Robustness Analysis of Feedback Linearisation with Robust State Estimation for a Nonlinear Missile Model

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Abstract—A nonlinear missile model with time-varying uncertain parameters is controlled with a simple feedback linearisation and time-scale separation design, with synthesis based on the nominal model and full state feedback. The closed-loop system is then represented as a linear fractional transformation (LFT). A robust $H_{\infty}$ filter is designed for the controlled plant, to estimate unknown states. Robust stability of the closed-loop system is then verified by using a scaled linear differential inclusion (LDI) technique.

I. INTRODUCTION

In the synthesis of a feedback linearising controller, it is common to assume that all the necessary states and controlled outputs are available to the controller. In practice, the available measurements will only be a subset of these. In aerospace systems, models are often highly nonlinear and contain several uncertainties. Therefore, robustness of the state observer and robust stability analysis of the closed-loop system is important. Even if inversion was performed exactly, the system may still be unstable due to the nonminimum phase nature of its (generally nonlinear) internal dynamics [12]. For a robust stability guarantee in the presence of uncertainties in the plant model, we cannot assume that inversion is performed exactly.

Robust stability analysis of systems controlled using feedback linearisation is generally based on simulations e.g. [2], [13], [5]. This also applies to linear parameter varying (LPV) systems e.g. [18]. A systematic, stochastic approach to the nonlinear dynamic synthesis is the focus of [20]. It is possible with these approaches that there is some "worst-case" scenario that is missed, which is why we aim for an analysis technique that will give a robust stability guarantee for all allowed combinations of uncertain parameters. Another interesting approach is observer-based feedback linearisation designed to alleviate the estimated disturbance [2], [10]. These place some restrictive assumptions on the form of the system e.g. full-state linearisable, or uncertainty only on the input channel. We aim for a simple controller design. In [1], integral quadratic constraints (IQCs) are used to perform a robust stability analysis for this system, controlled using feedback linearisation and time scale separation. However, there is no observer i.e. full state feedback is assumed. In [17] the stability analysis assumes the fast subsystem inversion is performed exactly. The assumption that control deflections affect only the moments is carried through from controller synthesis to stability analysis. Also, there is no robustness guarantee, which is the main aim of this paper.

We derive a method for the synthesis of a robust linear time invariant (LTI) filter to estimate unknown states, by solving a system of linear matrix inequalities (LMIs), which is based on [19]. The filter is designed to minimise the $L_2$ gain from an external input to the estimation error. This method requires that the plant is stable. Therefore, we assume that we are designing a filter for the controlled plant, because the model in question is only marginally stable. As we do not assume a priori knowledge of the reference signal, we cannot use the $H_2$ method given in [19]. We therefore derive the $H_{\infty}$ condition for a system with structured uncertainty. We also verify robust stability of the closed-loop system, including the filter, which is again a sufficient LMI condition using a linear differential inclusion (LDI) and scalings, based on a quasi-LPV/LFT form for the system.

This paper is organised as follows: In Section II we present a nonlinear missile model. In Section III we give a simple input-output linearising controller using time-scale separation. In Section IV, we move from a nonlinear to a quasi-LPV/LFT model, and derive an LMI condition for synthesis of a robust filter. In Section IV-B we show an LMI condition for giving a robust upper-bound on the $L_2$ gain of the closed-loop system. Nonlinear simulation with uncertainties and filter are given in Section V. Here we also give robust stability analysis using the LMI of Section IV-B. Conclusions are given in Section VI. Notation: $\|X\|$ means the maximum singular value of $X$. $F_i(X, \Delta)$ means the lower linear fractional transformation of $X$ with $\Delta$ [21].

