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Modeling and Predictive Control of Nonlinear Hybrid Systems Using Disaggregation of Variables – A Convex Formulation

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Abstract—The current work is motivated by the need of achieving global solution and better computational efficiency for control of any arbitrary nonlinear hybrid dynamical systems (NHDS). In this work, we present a novel modeling and corresponding model predictive control (MPC) formulation for NHDS. The proposed modeling approach relies on disaggregation of polynomials of binary variables that appear in the multiple partially linearized (MPL) model. In particular, we use auxiliary continuous variables and linear constraints to model these polynomials and represent the MPL model in a linear fashion. Subsequently, disaggregation of the variables based multiple models are used to formulate the MPC law for NHDS. The MPC formulation takes similar form as multiple mixed logical dynamical (MMLD) model based MPC and yields a *convex* MIQP optimization problem. Moreover, the proposed modeling approach results in a compact model than the corresponding MMLD model as it refrains from adding any extra binary variables. Therefore, offers certain computational advantage when used for the predictive control of NHDS. The efficacy of the proposed solution is demonstrated on a three-tank benchmark hybrid system.

I. Introduction

Many practical applications exhibit hybrid character, where continuous dynamics interact with discrete events. Such systems are known as hybrid systems [1]-[3]. Examples of hybrid systems are encountered in manufacturing industries, robotics, biological systems and in chemical process industries among others [3]. In particular, hybrid characters arise in chemical process industries due to various safety interlocks, grade transitions, on-off valves, etc. Although the use of hybrid systems framework for modeling and control of chemical processes is not very well-known among various process industries, increasing demand for high quality products and cut-throat competition calls for a flexible and more accurate control and scheduling solution that also incorporate various logical conditions. Moreover, nonlinearity makes model based application, such as model predictive control (MPC), far complex and practically intractable.

Significant efforts have been made by various researchers to simplify modeling of hybrid systems for various applications such as verification [4], control [1],[3], stability [5] and optimization [6]. These efforts led to various modeling formalisms, which include mixed logical dynamical (MLD) framework [1], linear complementarity (LC) modeling [7], piecewise affine (PWA) modeling formalism [8]. These formalisms consider linear continuous dynamics with discrete-decisions and logical conditions. In addition, it has been proven that all of these formalisms are equivalent and can be represented using the MLD framework [9]. The key advantage of MLD framework is that it can be used to formulate MPC problem. The MLD model based predictive control leads to a mixed integer quadratic/linear program (MIQP/MILP).

Thus, numerous literatures are available for linear hybrid dynamical systems. However, efforts towards nonlinear hybrid dynamical systems (NHDS) are limited. Wang et al. [10] have presented a robust state estimation and fault diagnosis approach for nonlinear hybrid systems with unknown mode transition functions, model uncertainty and unmeasured disturbances. A fuzzy modeling approach and corresponding MPC formulation have also been developed [11]. Nandola and Bhartiya [3] have proposed a multiple partially linearized (MPL) modeling and corresponding predictive control approach for nonlinear hybrid dynamical systems. The MPL based MPC results in a *nonconvex* mixed integer nonlinear program (MINLP) as compared to MIQP in case of multiple-MLD based approach [1]. However, it has been shown that the MPL approach is computationally superior than multiple MLD model based approach [3]. This is primarily because of the MLD model masks multiplicative terms by adding auxiliary binary variables, continuous variables as well as mixed integer linear constraints. Thus, significantly increases the size of the MPC problem. On the other hand, the MPL modeling retains nonlinear polynomial terms of binary variables and refrains from adding extra variables. In addition, the MPC using MPL model results in a canonical form, which has certain advantage. The authors have further discovered this canonical form [12]. In particular, they tailored a generalized outer approximation (GOA) algorithm, which is computationally superior. They have also validated its computational efficiency via

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experimentations on a lab-scale three-tank hybrid system setup [13]. However, GOA is for *convex* MINLP, hence it may results in a sub-optimal solution, in this case.

