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On Fourier transform, Parseval equality, and the inversion formula in idempotent analysis

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***Abstract*— In the paper we revisit some known remarkable formulas and their idempotent versions highlighting the role of functional spaces. More precisely, we focus on the space where the analogue of the Fourier transform takes place, the Legendre transform in $\mathbb{R}_{\min}$ and the concave conjugate transform in $\mathbb{R}_{\max}$, then we give the analogues of some classical theorems.**

I. Introduction

The Legendre transform takes a privilegiate role in the optimization theory. It transforms a function into a function of different variable, using maximization as the transformation procedure. This transform is closely related to the idempotent analogues of the Fourier transform, (see [8], [7] and reference therein). In the paper we revisit some known formulas, and we are going to write idempotent versions of some remarkable formulas related to the Fourier transform. We highlight the difference between the Fourier transform in $\mathbb{R}_{\min}$ and the Fourier transform in $\mathbb{R}_{\max}$, which are closely related to the Legendre transform and the concave conjugate transform, respectively. Thought the analogues of the Fourier transform can be developed in a more general space [8], in ths paper we focus on the spaces of concave upper semicontinuos functions *Conc*$(\mathbb{R}^N, \mathbb{R}_{\max})$ and to the spaces of convex lower semicontinuos functions *Conv*$(\mathbb{R}^N, \mathbb{R}_{\min})$. The different role of the space leads to give a different definition of Fourier transform in $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$, and to give relevance to the sign highlighting the analogues with the Legendre transform $\mathbb{R}_{\min}$ and the concave conjugate transform in $\mathbb{R}_{\max}$.

If $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty, +\infty\}$ the Legendre transform is defined as

$$f^\star(\xi) = \sup_{x \in \mathbb{R}^N} \{x\xi - f(x)\},$$

and the concave conjugate of f

$$f_\star(p) = \inf_{y \in \mathbb{R}^N} [py - f(y)].$$

It is well known that Legendre transform of the Legendre transform of a function f is the largest lower semi-continuous convex function such that $(f^\star)^\star \leq f$, and the concave biconjugate of f is the smallest upper semi-continuous concave function such that $(f_\star)_\star \geq f$. Here we revise some basic facts on Fourier transform, the analogues in the context of idempotent analysis, and we give some semplified proof of known theorems, and we open the discussion to generalization and related research problems. The functions we are discussing take their values in an idempotent semiring.

We briefly recall the definition of idempotent semiring.

Generally a set S endowed by two algebric operations $\oplus$ (addition) and $\odot$ (multiplication) is a semiring if

- the addition $\oplus$ and the multiplication $\odot$ are associative,
- the addition $\oplus$ is commutative,
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$.
- Exists the unity $\mathbf{1}$, that is an element belonging to S such that

$$x \odot \mathbf{1} = \mathbf{1} \odot x = x, , \quad \forall x \in S$$

- Exists $0 \in S$ such that

$$x \oplus \mathbf{0} = \mathbf{0} \oplus x = x, \quad \forall x \in S.$$

A semiring S is an idempotent semiring if

$$x \oplus x = x, \quad \forall x \in S.$$

As special case, here will consider the semiring $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ with the operations

$$\oplus := \min, \qquad \odot := +$$

where

$$\mathbf{0} = +\infty, \quad \mathbf{1} = 0.$$

In $\mathbb{R}_{\min}$ the idempotent analogues of integration on $\mathbb{R}^N$ is defined by the formula

$$I(\phi) = \int_{\mathbb{R}^N}^{\oplus} \phi(x) dx = \inf_{x \in \mathbb{R}^N} \phi(x),$$

for any function $\in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\min})$, that is the set of functions defined in $\mathbb{R}^N$ lower bounded in $\mathbb{R}$.

We will also consider the semiring $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with the operations

$$\oplus := \max, \qquad \odot := +$$

where

$$\mathbf{0} = -\infty, \quad \mathbf{1} = 0.$$

Here $\mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\max})$ is the set of functions which are bounded in $\mathbb{R}_{\max}$, that is the set of functions defined in $\mathbb{R}^N$ upper bounded in $\mathbb{R}$.