II. THE PLANT

The nonlinear missile model from [15], has states angle of attack and pitch rate $\alpha$ (rad) and $q$ (rad/s), output normal acceleration $\eta$ (m/s$^2$) and input tail fin deflection $u$ (rad):

$$\dot{\alpha}(t) = K_1 M(t) C_z(\alpha(t), M(t), u(t)) \cos(\alpha) + q(t) \quad (1a)$$
$$\dot{q}(t) = K_2 M^2(t) C_m(\alpha(t), M(t), u(t)) \quad (1b)$$
$$\eta(t) = K_3 M^2(t) C_z(\alpha(t), M(t), u(t)) \quad (1c)$$

The Mach number $M(t)$ is treated as an exogenous variable. The term $\cos(\alpha) \approx 1$ for the operating range and is therefore neglected from hereon. The aerodynamic coefficients are
given by:
\[
C_z(\alpha, M, u) = z_3|\alpha|^2 + z_2|\alpha|\alpha + z_1(2 - M/3)\alpha + zu \tag{2}
\]
\[
C_m(\alpha, M, u) = m_3|\alpha|^2 + m_2|\alpha|\alpha + m_1(-7 + 8M/3)\alpha + m_0 u
\]
Measurements available to the controller are \( q \) and \( \eta \). Physical data is given in Table II.

The plant is augmented with a second-order actuator with input commanded tail fin deflection \( u_c(t) \) (rad) and output \( u(t) \):
\[
\begin{bmatrix}
\dot{u}(t) \\
\ddot{u}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta\omega_n
\end{bmatrix}
\begin{bmatrix}
u(t) \\
\dot{u}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
\omega_n^2
\end{bmatrix}
u_c(t) \tag{3}
\]
The operating range is given by \(|\alpha(t)| \leq 20^\circ\) and \(1.5 \leq M(t) \leq 3\). The controller should achieve robust stability over the operating range, to uncertainty in \(\alpha\) and \(u\) dependent parts of \(C_m\) that can vary independently by \(\pm 25\%\). Performance specifications are that the controller should track step commands \(\eta_c\) with maximum time constant 350ms, overshoot 10\% and steady-state error 1\%. The maximum tail fin deflection rate should meet \(|\dot{u}(t)| \leq 25^\circ/\text{s}\) for step command \(\dot{\eta}_c = 1g\).

### III. Controller Synthesis

We follow the method in [1] for controller synthesis. The model is nonminimum phase, hence a time-scale separation technique is used. Neglecting the actuator, the plant is split into slow and fast subsystems. The \(u\)-dependent term in \(C_z\) is neglected in the slow subsystem.

**Slow subsystem:** The slow subsystem has one state \(\alpha\), input \(q_c\) (commanded value for pitch rate) and output \(q\). Defining \(\dot{C}_z := C_z(\alpha, M, 0), \eta_q := \eta - \eta_c\), and following standard input-output linearising controller synthesis [12], the slow subsystem controller is
\[
q_c = -K_1M\dot{C}_z + \left( \frac{K_3M^2}{\partial C_z / \partial \alpha} \right)^{-1}(-k_1\eta_q + \dot{\eta}_c) \tag{4}
\]
which will, for the approximate slow subsystem, achieve asymptotic tracking of \(\eta_c\) with first-order dynamics. The term \(\partial C_z(\alpha, M)/\partial \alpha\) has one positive root \(|\alpha| = 73 - 69^\circ\) for \(M = 1.5 - 3\), which is far outside the operating range, hence the controller is well-defined.

**Fast subsystem:** The fast subsystem has one state \(q\), input \(u_c\) and output \(q\). Defining \(\dot{C}_m := C_m(\alpha, M, 0), \epsilon_q := q - q_c\) an input-output linearising controller with second-order dynamics is given by
\[
u_c = -\frac{\dot{C}_m}{m_0} + \left( \frac{K_2M^2m_0}{\partial C_m / \partial \alpha} \right)^{-1}(-k_2\epsilon_q - k_3\int \epsilon_q dt) \tag{5}
\]
to achieve asymptotic tracking of \(q_c\).

The controller gains \(k_1, k_2\) and \(k_3\) were tuned in [1] using a genetic algorithm, for particular values of Mach. Here, we use the gains calculated in [1] for the nominal Mach value \(M = 2.25\): \(k_1 = 4.69, k_2 = 18.3, k_3 = 211\).