Thus, there is a tradeoff between computational efficiency and quality of the solution. The MLD approach, due to its *convex* nature, provides better solution at the expense of computational efficiency, while the MPL provides faster solution at the risk of sub-optimality. The current work is an attempt to obtain the optimal solution for the MPL based MPC problem. In order to achieve this, one may use global optimization algorithms of *nonconvex* MINLP [14]. Alternatively, one may consider various *convexification* approaches for MINLP [15]. In this work, we tailored *variables disaggregation* technique [16] to convert the MPL based MPC problem into a *convex* MIQP. In order to achieve this, we introduce additional linear constraints and continuous variables to the MPL model. This approach is expected to outperform multiple-MLD (MMLD) models based approach with respect to computational efficiency, while providing solution of the similar quality. This is primarily because MLD approach adds extra binary variables and associated linear constraints, to mask the polynomials of binary variables, in addition to the extra continuous variables and their associated constraints. On the other hand, the proposed approach only adds continuous variables. The advantage of the proposed modeling and control is demonstrated on a benchmark three-spherical tank system via a simulation case study.

The paper is organized as follows: Section II presents an overview of the MPL modeling and consequently develops the variables disaggregation based modeling approach for the NHDS. Model predictive control formulation using the proposed modeling is documented in Section III. Section IV demonstrate efficacy of the proposed modeling and control approach on a three-tank benchmark hybrid systems. A summary and conclusion is presented in Section V.

II. MODELING OF NONLINEAR HYBRID SYSTEMS

Since the proposed modeling approach is a significant improvement over earlier presented MPL modeling [3] of the nonlinear hybrid dynamical systems, in this section, we present an overview of the MPL modeling approach followed by the proposed variable disaggregation based modeling.

A. MPL modeling of hybrid systems-A brief overview

The MPL model is a simplified version of the hybrid state model (HSM) [17] of NHDS. The mathematical form of the HSM can be represented as follows [3]:

$$\dot{x}^c = f_g(x^c, u^c, \delta) \quad (1)$$

$$x^d(t) = b_d\big(\delta(t)\big) \quad (2)$$

$$E_1 u^c(t) + E_2\delta(t) + E_3 x(t) \le E_4 \quad (3)$$

$$x^c(t^+) = x^c(t^-) \quad (4)$$

where x^c represent continuous states, x^d stands for discrete states, u^c is continuous inputs, δ represent binary indicator variables, which capture discrete decisions due to the discontinuities in states (*i.e.* discrete states) and inputs (*i.e.* discrete inputs). The evolution of binary variables, δ, is governed by inequality constraints in (3). Moreover, a change in the status of the elements of δ, indicates a change in the location (mode) of the hybrid systems. Thus, (1)-(4) represents mathematical model for general nonlinear hybrid dynamical systems that can be used to formulate MPC. However, it requires numerous integrations of (1) as well as online solution of a complex and *nonconvex* MINLP optimization. Therefore, to simplify the problem, Nandola and Bhartiya [3] have adopted a multiple partial linearized (MPL) models based approach. In the reminder of this section, the MPL approach is briefly reviewed.

The MPL model begins with a Taylor series expansion of (1) around an operating point of continuous variables (x^c, u^c), while keeping binary variables, δ, constant as parameters. Thus, obtain a continuous-time linearized model as follows,

$$\dot{x}^c = A(\delta)x^c + B(\delta)u^c + f(\delta) \quad (5)$$

$$x^d(t) = b(\delta), \quad i = 1,2,\dots,2^{n_s} \quad (6)$$

$$E_1 u^c(t) + E_2\delta(t) + E_3 x(t) \le E_4 \quad (7)$$

Here it should be noted that the system matrices of (5) are function of binary variables δ and a fixed value of vector δ represents an unique location of the NHDS. Next, the continuous-time model (5) is discretized by substituting values of binary variables for all possible instantiations. As n_s binary variables result in 2^{n_s} instantiations, one can obtain 2^{n_s} discrete-time models as follows,