The idempotent analogue of integration on $\mathbb{R}^N$ is defined by the formula

$$I(\phi) = \int_{\mathbb{R}^N}^{\oplus} \phi(x)dx = \sup_{x\in\mathbb{R}^N} \phi(x),$$

for any function $\in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\max})$.
In both cases the invariance of the integral with respect to translation holds true

$$\int_{\mathbb{R}^N}^{\oplus} \phi(x+t)dx = \int_{\mathbb{R}^N}^{\oplus} \phi(y)dy.$$

The Fourier transform of a function $\phi \in L^1(\mathbb{R}^N)$, is given by

$$\mathcal{F}\phi(\xi_1, \ldots, \xi_N) =$$

$$\int_{\mathbb{R}^N} \phi(x_1, \ldots, x_N) e^{-2\pi i(x_1\xi_1 + \cdots + x_N\xi_N)} dx_1 \ldots dx_N$$

or, shortly,

$$\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^N} \phi(x)e^{-2\pi i x\cdot\xi}dx,$$

and the inverse transform of $\mathcal{F}$ (if $\psi = \mathcal{F}\phi \in L^1(\mathbb{R}^N)$) is

$$\mathcal{F}^{-1}\psi(\xi) = \int_{\mathbb{R}^N} \psi(x)e^{2\pi i x\cdot\xi}dx.$$

Notice that

$$\mathcal{F}\phi(-\xi) = \mathcal{F}^{-1}\phi(\xi),$$

stating a duality between the Fourier transform and its inverse. Moreover the inversion formula states

$$(\mathcal{F}^{-1})^{-1}\phi(\xi) = \phi(-\xi).$$

In the paper we mainly deal with idempotent aspects of the well known formula on the Fourier transform of the convolution product.
More precisely if

$$f = f_1 * f_2,$$

is the convolution product in $L^1(\mathbb{R}^N)$ of f_1 and f_2, then

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2).$$

For basic notions and relationships in idempotent analysis see [3], [4], [5], [6], [8], [7], [9], and reference therein. For the results and detailed analysis on convex functions we refer to [1], [2]

II. The convolution product

To illustrate the remarkable analogues between classical analysis, let us recall the idempotent analogue of the property that the Fourier transform of the convolution product of two functions is the product of their transforms.

Theorem 1. Let ψ and $\phi \in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\min})$ [or ψ and $\phi \in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\max})$] be given. We denote by $\phi \otimes \psi$ the convolution product, and by $\hat{f}$ the Fourier transform of f. Then

$$\left[\widehat{(\phi \otimes \psi)}\right](\xi) = \hat{\phi}(\xi) \odot \hat{\psi}(\xi).$$

The definition of convolution product and of Fourier transform are different in $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$ These definitions and the proof of Theorem 1 will be given in the next subsections.

A. *The formula in* $\mathbb{R}_{\min}$

Definition 1. The Fourier transform of a function $\phi \in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\min})$ is defined as

$$\hat{\phi}(\xi) = \sup_{x\in\mathbb{R}^N} [\xi x - \phi(x)].$$

Observe that

$$\sup_{x\in\mathbb{R}^N} [\xi x - \phi(x)] = -\int_{\mathbb{R}^N}^{\oplus} -\xi x \odot \phi(x)dx.$$

In the semiring $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ the definition of the convolution product between two real valued functions ϕ and $\psi \in \mathcal{B}(\mathbb{R}^N, \mathbb{R}_{\min})$ is given by the formula

$$(\phi \otimes \psi)(y) = \int_{\mathbb{R}^N}^{\oplus} \phi(x) \odot \psi(y-x)dx$$

and the idempotent integral, by definition, is given by

$$\int_{\mathbb{R}^N}^{\oplus} \phi(x) \odot \psi(y-x)dx = \inf_{x\in\mathbb{R}^N} [\phi(x) + \psi(y-x)].$$

To prove the Theorem 1 we first show the following
Lemma 1.

$$\sup_{y\in\mathbb{R}^N} \left[\xi(y-x) - \psi(y-x)\right] = \widehat{\psi}(\xi)$$