### IV. Filter Synthesis and Stability Analysis

#### A. Robust Filter Synthesis

In [19], LMI conditions are derived for synthesis of an LTI filter for a system with uncertainty that can be represented as an LFT. For a system with structured uncertainty, only the \(H_2\) result is given in [19]. However, it is stated in [19] that the \(H_\infty\) result can be derived using the methods given in that paper, which is what we do here.

The controlled plant with actuator can be written in quasi-LPV form, treating a reference demand \(r\) as the external input to the system and measured output \(y_m\):
\[
\begin{bmatrix}
\dot{x}(t) \\
y_m(t)
\end{bmatrix} =
\begin{bmatrix}
A(q_c(t)) & B_c \\
C(q_c(t)) & D
\end{bmatrix}
\begin{bmatrix}
x(t) \\
r(t)
\end{bmatrix} := M(\Delta(t))
\begin{bmatrix}
x(t) \\
r(t)
\end{bmatrix} \tag{6}
\]
where
\[
M(\Delta(t)) = F_q(H, \Delta)
\]
\[
\begin{bmatrix}
A & B_c \\
C_y & D_{yr}
\end{bmatrix} +
\begin{bmatrix}
B_p \\
D_{yp}
\end{bmatrix} \Delta(t)(I - D_{qp}\Delta(t))^{-1} \begin{bmatrix}
C_q & 0
\end{bmatrix}
\]
and \(\Delta(t)\) is a block-diagonal matrix with \(n_\theta\) blocks. Here, each diagonal block consists of a repeated scalar \(\delta_i(t)\); the normalised variation in one of the elements of \(\theta(t)\). The size of each block, \(k_i\), depends on the degree of nonlinearity with that scalar appears in the system equations. \(\Delta(t)\) therefore represents structured uncertainty and is norm-bounded, with allowed values in the set (here \(l = n_\theta\)):
\[
\Delta := \{\text{diag}(\delta_1 I_{k_1}, \ldots, \delta_l I_{k_l}) : ||\Delta|| \leq \sigma^{-1},
\delta_i \in \mathbb{R}, \sigma > 0\} \subset \mathbb{R}^{n_p \times n_p} \tag{9}
\]

\[
\begin{bmatrix}
\dot{x}(t) \\
y_m(t)
\end{bmatrix} =
\begin{bmatrix}
A & B_r \\
C_y & D_{yr}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
r(t)
\end{bmatrix}
\]

Fig. 1. Linear fractional representation of the controlled plant (6).
\[ \dot{x}(t) = Ax(t) + Br(t) + B_p p(t) \quad (10a) \]
\[ y_m(t) = C_y x(t) + D_{yr} r(t) + D_{yp} p(t) \quad (10b) \]
\[ q(t) = C_q x(t) + D_{qp} p(t) \quad (10c) \]
\[ p(t) = \Delta(t) q(t) \quad (10d) \]

with \( p, q \in \mathbb{R}^{n_p} \). We want to estimate

\[ z(t) := Lx(t), \quad L \in \mathbb{R}^{n_x \times n} \]

with a full-order LTI filter of the form

\[ \dot{\hat{x}}(t) = A_f \dot{x}(t) + B_f y_m(t) \quad (11a) \]
\[ \dot{\hat{z}}(t) = L_f \dot{x}(t) \quad (11b) \]

where \( A_f \in \mathbb{R}^{n_x \times n_x}, B_f \in \mathbb{R}^{n_x \times n_y} \) and \( L_f \in \mathbb{R}^{n_x \times n} \) are the state-space matrices of the filter, to be found. Substitute (10b), in (11a):

\[ \dot{\hat{x}} = A_f \dot{x}(t) + B_f [C_y x(t) + D_{yr} r(t) + D_{yp} p(t)] \quad (12) \]