$$x_{k+1} = \Phi_i x_k + \Gamma_i u_k^c + f_{di}, \quad i = 1,2,\dots,2^{n_s} \quad (8)$$

where $x_k = \begin{bmatrix} x_k^c \\ x_k^d \end{bmatrix}$ is a state vector of the NHDS, Φ_i, Γ_i and f_{di} are discrete-time equivalent system matrices representing i^{th} location of the NHDS. The above models are then combined using logical multipliers, l_i, to obtain an unified discrete time representation of (5) - (7) as follows,

$$x_{k+1} = \left(\textstyle\sum_{i=1}^{2^{n_s}} l_{i,k}\Phi_i\right)x_k + \left(\textstyle\sum_{i=1}^{2^{n_s}} l_{i,k}\Gamma_i\right)u_k^c + \left(\textstyle\sum_{i=1}^{2^{n_s}} l_{i,k} f_{di}\right) \quad (9)$$

$$E_1 u_k^c + E_2\delta_k + E_3 x_k \le E_4 \quad (10)$$

Note that the above model is linear in continuous variables, while nonlinear in binary variables due to the fact that the logical multipliers are polynomials of binary variables. The logical multiplier, l_i, is defined such that it takes value 1 if and only if i^{th} combination of binary variables is encountered (i.e. i^{th} location of NHDS becomes active) and 0, otherwise. For example, the NHDS with two binary variables, *i.e.*, $\delta = [\delta_1 \quad \delta_2]^T$, result in four logical multipliers [3]: $l_1 = (1-\delta_1)(1-\delta_2)$, $l_2 = (1-\delta_1)\delta_2$, $l_3 = \delta_1(1-\delta_2)$ and $l_4 = \delta_1\delta_2$ corresponding to $\delta = [0 \quad 0]^T$, $\delta = [0 \quad 1]^T$, $\delta = [1 \quad 0]^T$ and $\delta = [1 \quad 1]^T$, respectively. Here it should be noted that (9) retain the polynomials of binary variables, δ, and $l_{i,k}$ are used only for the ease of representation; they should not be considered as extra variables.

The discrete-time linearized model, (9)-(10), can be represented in a compact form as follows,

$$x_{k+1} = (L_k\bar{\Phi})x_k + (L_k\bar{\Gamma})u_k^c + L_k\bar{f} \tag{11}$$

$$E_1u_k^c + E_2\delta_k + E_3x_k \leq E_4 \tag{12}$$

where L_k, $\bar{\Phi}$, $\bar{\Gamma}$ and $\bar{f}$ are constituted from $l_{i,k}$, Φ_i, Γ_i and f_{di}, respectively. The output of the linearized model may be written as:

$$y_k = Cx_k \tag{13}$$

Equations (11) - (13) represent final form of the linearized NHDS around an operating point. This model represents all locations of the NHDS in the neighborhood of a particular operating point. Similar linearized models can be obtained around n_l different operating points. These models are then combined using multiple model approach, such as Bayesian weighting [18], to reconstitute original nonlinear model. The resulting weighted model is referred to as the MPL model and can be represented as follows [3],

$$x_{k+1} = (L_k\bar{\Phi}_w)x_k + (L_k\bar{\Gamma}_w)u_k^c + L_k\bar{f}_w \tag{14}$$

$$E_1u_k^c + E_2\delta_k + E_3x_k \leq E_4 \tag{15}$$

$$y_k = Cx_k \tag{16}$$

where $\bar{\Phi}_w$, $\bar{\Gamma}_w$ and $\bar{f}_w$ are blended system matrices constituted by taking weighted average of n_l linearized models as follows,

$$*_w = \sum_{j=1}^{n_l} w_j *_j \tag{17}$$

where w_j represents weight of j^{th} linearized model, n_l is number of local multiple models, such as (11)-(13), and $*$ represent system matrices $\bar{\Phi}$, $\bar{\Gamma}$ and $\bar{f}$. These models describe overall operating range of the NHDS. The weights w_j are updated using Bayesian approach [18], at each sampling instances, based on the distance of the local model prediction from the current measurements.