Proof of the Lemma 1

The proof is a direct consequence of the invariance of the idempotent measure. Indeed

$$\sup_{y\in\mathbb{R}^N} \left[\xi(y-x) - \psi(y-x)\right] =$$

$$-\inf_{y\in\mathbb{R}^N} \left[-\xi(y-x) + \psi(y-x)\right] =$$

$$= -\inf_{t\in\mathbb{R}^N} \left[-\xi t + \psi(t)\right] =$$

$$= \sup_{t\in\mathbb{R}^N} \left[\xi t - \psi(t)\right] = \widehat{\psi}(\xi).$$

Proof of the Theorem 1 related to $\mathbb{R}_{\min}$**.** To show the property we compute

$$\left[\widehat{(\phi \otimes \psi)}\right].$$

By the definition, this is equal to

$$\sup_{y\in\mathbb{R}^N} \big[\xi y - \big[(\phi\otimes\psi)(y)\big]\big]$$

Recalling the definition of $(\phi\otimes\psi)$ this is equal to

$$\sup_{y\in\mathbb{R}^N} \big[\xi y - \inf_{x\in\mathbb{R}^N} \big[\phi(x)+\psi(y-x)\big]\big]$$

Changing the sup to the inf, this is equal to the

$$\sup_{y\in\mathbb{R}^N} \big[\xi y + \sup_{x\in\mathbb{R}^N} \big[-\phi(x)-\psi(y-x)\big]\big] =$$

$$\sup_{y\in\mathbb{R}^N} \sup_{x\in\mathbb{R}^N} \big[\xi y - \phi(x)-\psi(y-x)\big]$$

Changing the sup respect to y to the sup with respect to x this is equal to

$$\sup_{x\in\mathbb{R}^N} \sup_{y\in\mathbb{R}^N} \big[\xi x - \phi(x)+\xi(y-x)-\psi(y-x)\big] =$$

$$\sup_{x\in\mathbb{R}^N} \big[\xi x - \phi(x)+ \sup_{y\in\mathbb{R}^N} [\xi(y-x)-\psi(y-x)]\big]$$

Applying the Lemma 1 we have

$$= \sup_{x\in\mathbb{R}^N} \big[\xi x - \phi(x)+\hat{\psi}(\xi)\big] =$$

$$\widehat{\phi}(\xi)\odot\widehat{\psi}(\xi),$$

Then

$$\big[\widehat{(\phi\otimes\psi)}\big](\xi) = \widehat{\phi}(\xi)\odot\widehat{\psi}(\xi).$$

This ends the proof of the first part of the theorem.
We are going to establish the formula in $\mathbb{R}_{\max}$. As we will see we change the definition of the Fourier transform in $\mathbb{R}_{\max}$ with respect to $\mathbb{R}_{\min}$. Also we repeat the proof, since we need to use the operations defined in $\mathbb{R}_{\max}$.

B. The formula in $\mathbb{R}_{\max}$

In literature (see for example the reference [7]) the Fourier transform is given by the tranform

$$\hat{\phi}(\xi) = \int^{\oplus}_{\mathbb{R}^N} \xi x\odot\phi(x)dx = \sup_{x\in\mathbb{R}^N} [\phi(x)+\xi x]$$

Here we slightly change the definition of the Fourier transform.

Definition 2. The Fourier transform of a function $\phi\in\mathcal{B}(\mathbb{R}^N,\mathbb{R}_{\max})$ is defined as

$$\hat{\phi}(\xi) = \inf_{x\in\mathbb{R}^N} [\xi x - \phi(x)],$$

that is the concave conjugate tranform of the function ϕ. Notice that

$$\inf_{x\in\mathbb{R}^N} [\xi x - \phi(x)] = -\int^{\oplus}_{\mathbb{R}^N} \phi(x)\odot(-\xi x)dx.$$

Proof of the Theorem 1 related to $\mathbb{R}_{\max}$.