Define the augmented state vector \( \mu := [x^T \quad \tilde{x}^T]^T \in \mathbb{R}^{2n} \), then:

\[ \dot{\mu} = \begin{bmatrix} A & 0 \\ B_f C_y & A_f \end{bmatrix} x + \begin{bmatrix} B_r \\ B_f D_{yr} \end{bmatrix} r + \begin{bmatrix} B_p \\ B_f D_{yp} \end{bmatrix} p \]
\[ =: \dot{\hat{\mu}} + \hat{B}r + \hat{L}p \quad (13) \]

Define the output of the augmented system as the estimation error:

\[ e_z := z - \hat{z} = Lx - L_f \dot{x} = L [I \quad -L_f] \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} =: \hat{C}\mu \quad (14) \]

Then, defined in terms of \( \mu \), (10c) becomes:

\[ q = [C_q \quad 0] \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + D_{qp} p =: \hat{E}\mu + D_{qp} p \quad (15) \]

and (10d), as before (augmenting with the filter does not change the \( p, q \) relation). We can represent the augmented system as an LFT, analogously to the controlled plant, however as \( B \) and \( D \) are known constant matrices, we can represent the system in the following LFT:

\[ \tilde{A}_\Delta := F_l \begin{bmatrix} \tilde{A} \\ \tilde{E} \\ D_{qp} \end{bmatrix}, \Delta(t) \quad (16) \]

This means that (13), (14), (15), (10d) are assumed to be equivalent to:

\[ \dot{\mu}(t) = \hat{A}_\Delta \mu(t) + \hat{B}r(t) \quad (17a) \]
\[ e_z(t) = \hat{C}\eta(t) \quad (17b) \]

We would like to find the state-space filter matrices \( A_f, B_f \) and \( L_f \) that minimise, in a \( H_\infty \) sense, the \( L_2 \) gain from \( r \) to \( e_z \).

**Theorem 1:** For the LDI (17) and a given \( \sigma > 0 \) and assuming the energy of the input is such that the bound on the endogenous part of \( \theta(t) \) is valid, minimise \( \beta^2 \) subject to the LMIs \( P_0 > 0, P_1 - P_0 > 0 \) and (18) (below), then a robust LTI filter is given by state-space matrices \( A_f = P_3^{-1} M_A P_3^{-1}, B_f = P_3^{-1} M_B, L_f = M_L P_3^{-1} \) where \( P_3 = P_0/\beta^2 \) and \( \beta \) is an upper bound on the \( L_2 \) gain from input \( r \) to estimation error \( e_z \). Defining \( P = \begin{bmatrix} P_1 & P_3 \\ * & I_n \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, V(\mu) = \mu^T P \mu \) is a Lyapunov function that proves it. Also, the LDI is well-posed. The variables in (18) are \( P_1, P_0, M_A \in \mathbb{R}^{n \times n}, M_B \in \mathbb{R}^{n \times n_p}, M_L \in \mathbb{R}^{n_x \times n} \) and \( \beta \in \mathbb{R} \).

\[ X := \begin{bmatrix} X_1 & X_3 \\ * & X_2 \end{bmatrix} < 0 \quad (18) \]

where

\[ X_1 := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} \]

\[ \Psi_{11} = P_1 A + M_B C_q + A^T P_1 + C_q^T M_B^T + C_q^T S C_q \]
\[ \Psi_{12} = M_A + A^T P_0 + C_q^T M_B^T \]
\[ \Psi_{13} = P_1 B_p + M_B D_{qp} + C_q^T S D_{qp} + C_q^T G \]
\[ \Psi_{22} = M_A + M_B^T, \quad \Psi_{23} = P_0 B_p + M_B D_{qp} \]
\[ \Psi_{33} = D_{qp}^T S D_{qp} + D_{qp}^T G - G D_{qp} - \sigma^2 S \]

\[ X_2 := \begin{bmatrix} -\beta^2 I_{n_r} \quad 0 \\ 0 \quad -I_{n_x} \end{bmatrix} \]

\[ X_3 := \begin{bmatrix} P_1 B_r + M_B D_{yr} & L^T \\ P_0 B_r + M_B D_{yp} & -M_L^T \end{bmatrix} < 0, \forall \Delta(t) \in \Delta \quad (19) \]

Then the induced \( L_2 \) gain from reference \( r \) to estimation error \( e_z \) for the augmented system (17) is less than \( \beta \) for all permitted values of \( \Delta \).