As the MPL model (14)-(16) retain polynomial terms of binary variables, the MPC using MPL model turns out to be a *nonconvex* MINLP, hence, vulnerable to sub-optimality. Alternatively, following the approach presented by Colmenares et al. [19] , one may expand polynomials of binary variables in (9), followed by masking of the multiplicative terms between binary variables and the multiplicative terms between binary and continuous variables, using additional binary and continuous variables as well as their corresponding constraints, respectively. Thus, obtain a well-known discrete-time MLD model for the local operating regime. Similarly, all n_l linearized model can be converted into MLD models to obtain multiple-MLD (MMLD) model. An advantage of the multiple-MLD model is that, because of its linear structure, MLD based MPC leads to a MIQP optimization problem, which is a *convex* optimization problem. However, in the process of converting (9) into the MLD framework, it adds large number of binary variables, continuous variables and constraints; hence increase in the computational burden. On the other hand, it has been shown that the MPL model is computationally more efficient than the multiple-MLD based MPC, typically due to the smaller size of optimization problem. Thus, it is clear that one may want to have such an approach, which enables global optima as well as better computational efficiency. In order to achieve this objective, we present a modeling approach for NHDS based on disaggregation of variables, which is discussed next.

B. Modeling of NHDS using disaggregation of variables

This section presents modification in the MPL model that eliminates polynomials of binary variables. In particular, polynomials of binary variables in (9) are modeled using additional linear constraints and continuous variables as follows,

$$x_{k+1} = \sum_{i=1}^{2^{n_s}} \Phi_i x_{i,k} + \sum_{i=1}^{2^{n_s}} \Gamma_i u_{i,k} + \sum_{i=1}^{2^{n_s}} f_{di} z_{i,k} \tag{18}$$

$$x_L z_{i,k} \leq x_{i,k} \leq x_U z_{i,k} \tag{19}$$

$$u_L z_{i,k} \leq u_{i,k} \leq u_U z_{i,k} \tag{20}$$

$$\textstyle\sum_{i=1}^{2^{n_s}} z_{i,k} = 1 \tag{21}$$

$$\textstyle\sum_{i=1}^{2^{n_s}} x_{i,k} = x_k \tag{22}$$

$$\textstyle\sum_{i=1}^{2^{n_s}} u_{i,k} = u_k \tag{23}$$

where x_L, x_U and u_L, u_U are lower and upper bounds on states and inputs of the system, respectively and $z_{i,k}$ $(i = 1,2,\dots,2^{n_s})$ are non-negative auxiliary continuous variables. Each of these variables is determined using n_s linear constraints, where n_s is the number of original binary indicator variables. Thus, it adds $n_s(2^{n_s})$ linear constraints as given below,

$$\begin{array}{c} 1-\delta_{1,k} \geq z_{1,k}\,; \dots; 1-\delta_{n_s,k} \geq z_{1,k} \\ 1-\delta_{1,k} \geq z_{2,k}\,;\ 1-\delta_{2,k} \geq z_{2,k}\,; \dots; \delta_{n_s,k} \geq z_{2,k} \\ 1-\delta_{1,k} \geq z_{3,k}\,; \dots; \delta_{n_s-1,k} \geq z_{3,k};\ \delta_{n_s,k} \geq z_{3,k} \\ \vdots \\ \delta_{1,k} \geq z_{2^{n_s},k}\,; \dots;\ \delta_{n_s,k} \geq z_{2^{n_s},k} \end{array} \tag{24}$$

Above constraints are defined such that $z_{i,k}$ takes value 1 *if and only if* i^{th} combination of binary variables is encountered and 0, otherwise. Equations (18)-(24) are linear representation of (9), while (10) can be adopted without any change. Thus, at particular time instance, location of the hybrid system is determined by (10), (19)-(21) and (24).

