti-windup algorithms without largely increasing the complexity of its structure. The stability properties of the controller for a given set of design parameters constitute our future research scope.

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