By Schur complement [4], (19) is equivalent to \(-\beta^2 I < 0\) (which is obvious), together with

\[ T_1 + T_2 \Delta(I - T_4 \Delta)^{-1} T_3 + T_3^T(I - T_4 \Delta)^{-T} T_2 T_3^T T_2 < 0 \quad (20) \]

where

\[ T_1 := \tilde{A}_\Delta P + P \tilde{A}_\Delta + \tilde{C}^T \tilde{C} + \frac{1}{\beta^2} P \tilde{B} \tilde{B}^T P \]
\[ T_2 := P \tilde{L}, \quad T_3 := \tilde{E}, \quad T_4 := D_{qp} \quad (21) \]

We now associate with \( \Delta \) the subspaces of block-diagonal scaling matrices (representing relationships for real, structured uncertainty):

\[ \Sigma := \{ \text{diag}(S_1, \ldots, S_l) : 0 < S_i \in \mathbb{R}^{k_i \times k_i} \} \subset \mathbb{R}^{n_p \times n_p} \]
\[ \Gamma := \{ \text{diag}(G_1, \ldots, G_l) : G_i = -G_i^T \in \mathbb{R}^{k_i \times k_i} \} \subset \mathbb{R}^{n_p \times n_p} \]

Then [8], [9] (20) holds and is well-posed \( \forall \Delta(t) \in \Delta \), i.e.

\[ \det(I - T_4 \Delta) \neq 0, \]
This is a sufficient condition and therefore introduces conservatism.

The augmented system matrices, as defined in (13), (14) and (15), are substituted into (22), with \( P \) partitioned as

\[
\begin{bmatrix}
  P_1 & P_3 \\
  * & P_2 
\end{bmatrix}
\]

The filter matrices are not fixed and therefore, without loss of generality [6], we can assume that \( P_2 = I_n \). This results in a nonaffine matrix inequality \( Z < 0 \), with variables \( P_1, P_3, S, G, A_f, B_f, L_f \) and \( \beta \). The goal is to find the filter matrices that minimise \( \beta \), subject to \( Z < 0 \), \( P > 0 \), \( \beta > 0 \), \( S \in S \) and \( G \in G \).

Making \( Z \) affine requires a nonlinear change of variables and the use of Schur complement. First, define \( P_0 := P_3P_3^{-1} \), so \( P_3 = P_0^{1/2} \). By Schur complement on \( P \), we require \( P_0 > 0 \) and \( P_0 - P_3 > 0 \). Next, we define the new variables \( M_A := P_3A_fP_{3}^{-1}, M_B := P_3B_f \) and \( M_L := L_fP_3^{-1} \). Then, we pre and post multiply \( Z \) by \( J^T \) and \( J \) respectively, where \( J := \text{diag}(I_n, P_3^T, I_{n_p}) \). This results in the new matrix inequality

\[
\tilde{Z} := J^T Z J = \begin{bmatrix}
  Z_{11} & Z_{12} & Z_{13} \\
  * & Z_{22} & Z_{23} \\
  * & * & Z_{33}
\end{bmatrix} < 0 \quad (23)
\]