The discrete-time linear model (18)-(24) and (10) can be represented in a compact form as follows,

$$x_{k+1} = \underline{\Phi} X_k + \underline{\Gamma} U_k + \underline{f} Z_k \quad (25)$$

$$E_{1a} U_k + E_{2a} \delta_k + E_{3a} X_k + E_{5a} Z_k \leq E_{4a} \quad (26)$$

$$E_{3a}^{eq} X_k = x_k \quad (27)$$

$$E_{1a}^{eq} U_k = u_k \quad (28)$$

where X_k, U_k and Z_k are constituted from $x_{i,k}$, $u_{i,k}$ and $z_{i,k}$, respectively, (26) is a compact representation of the constraints defined in (10),(19)-(21) and (24) while, (27) and (28) are compact representation of the equalities defined in (22) and (23). Matrices E_{1a}, E_{2a}, E_{3a}, E_{4a}, E_{5a}, E_{3a}^{eq} and E_{1a}^{eq} are appropriately defined coefficient matrices, which are constituted by augmenting coefficients of respective constraints together and matrices $\underline{\Phi}$, $\underline{\Gamma}$ and $\underline{f}$ are constituted from Φ_i, Γ_i and f_{di} $(i = 1,2,\ldots,2^{n_s})$. These matrices are not shown here due to brevity. The output can be written as in (13), which is reiterated below:

$$y_k = C x_k \quad (29)$$

Thus, (25)-(29) is the proposed modification in the partially linearized discrete-time model (11)-(13), which represents dynamic of the NHDS in the vicinity of an operating point. Similar models are obtained for multiple operating points. These models are then blended using Bayesian weighting, as explained in the previous section, to reconstitute the dynamics of the NHDS over entire operating rang. The weighted form of (25) can be represented as,

$$x_{k+1} = \underline{\Phi_\mathrm{w}} X_k + \underline{\Gamma_w} U_k + \underline{f_w} Z_k \quad (30)$$

where $\underline{\Phi_\mathrm{w}}$, $\underline{\Gamma_w}$ and $\underline{f_w}$ are blended system matrices, which are constituted from $\underline{\Phi}$, $\underline{\Gamma}$ and $\underline{f}$. Here it should be noted that the constraints and output of the weighted model remains unchanged from (26)-(28). Thus, (26)-(30) represent the proposed variable disaggregation based modeling for the nonlinear hybrid dynamical systems. The structure of the proposed model remains fixed for any arbitrary NHDS. Moreover, it yields a MIQP when used for the MPC formulation, which is discussed in Section III.

III. Model Predictive Control for Nonlinear hybrid dynamical systems

In this work, we use a quadratic objective function as given below,

$$\min_{U_{(\cdot)}\,\delta_{(\cdot)}\,Z_{(\cdot)}\,X_{(\cdot)}} \sum_{i=1}^{p} \Big(\left\| y_{k+i} - y_{sp} \right\|_{\Lambda_y}^2 + \left\| X_{k+i-1} - X_{sp} \right\|_{\Lambda_\mathrm{x}}^2 + \left\| \delta_{k+i-1} - \delta_{sp} \right\|_{\Lambda_\mathrm{d}}^2 + \left\| Z_{k+i-1} - Z_{sp} \right\|_{\Lambda_z}^2 \Big) + \sum_{i=0}^{m-1} \Big(\left\| \Delta U_{k+i} \right\|_{\Lambda_u}^2 + \left\| U_{k+i} - U_{sp} \right\|_{\Lambda_\mathrm{su}}^2 \Big) \quad (31)$$

subject to constraints (26) and bound constraints on outputs,

$$y_L \leq y \leq y_U \quad (32)$$

Here p and m are the prediction horizon and control horizon, respectively, Λ_* represent weight matrices on corresponding error terms in (31), $*_{sp}$ $(* = y, X, \delta, Z, U)$ represent setpoints for corresponding variables, y_L and y_U are lower and upper bounds on the outputs.