In the semiring $\mathbb{R}_{\max} = \mathbb{R}\cup\{-\infty\}$ we recall the definition of the convolution product between two real valued functions ϕ and ψ on the additive group $\mathbb{R}^N$, by the formula

$$(\phi\otimes\psi)(y) = \int^{\oplus}_{\mathbb{R}^N} \phi(x)\odot\psi(y-x)dx$$

By the definition of idempotent integral

$$\int^{\oplus}_{\mathbb{R}^N} \phi(x)\odot\psi(y-x)dx = \sup_{x\in\mathbb{R}^N} [\phi(x)+\psi(y-x)].$$

We compute using our definition

$$\big[\widehat{(\phi\otimes\psi)}\big](\xi)$$

which is equal to

$$\inf_{y\in\mathbb{R}^N} \big[\xi y - \big[(\phi\otimes\psi)(y)\big]\big] =$$

$$\inf_{y\in\mathbb{R}^N} \big[\xi y + \inf_{x\in\mathbb{R}^N} \big[-\phi(x)-\psi(y-x)\big]\big] =$$

$$\inf_{y\in\mathbb{R}^N} \inf_{x\in\mathbb{R}^N} \big[\xi y - \phi(x)-\psi(y-x)\big] =$$

$$\inf_{x\in\mathbb{R}^N} \inf_{y\in\mathbb{R}^N} \big[\xi x - \phi(x)+\xi(y-x)-\psi(y-x)\big] =$$

$$\inf_{x\in\mathbb{R}^N} \big[\xi x - \phi(x)+ \inf_{y\in\mathbb{R}^N} [\xi(y-x)-\psi(y-x)]\big]$$

$$= \inf_{x\in\mathbb{R}^N} \big[\xi x - \phi(x)+\hat{\psi}(\xi)\big] = \hat{\phi}(\xi)\odot\hat{\psi}(\xi).$$

This ends the second part of the proof, and the proof of the theorem is complete.

III. Fenchel duality theorem, Parseval equality in $\mathbb{R}_{\min}$ and the inversion formula

Let us recall the Fenchel's duality theorem. Given f, $g:\mathbb{R}^N\to\mathbb{R}$ with f convex and g concave. Then, under regularity assumptions, we have

$$\min_{x\in\mathbb{R}^N} \big(f(x)-g(x)\big) = \max_{x\in\mathbb{R}^N} \big(g_\star(x)-f^\star(x)\big)$$

with

$$f^\star(p) = \max_{x\in\mathbb{R}^N} [px-f(x)]$$

$$g_\star(p) = \min_{x\in\mathbb{R}^N} [px-g(x)].$$

We sketch briefly here the main idea of the proof of the Fenchel's duality theorem. By assumption the function $f-g$ is convex and lower semicontinuous in $\mathbb{R}^N$. Then

$$(f(\cdot)-g(\cdot))^\star(0) = -\inf_{x\in\mathbb{R}^N} \big(f(x)-g(x)\big) =$$

$$-\min_{x\in\mathbb{R}^N} \big(f(x)-g(x)\big).$$

On the other hand

$$(f(\cdot)-g(\cdot))^\star(0) = \sup_{x\in\mathbb{R}^N} \big[x\cdot 0 - \big(f(x)-g(x)\big)\big] =$$

$$\sup_{x\in\mathbb{R}^N} \big[\big(xp-f(x)\big)-\big(xp-g(x)\big)\big] \quad \forall p\in\mathbb{R}^N.$$

For any $p \in \mathbb{R}^N$ it follows

$$\min_{x\in\mathbb{R}^N}\big(f(x)-g(x)\big)=\inf_{x\in\mathbb{R}^N}\big[\big(xp-g(x)\big)-\big(xp-f(x)\big)\big]$$

Then

$$\min_{x\in\mathbb{R}^N}\big(f(x)-g(x)\big)\geq g_\star(p)-f^\star(p)\quad\forall p\in\mathbb{R}^N.$$

Since $g_\star - f^\star$ is concave and upper semicontinuous it has at least a maximum point $p^\star$. The achievement of the equality in last formula is the main point of the proof.
For the complete proof we refer to [2] pag. 228. The Fenchel duality theorem, also kwown as the min-max theorem, gives a *certificate of optimality* in the following sense: if we wish to minimize $f(x)-g(x)$ and $x^\star$ is a possible minimizer then it is enough to find a vector $p^\star$ such that

$$\big(f(x^\star)-g(x^\star)\big)=\big(g_\star(p^\star)-f^\star(p^\star)\big).$$

The vector $p^\star$ (dual optimal solution) is the *certificate* of $x^\star$.