where

\[
\begin{align*}
Z_{11} &= P_1 A + M_B C_y + A^T P_1 + C_y^T M_B^T + L^T L + \ldots \\
&\quad + C_q^T S C_q + \beta^{-2} (P_1 B_r + M_B D_{yr})(B_r^T P_1 + D_{yr}^T M_B^T) \\
Z_{12} &= M_A + A^T P_0 + C_y^T M_B^T - L^T M_L + \ldots \\
&\quad + \beta^{-2} (P_0 B_r + M_B D_{yr})(B_r^T P_0 + D_{yr}^T M_B^T) \\
Z_{13} &= P_1 B_p + M_B D_{yp} + C_q^T S D_{yp} + C_q^T G \\
Z_{22} &= M_A + M_B^T + M_L^T M_L + \ldots \\
&\quad + \beta^{-2} (P_0 B_r + M_B D_{yr})(B_r^T P_0 + D_{yr}^T M_B^T) \\
Z_{23} &= P_1 B_p + M_B D_{yp} \quad \text{and} \quad Z_{33} = \Psi_{33}
\end{align*}
\]

It can be seen that \( \tilde{Z} \) is still not affine in the variables, so we use Schur complement. We have \( \tilde{Z} = X_1 - X_3 X_2^{-1} X_3^T < 0 \), with \( X_1, X_2 \) and \( X_3 \) defined as in (18) and \( X_2 < 0 \), hence by Schur complement \( \tilde{Z} < 0 \) is equivalent to \( X_1 < X_2 < 0 \).

\section{Robust \( L_2 \) Gain}

In this section, we present a method for robust stability analysis, by solving a system of LMIs. The proof is straightforward to derive from the literature [4], [9], [7]. In Section V, we will apply this analysis to the closed-loop system formed by the plant, controller, actuator and filter.

The system under consideration is again an LDI, similarly to the case for filter synthesis, with \( \Delta, S \) and \( G \) defined analogously. The external input is \( r \in \mathbb{R}^{n_r} \), output for performance analysis is \( e \in \mathbb{R}^{n_p} \) and states \( x \in \mathbb{R}^n \):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_r r(t) + B_p p(t) \\
e(t) &= C_q x(t) + D_{er} r(t) + D_{ep} p(t) \\
q(t) &= C_q x(t) + D_{qr} r(t) + D_{qp} p(t) \\
p(t) &= \Delta(t) q(t)
\end{align*}
\]

where \( q, p \in \mathbb{R}^{n_p} \).

\textbf{Theorem 2 :} For a given \( \sigma > 0 \) and assuming the energy of the input is such that the bound on the endogenous part of \( \theta(t) \) is valid, if \( \exists P > 0, S \in S, G \in G \) and \( \gamma > 0 \), such that LMI (26) holds, then the LFT system (25) has a finite \( L_2 \) gain from input \( r \) to output \( e \), with upper bound \( \gamma, \forall \Delta \in \Delta \). Moreover, the LDI is well-posed, i.e. \( \det(I - D_{qp} \Delta(t)) \neq 0 \).

\[
\Pi := \begin{bmatrix}
  \Pi_{11} & \Pi_{12} & \Pi_{13} \\
  * & \Pi_{22} & \Pi_{23} \\
  * & * & \Pi_{33}
\end{bmatrix} < 0 \quad (26)
\]

where

\[
\begin{align*}
\Pi_{11} &= PA + A^T P + C_e^T C_e + C_q^T S C_q \\
\Pi_{12} &= PB_r + C_e^T D_{er} + C_q^T S D_{qr} \\
\Pi_{13} &= PB_p + C_e^T D_{ep} + C_q^T S D_{qp} + C_q^T G \\
\Pi_{22} &= D_{er}^T D_{er} - \gamma^2 I_{n_r} + D_{qr}^T S D_{qr} \\
\Pi_{23} &= D_{er}^T D_{ep} + D_{qr}^T S D_{qp} + D_{qr}^T G \\
\Pi_{33} &= D_{ep}^T D_{ep} + D_{qp}^T S D_{qp} - \sigma^2 S + D_{qp}^T G - G D_{qp}
\end{align*}
\]

Given that we only have a sufficient condition for stability, we may find that the LMIs are not feasible with \( \sigma = 1 \). This means that we cannot find, by this method, a single quadratic Lyapunov function that guarantees stability over all allowed values of \( \Delta(t) \). By increasing \( \sigma \) iteratively, we may be able to satisfy the LMIs, at the cost of reducing the bound on \( \Delta \).