The MPC problem (31) requires future predictions of the outputs as well as constraints in (26) and (32). These prediction can be obtained by propagating (30), (29) and (26) for p steps in future. Thus, obtain prediction equations as follows,

$$\chi_k = H_1 \bar{X}_k + H_2 \bar{U}_k + H_3 \bar{Z}_k \quad (33)$$

$$\psi_k = \bar{H}_1 \bar{X}_k + \bar{H}_2 \bar{U}_k + \bar{H}_3 \bar{Z}_k \quad (34)$$

$$\bar{E}_{1a} \bar{U}_k + \bar{E}_{2a} \bar{\delta}_k + \bar{E}_{3a} \bar{X}_k + \bar{E}_{5a} \bar{Z}_k \leq \bar{E}_{4a} \quad (35)$$

$$\psi_L \leq \psi_k \leq \psi_U \quad (36)$$

The various prediction vectors in (33)-(35) are defined as,

$$\chi_k = \left[x_{k+1}^T \, x_{k+2}^T \ldots \, x_{k+p}^T \right]^T; \psi_k = \left[y_{k+1}^T \, y_{k+2}^T \ldots \, y_{k+p}^T \right]^T$$

$$\bar{X}_k = \left[X_k^T \, X_{k+1}^T \ldots \, X_{k+p-1}^T \right]^T; \bar{\delta}_k = \left[\delta_k^T \, \delta_{k+1}^T \ldots \, \delta_{k+p-1}^T \right]^T$$

$$\bar{Z}_k = \left[Z_k^T \, Z_{k+1}^T \ldots \, Z_{k+p-1}^T \right]^T; \bar{U}_k = \left[U_k^T \, U_{k+1}^T \ldots \, X_{k+m-1}^T \right]^T$$

$H_i, \bar{H}_i$ and $\bar{E}_{ja}$ $(i = 1,2,3;\ j = 1,2,3,4,5)$ are the appropriately defined coefficient matrices, which are constituted from $\underline{\Phi_\mathrm{w}}$, $\underline{\Gamma_w}$, $\underline{f_w}$, C and E_{ja}, respectively.

Substituting (33),(34) and various prediction vectors into (31), followed by expansion and rearrangement, we obtain the standard quadratic objective function as follows:

$$\min_{\xi} J = 0.5\, \xi^T Q \xi + g^T \xi \quad (37)$$

Similarly, augmentation and rearrangement of constraints (35) and (36) yields following compact form:

$$S\xi \leq b \quad (38)$$

where $\xi = [\bar{U}_k^T \quad \bar{X}_k^T \quad \bar{Z}_k^T \quad \bar{\delta}_k^T]^T$, Q, g and S, ξ are appropriately defined matrices, which are constituted from $H_i, \bar{H}_i$, $*_{sp}$ and $\bar{E}_{ja}$ $(j = 1,2,3,4,5)$, y_{sp}, ψ_L and ψ_U, respectively. Thus, (37)-(38) represent MPC formulation for the NHDS, which is a MIQP optimization problem.

IV. Application

We demonstrate efficacy of the proposed variables disaggregation based modeling and control strategy using a three- spherical tanks benchmark system. The schematic of the three-tank system is shown in the Fig. 1. The three-tank system consist of three spherical tanks of $0.6m$ diameter, two continuous manipulated inputs correspond to two control valves, six discrete (binary) manipulated inputs, characterized by on/off valves. Tank-1 and Tank-2 are connected to Tank-3 (middle tank) from the bottom as well as from the center of the tanks (*i.e.* $0.3m$ from the bottom).

The first principles model of the three-tank system can be represented as,

$$a(h_1)\dot{h}_1 = (u_1 - V_{13}Q_{13V_{13}} - V_1Q_{13V_1} - V_{L1}Q_{L1}) \quad (39)$$

$$a(h_2)\,\dot{h}_2 = (u_2 - V_{23}Q_{23V_{23}} - V_2Q_{23V_2}) \quad (40)$$

$$a(h_3)\dot{h}_3 = (V_{13}Q_{13V_{13}} + V_1Q_{13V_1} + V_{23}Q_{23V_{23}} + V_2Q_{23V_2} + V_{N3}Q_{N3}) \quad (41)$$

where $a(h_i) = \pi h_i(h_{max} - h_i)$, h_{max} is diameter of tanks ($0.6m$), h_i, $i = 1,2,3$, are liquid levels of three tanks, u_1, u_2 are continuous manipulated variables representing liquid inflows to the Tank-1 and Tank-2, respectively, and V_i, $i = 1, 2, 13, 23, L1, N3$, are binary variables that represent on/off valves. Variables Q_i represent flows through V_i.