Setting

$$h(x)=-g(x)\quad\forall x\in\mathbb{R}^N,$$

then

$$h^\star(-p)=-g_\star(p).$$

Indeed

$$g_\star(p)=\min_{x\in\mathbb{R}^N}[px-g(x)]=\min_{x\in\mathbb{R}^N}[px+h(x)]=$$

$$-\max_{x\in\mathbb{R}^N}[-px-h(x)]=-\max_{x\in\mathbb{R}^N}\big((-p)x-h(x)\big)=$$

$$-h^\star(-p).$$

Then, substituting

$$\min_{x\in\mathbb{R}^N}\big(f(x)+h(x)\big)=-\min_{p\in\mathbb{R}^N}\big(h^\star(-p)+f^\star(p)\big).$$

The last equality is also known as *Fenchel duality theorem* (see [1]). We refer to [2], Corollary 2.2.6, Theorems 2.2.5 and 2.2.7 for a more general result.
Then using Legendre inverse formula

$$\min_{x\in\mathbb{R}^N}\big(f^\star(x)+h(x)\big)=-\min_{p\in\mathbb{R}^N}\big(f(p)+h^\star(-p)\big),$$

that we call Parseval relation in $\mathbb{R}_{\min}$ since can be written out

$$< f^\star\,|\,h >= -< f\,|\,h^\star((-))>,$$

and we can prove that

Theorem 2. *Let $f, h \in C^1(\mathbb{R}^N)$, real valued and convex functions. Then the Parseval relation in $\mathbb{R}_{\min}$ is equivalent to the inversion formula for the Legendre (Fourier) formula in $\mathbb{R}_{\min}$.*

Proof of the Theorem 2. If we assume inversion formula for the Legendre (Fourier) formula in $\mathbb{R}_{\min}$ then by Fenchel's duality formula

$$\min_{x\in\mathbb{R}^N}\big(f(x)-g(x)\big)=\max_{x\in\mathbb{R}^N}\big(g_\star(x)-f^\star(x)\big),$$

and the result follows setting, as before, $-g = h$ and $f^\star$ instead of f. Let us prove the inversion formula by the Parseval relation in $\mathbb{R}_{\min}$. We wish to show

$$(f^\star)^\star = f.$$

We fix

$$h(x)=\frac{1}{2}\epsilon^2|x|^2-x\xi,$$

with $\xi\in\mathbb{R}^n$ and $\epsilon>0$ fixed.
To fill the formula we compute

$$h^\star(p)=\sup_{x\in\mathbb{R}^N}[px-h(x)]=$$

$$h^\star(p)=\sup_{x\in\mathbb{R}^N}[px-\frac{1}{2}\epsilon^2|x|^2+x\xi]$$

$$=\sup_{x\in\mathbb{R}^N}[x(p+\xi)-\frac{1}{2}|\epsilon x|^2]=$$

$$\frac{1}{2\epsilon^2}|p+\xi|^2.$$

Substituting in

$$\inf_{x\in\mathbb{R}^N}\big(f^\star(x)+h(x)\big)=-\inf_{p\in\mathbb{R}^N}\big(f(p)+h^\star(-p)\big),$$

$$\inf_{x\in\mathbb{R}^N}\big(f^\star(x)+\frac{1}{2}\epsilon^2|x|^2-x\xi\big)=$$

$$-\inf_{p\in\mathbb{R}^N}\big(f(p)+\frac{1}{2\epsilon^2}|\xi-p|^2\big).$$

Passing to the sup

$$\sup_{x\in\mathbb{R}^N}\big(-f^\star(x)-\frac{1}{2}\epsilon^2|x|^2+x\xi\big)=$$

$$\inf_{p\in\mathbb{R}^N}\big(f(p)+\frac{1}{2}\frac{1}{\epsilon^2}|\xi-p|^2\big)=$$

$$\inf_{y\in\mathbb{R}^N}\big(f(\xi+\epsilon y)+\frac{1}{2}|y|^2\big)$$

And

$$\sup_{x\in\mathbb{R}^N}\big(-f^\star(x)-\frac{1}{2}\epsilon^2|x|^2+x\xi\big)=$$