\section{Simulation & Results}

\textbf{Filter :} We aim for a filter that will give an estimate of \( \alpha \), using pitch rate \( q \) as the measured output from the plant. The controlled plant with actuator can be written in quasi- LPV form, treating the reference demand as the external input to the system, \( r := [y_e \ y_e]^T \) and measured output \( y_m := q \). In order to do this, we define \( [\alpha(t)] \in [0, 0.349][\text{rad}] \), \( C_m = d_1(t) C_m(\alpha(t), M(t)) + d_2(t) m_0 u(t) \), with \( d_1 \) and \( d_2 \in [0.75, 1.25] \). Then \( x := [\alpha \ q \ u \ \dot{u} \ e_q]^T \) and
\( \theta(t) = [|\alpha| \ M \ d_1 \ d_2]^T \). We create the LFT using the free MATLAB toolbox [14] and the LFT is normalised such that \( \sigma = 1 \iff ||\Delta|| = 1 \). We obtain an LFT with \( \Delta(t) \in \mathbb{R}^{17 \times 17} \): 4 in \( \delta_\alpha(t) \), 11 in \( \delta_\gamma(t) \), 1 in \( \delta d_1(t) \) and 1 in \( \delta d_2(t) \), where these are the normalised variations in the elements of \( \theta(t) \) about their nominal values.

Theorem 1 is used to obtain the filter state-space matrices. We find that the LMIs are not feasible with \( \sigma^2 = 1 \), which means we cannot find a filter that guarantees a robust \( L_2 \) gain from \( r \) to \( e_z \) for all allowed values of \( \theta(t) \). We therefore increase \( \sigma^2 \) iteratively, until we find the smallest value for which the LMIs are feasible. The smallest value is \( \sigma^2 = 1.36 \), which implies \( ||\Delta|| = 0.86 \), with a corresponding decrease in the range of \( \theta(t) \) over which robust stability is guaranteed. The filter matrices obtained are:

\[
A_f = \begin{bmatrix}
-8245 & 3127 & 1492 & -155.5 & 43700 \\
4003 & -1692 & -797.0 & 58.93 & -21200 \\
1190 & -429.6 & -313.9 & 85.77 & -6308 \\
182.3 & -126.5 & -51.79 & -18.87 & -960.1 \\
43590 & -16500 & -7873 & 821.6 & -231100
\end{bmatrix}
\]

\[
B_f = \begin{bmatrix}
-35150 & 17050 & 5074 & 773.4 & 185900 \\
9901 & -0.2175 & 0.2883 & 0.1374 & 0.1906
\end{bmatrix}^T
\]

**Robust Stability & Performance:** Results for the \( L_2 \) performance analysis using Theorem 2 are given in Table I. The output for performance analysis is \( e := \eta - \eta_c \). We minimise \( \gamma \), for a given value of \( \sigma^2 \). As with the filter synthesis, this is a linear objective with LMI constraints and is solved using the MATLAB toolbox [3]. Our goal is to minimise \( \sigma^2 \) whilst still being able to find a finite \( L_2 \) gain (much like the approach in [11] for robust stability analysis). We find that the LMIs are not feasible with \( \sigma^2 = 1 \). The smallest value for which we can obtain a result is \( \sigma^2 = 1.42 \), which implies a robust \( L_2 \) gain, not over all allowed values of \( \theta(t) \), but for \( M \in [1.62, 2.88], |\alpha| \in [1.61, 18.4^\circ] \) and \( d_1, d_2 \) represent \( \pm 21\% \) on the \( \alpha \) and \( u \) dependent parts of \( C_m \), respectively. We obtain \( \gamma = 198 \), which of course does not guarantee good robust performance in tracking \( \eta_c \). By increasing \( \sigma^2 \) we can obtain better values of \( \gamma \), at the cost of further reducing the region over which we can give a robustness guarantee. We note that increasing \( \sigma^2 \) beyond about 10 does not reduce \( \gamma \) significantly.