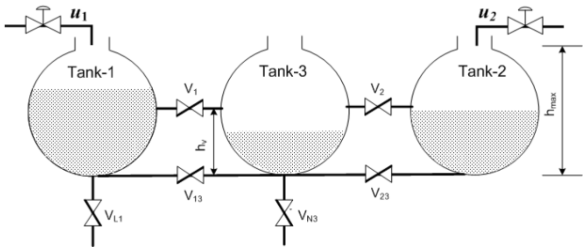

Fig. 1. Schematic of three-tank system (adopted from [3])

In order to develop MPC control law for the aforementioned three tank system, we linearized (39)-(41) around three different operating points. Consequently, three linearized models, of the form (25)-(29), are obtained. The linearization points are considered as in [3], and are listed below: (i) $h_1 = h_2 = 0.15$, $h_3 = 0.14$ and $Q_1 = Q_2 = 0$ (ii) $h_1 = h_2 = 0.25$, $h_3 = 0.24$ and $Q_1 = Q_2 = 0$ (iii) $h_1 = h_2 = 0.35$, $h_3 = 0.34$ and $Q_1 = Q_2 = 0$. These models are then used to develop MPC law of the form (37)-(38). The proposed multiple-model based MPC law is used for the level control of the three-tank system shown in Fig. 1. The control of the three-tank system involves tracking of various setpoints of levels in three tanks.

To demonstrate the computational efficiency of the proposed modeling and control approach, the results are compared with the multiple MLD (MMLD) model based control approach. The MMLD models are developed at the same operating points. The sampling time T_s, for both the methods, is considered as $3sec$, prediction and control horizons are considered as $5(15sec)$ and $2(6sec)$, respectively. In this study, the nonlinear dynamic (39)-(41) is considered as the "plant" model while the proposed variables disaggregation based weighted model or MMLD model are considered as the predication models, respectively. The model predictions are corrected by adding plant-model mismatch, which is assumed to be constant over the prediction horizon. Note that the proposed model as well as the MMLD model based MPC yield MIQP optimization problem, which has to be solved online. Many commercial solvers for MIQP are available. In this work, we rely on the IBM ILOG CPLEX solver using the MATLAB interface. All simulations have been performed on a 2.27 GHz Intel core i3 machine with 4 GB RAM.

Fig. 2 documents response of the controlled variables of the three-tank hybrid system. Solid blue line denotes output response using the variables disaggregation based modeling and control approach, dashed red line represents the corresponding results using the MMLD based control and dashed-dotted black line stands for setpoints in the three-tanks. Corresponding manipulated inputs are represented in Fig. 3.

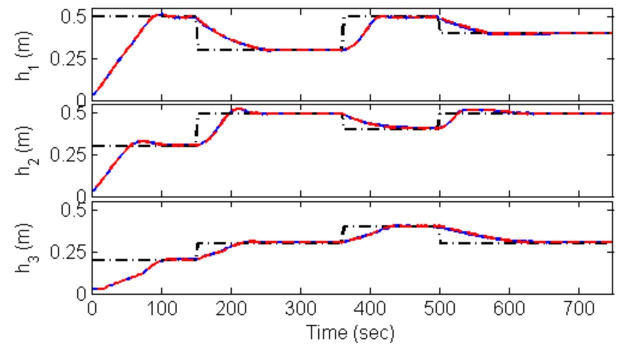

Fig. 2 Response of controlled variables: $\boldsymbol{h_1}, \boldsymbol{h_2}, \boldsymbol{h_3}$ using variables disaggregation (solid blue line) & MMLD (red dashed line) model.