$$\inf_{y\in\mathbb{R}^N}\big(f(\xi+\epsilon y)+\frac{1}{2}|y|^2\big)$$

To show the existence of the limit as as $\epsilon\to 0^+$, we firstly observe that the infimum of $f(\xi+\epsilon y)+\frac{1}{2}|y|^2$ is a minimum. Moreover the minimum point $\overline{y}$ satisfies the conditions

$$\epsilon(Df)(\xi+\epsilon\overline{y})+\overline{y}=0,$$

which implies in a neighbourhood of ξ

$$\overline{y}(\xi,\epsilon)=O(\epsilon),\ \text{locally}$$

then

$$\lim_{\epsilon\to 0^+}\inf_{y\in\mathbb{R}^N}\big(f(\xi+\epsilon y)+\frac{1}{2}|y|^2\big)=f(\xi).$$

Next we observe that by setting in the right hand side of Parseval relation $f^\star$ instead of f we get

$$\inf_{p\in\mathbb{R}^N}\big(f^\star(p)+\gamma^\star(-p)\big)=\sup_{x\in\mathbb{R}^N}\big(-(f^\star)^\star(x)-\gamma(x)\big)$$

for any $\gamma\in C^1(\mathbb{R}^N)$, real valued and convex. Here we take

$$\gamma(x)=\frac{1}{2\epsilon^2}|x-\xi|^2,$$

with the same ξ and ϵ of the function h selected before. With an easy computation we get

$$\gamma^\star(-p)=\frac{1}{2}\epsilon^2|p|^2-p\xi,$$

Then, applying the last result

$$\lim_{\epsilon\to 0^+}\sup_{x\in\mathbb{R}^N}\big(-f^\star(x)-\frac{1}{2}\epsilon^2|x|^2+x\xi\big)=$$

$$\lim_{\epsilon\to 0^+}\inf_{y\in\mathbb{R}^N}\big((f^\star)^\star(\xi+\epsilon y)+\frac{1}{2}|y|^2\big)=$$

$$=(f^\star)^\star(\xi)$$

and the result

$$f=(f^\star)^\star$$

follows by the equality of the two limits.

IV. Inverse formula in $\mathbb{R}_{\max}$ for the conjugate transform

In $\mathbb{R}_{\max}$ given two concave, upper semicontinuous functions h and g, not identically equal to $-\infty$, the Fenchel duality formula reads

$$\max_{x\in\mathbb{R}^N}\big(g(x)+h(x)\big)=-\max_{p\in\mathbb{R}^N}\big(h_\star(p)+g_\star(-p)\big).$$

and the Parseval formulas become

$$<h\,|\,g>=-<h_\star\,|\,g_\star(-(\cdot))>,$$

and

$$<h_\star\,|\,g>=-<(h_\star)_\star\,|\,g_\star(\cdot)>.$$

Notice that

$$(h_\star)_\star(\xi)=\inf_{x\in\mathbb{R}^N}\big(x\xi-h_\star(x)\big)=$$

By the definition of $h_\star$ we have that this equal to

$$\inf_{x\in\mathbb{R}^N}\big(x\xi-\inf_{y\in\mathbb{R}^N}\big(xy-h(y)\big)\big)=$$

Changing the inf to the sup

$$-\sup_{x\in\mathbb{R}^N}\big(x(-\xi)-\sup_{x\in\mathbb{R}^N}\big(h(y)-xy\big)\big)=$$

$$-\sup_{x\in\mathbb{R}^N}\big(x(-\xi)-\sup_{y\in\mathbb{R}^N}\big(-(-h(y))+y(-x)\big)\big)=$$

$$-\sup_{x\in\mathbb{R}^N}\big(x(-\xi)-(-h)^\star(-x)\big)=$$

$$-\sup_{z\in\mathbb{R}^N}\big(\xi(-x)-(-h)^\star(-x)\big)=h(\xi).$$

Hence

$$(h_\star)_\star=h,$$

since we apply the the inverse Legendre formula to the convex function $-h$.