**TABLE I**

| \( \sigma^2 \) | \( ||\Delta|| \) | \( \gamma \) |
|---|---|---|
| 1.42 | 0.84 | 198 |
| 2 | 0.71 | 4.57 |
| 10 | 0.32 | 1.77 |
| 100 | 0.1 | 1.47 |
| 1000 | 0.052 | 1.42 |

**Nonlinear simulation:** The simulation results are given in figs. 2 to 5 for a series of constant step demands in \( \eta_c \). This is the same series of steps carried out in [15] and [1]. There are four simulations shown: case 1 is the nominal model without filter; cases 2 to 4 include the filter. All simulations were done at \( M = 3 \) (constant) and included the actuator. Case 2 is the nominal model with the filter (no uncertainty on \( C_m \), i.e. \( d_1 = d_2 = 1 \)). Cases 3 and 4 include independent time-varying uncertainties \( d_1(t) \) and \( d_2(t) \). For case 3 \( d_1 = 1+0.25\sin(2\pi t/4.5+0.1), d_2 = 1+0.25\sin(2\pi t/4.5+0.2) \). For case 4 \( d_1 = 1+0.25\sin(2\pi t/4.5+0.3), d_2 = 1+0.25\sin(2\pi t/9 - 0.8) \). These were chosen such that for case 3 the initial values of the uncertainties are small, whereas for case 4 the initial values are larger. It can be seen that for case 4, the performance for the initial 30g step is significantly poorer.

The approximation made in the slow subsystem control synthesis, produces a steady-state error, which is quite noticeable for the larger step commands. This error essentially comes from the fact that the slow subsystem command for \( q_c \) is not the correct value in order to achieve \( \dot{\eta} = -k_e \eta_c \). This means that there is an equilibrium \( \dot{\eta}_c = 0 \) when \( \eta_c \neq 0 \). This error is present even for the nominal case and is given by (for constant \( \text{Mach} \)):

\[
e_{\eta,ss} = \frac{K_1 K_3 M^3 z_0}{k_1 m_0} \frac{\partial C_e}{\partial \alpha} (|\alpha_{ss}|, M) \tilde{C}_m(\alpha_{ss}, M)
\]

**VI. CONCLUSIONS**

The benefit of the methods presented here is that they require similar LFT system representations for the robust
filter synthesis and for the robust performance analysis. We have shown that these methods can be applied successfully to a nonlinear, uncertain system with a simple feedback linearisation controller. Also, we have found that the filter performs well in all simulations, even in the presence of time-varying uncertainty.

Future work will focus on 1) design of robust observers for better state estimation (the filter does not calculate the residual in the output) and 2) reducing conservatism in the analysis of the overall system, by IQC or sum-of-squares techniques.

REFERENCES


TABLE II

| Physical Data |
|---|---|
| P0 | 46602 Pa (static pressure at 20,000 ft) |
| v | 315.9 m/s (speed of sound at 20,000 ft) |
| S | 0.04 m² (surface area) |
| m | 204.02 kg (mass) |
| d | 0.23 m (diameter) |
| Iy | 247.44 kg·m² (pitch moment of inertia) |
| K1 | 0.7P0Slv/m |
| K2 | 0.7P0Slv/m |
| K3 | 0.7P0Slv/m |
| z3 | 19.3470 rad³ |
| z2 | -31.0084 rad² |
| z1 | -9.7174 rad¹ |
| z0 | -1.9481 rad⁻¹ |
| m3 | -40.4847 rad⁻³ |
| m2 | -64.1657 rad⁻² |
| m1 | 2.9221 rad⁻¹ |
| m0 | -11.8029 rad⁻¹ |
| ξ0 | 0.7 (actuator damping ratio) |
| ωn | 150 rad/s (actuator undamped natural frequency) |