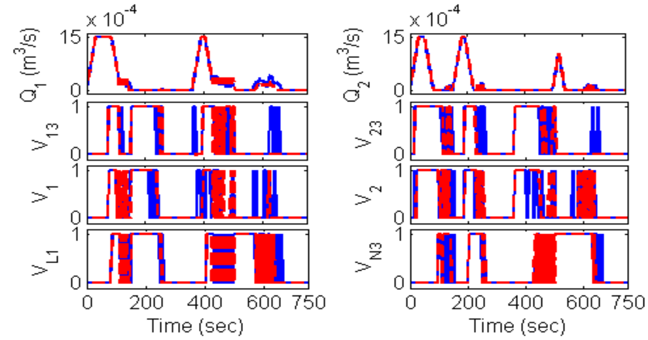

Fig. 3 Control moves using variables disaggregation (solid blue line) & MMLD (red dashed line) model.

From the figures, it can be seen that the response of the controlled variables are almost coincide with each other. Thus, the quality of the solution using the proposed method is identical to with that of the MMLD based solution. This is also evident from the average objective function values, which is 0.081 for the proposed approach whereas 0.105 for the MMLD based control. The minor difference may be attributed to the numerical methods. However, the advantage of the proposed approach lies in its computational efficiency. The computation time for each sampling instants are calculated using "tic-toc" function in the MATLAB. The mean and standard deviation of the computation time per optimization problem using the proposed modeling and control are 136.9 sec and 332.0 sec, respectively while for the MMLD based control, they are 241.1 sec and

340.1 sec, respectively. This can be explained by the fact that the proposed modeling approach yields a smaller MPC problem as compared to MMLD modeling based approach. The size of the MPC problem for both the cases are summarized in TABLE I.

TABLE I

SIZE OF THE MIQP (MPC) PROBLEM FOR 3-TANK SYSTEM ($p = 5$, $m = 2$)

	Proposed model based MPC	MMLD model based MPC
Binary variables	12	112
Continuous variables	1344	1189
Linear constraints	5175	5919

V. CONCLUSION

Most of the practical applications are inherently hybrid in nature. Recently, a multiple partially linearized (MPL) modeling for NHDS has been proposed [3]. The MPL based MPC yields a fixed structured MINLP optimization problem for any arbitrary NHDS. However, disadvantage with this approach is that the resulting optimization problem is *nonconvex*, hence global solution cannot be guaranteed. On the other hand, equivalent *convex* formulation based on multiple mixed logical dynamical (MMLD) model requires large number of additional binary variables, continuous variables and linear constraints that leads to computationally expensive optimization problem [3],[19]. In this work, we present a significant modification to the MPL model, where we use disaggregation of variables approach to model polynomials of binary variables using few additional continuous variables and linear constraints. The proposed approach produces linear representation of the MPL model of NHDS. Consequently, the MPC formulation results in a *convex* MIQP problem.

The proposed approach adds auxiliary continuous variables and constraints but refrains from adding additional binary variables and associate constraints therefore, yields relatively smaller optimization problem than the corresponding *convex* formulation based on MMLD model. Hence, significantly improves computational efficiency. The effectiveness of the proposed modeling and control of NHDS is demonstrated on the three-spherical tanks benchmark hybrid systems via simulation. From the results, it can be concluded that the proposed approach outperforms the MMLD based control of NHDS in terms of the computational efficiency, while producing identical results in terms of the solution quality. In addition, with increase in the binary variables, δ, in the NHDS (1)-(4), tree-size of the MMLD based MIQP increases exponentially due to the addition of large number of extra binary variables, which is not the case with the proposed approach. Therefore, the proposed approach may scale better than the corresponding MMLD approach. Moreover, the structure of the additional constraints, due to disaggregation of the polynomials of binary variables, remains sparse and can be exploited through sparse decomposition to further enhance the computational efficiency. For instance, barrier QP solvers (see [20]) allow a direct elimination of these inequalities.

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