The inverse formula for the conjugate transform of concave functions plays the same role as the Legendre transform for convex functions. The analogue of Theorem 2 in $\mathbb{R}_{\max}$ maybe proved by the same tools used in $\mathbb{R}_{\min}$.

V. Conclusion and Future work direction

We discussed the role of functional spaces where the analogue will take place, highlighting the difference between $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$, based on the definition of Fourier transform used in the computation. In literature of idempotent analysis the definition of Fourier transform of function ϕ is given by

$$\hat{\phi}(\xi)=\int_{\mathbb{R}^N}^{\oplus}\xi x\odot\phi(x)dx=\sup_{x\in\mathbb{R}^N}[\xi x+\phi(x)]$$

Despite the difference of a sign, the transform does not enjoy the peculiar aspects of the Legendre transform or of the concave conjugate. This suggests a further analysis on the spaces where the correspondence take places.

The Legendre transform, as well-known, has the property to change any function for which the transform makes sense in a convex function, as the sup of affine functions. This remark leads us to define the Fourier transform in $\mathbb{R}_{\max}$ as

$$\hat{\phi}(\xi)=-\int_{\mathbb{R}^N}^{\oplus}(-\xi x)\odot\phi(x)dx=\inf_{x\in\mathbb{R}^N}[\xi x-\phi(x)].$$

As well known $\hat{\phi}(\xi)$ in $\mathbb{R}_{\max}$ has the property to transform a function (for which the transform makes sense) in a concave function, as the inf of affine functions.

In $\mathbb{R}_{\min}$ the idempotent Fourier transform is defined as the Legendre transform

$$\hat{\phi}(\xi)=-\int_{\mathbb{R}^N}^{\oplus}(-\xi x)\odot\phi(x)dx=\sup_{x\in\mathbb{R}^N}[\xi x-\phi(x)]$$

These remarks suggest us to study other analogues in $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$, to confirm the appropriate role of the definitions of the Fourier transform in $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$.

In an equivalent way we may define the Fourier transform in $\mathbb{R}_{\min}$, and $\mathbb{R}_{\max}$, as

$$\hat{\phi}(\xi)=-\int_{\mathbb{R}^N}^{\oplus}(-\xi x)\odot\phi(x)dx,$$

with a different meaning of the integrals.

Remark Provided ϕ and ψ are real valued convex and enough regular functions, in $\mathbb{R}_{\min}$ the analogue of the result of Theorem 1 gives us the formula

$$[\phi^\star+\psi^\star]^\star(\xi)=\inf_{x\in\mathbb{R}^N}[\phi(x)+\psi(\xi-x)]=$$

$$\inf_{x+\eta=\xi}[\phi(x)+\psi(\eta)].$$

Similar arguments apply to $\mathbb{R}_{\max}$. See [1] for references to these formulas.

A. *Future work direction.*

1. Fourier transform properties and its analogues in functional analysis, and Legendre transform and its analogues in harmonic analysis. Analysis of the properties of the two transfoms, some of them could be easier to compare while others requires a detailed study. We wish to point out that this note could open a discussion about this topic.

2. Study of the spaces. Looking at the role of Parseval equality in linear analysis, we may consider the extension of the Fourier transform to the dual of the space $\mathcal{S}(\mathbb{R}^N)$ (Schwartz spaces). Let $\mathcal{T} \in \mathcal{S}'(\mathbb{R}^N)$ represented by

$$\phi \in \mathcal{S}(\mathbb{R}^N) \to < \mathcal{T}, \phi > \in \mathbb{C},$$

then the Fourier transform of $\mathcal{T}$ is defined by

$$< \hat{T}|\phi) = < \mathcal{T}|\hat{\phi} > \quad \forall \phi \in \mathcal{S}(\mathbb{R}^N).$$

The research line is to find suitable classes of enough regular convex or concave functions corresponding, from the point of view of idempotent analysis, to $\mathcal{S}(\mathbb{R}^N)$. A subsequent point is the construction of fundamental solution of linear partial differential operator (PDO) with constant coefficients. To this regard, the interest is also jointed to the fact that some PDO non linear in the classical theory could have a linear analogue in idempotent analysis.

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