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Solvability of the output regulation problem with a feedforward controller

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Abstract— **In this paper the solvability of the output regulation problem with an infinite-dimensional exosystem by using a linear feedforward controller is considered. New sum conditions that are necessary and sufficient for the solvability are found. In addition, the required smoothness properties of the reference signals are discussed in detail.**

I. INTRODUCTION

Different regulation problems have a central role in control theory. In the output regulation problem one wants the output of a system to follow a given signal. In this paper we consider the solvability of the output regulation problem where the signals to be tracked are generated by an infinite-dimensional exosystem. The controller is a feedforward controller using a state feedback to exponentially stabilize the plant.

Feedforward controllers are widely used in applications, if not by themselves, along with feedback controllers to improve performance. Thus, it is important to understand their possibilities and limitations. The most severe weakness of feedforward controllers is the lack of robustness properties. However, the output regulation problem without robustness is solvable by a feedforward controller if and only if it is solvable by a feedback controller [2]. Consequently, the solvability conditions of the output regulation problem with a feedforward controller are necessary for the robust output regulation problem too.

Necessary and sufficient conditions for the solvability of the output regulation problem with a feedforward controller in the finite-dimensional case have been known since the 70s [4]. Since then, many authors have generalized these results for infinite-dimensional plants with finite-dimensional exosystems, see for example [10], [8]. A case study with a feedforward controller and an infinite-dimensional exosystem in [1] revealed a connection between the solvability of the problem and the behavior of the plant transfer function at high frequencies. In principle, the faster the transfer function approaches zero the smoother the reference and the disturbance signals should be for the problem to be solvable. This connection was later studied in [6] for single output case and in [7] for finite-dimensional input and output spaces. Similar smoothness assumptions are also required with dynamic feedback controllers [9], [5].

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The purpose of this paper is to study the solvability of the output regulation problem with feedforward controllers. We have two main goals:

1) to characterize the solvability of the problem,
2) to study the required smoothness properties of the reference signals.

In this paper we give new sum conditions for the solvability that generalizes those in [6] and [7]. The earlier results are for finite-dimensional input and output spaces. Our sum conditions allow even infinite-dimensional input and output spaces. In addition, our sum conditions are given in terms of the exosystem and the original plant, in contrast to the results in [6] and [7], where the sum conditions were given in terms of the exosystem and the stabilized plant.

The results on the smoothness of the reference signals found in [7] are extended in the following way. We are able to deal with a more general class of signals, since the smoothness properties of the reference signals are not controlled through a sequence, but a set of sequences. This leads to a characterization of all the reference signals generated by the proposed exosystem for which the output regulation problem is solvable with a feedforward controller. In [7] this question was answered only partially.

The rest of this paper is organized as follows. In Section II we present required notations and preliminary results. In Section III we present the output regulation problem and in Section IV we give new solvability conditions for the problem. Section V is dedicated for studying the required smoothness properties of the reference signals.

II. NOTATIONS AND PRELIMINARY RESULTS

From now on a left inverse of $P(s)$ at $s = i\omega_n$ is denoted by $P^l(i\omega_n)$ and a right inverse by $P^r(i\omega_n)$ whenever they exist. The domain and the range of an operator are denoted by $\mathscr{D}(\cdot)$ and $\mathscr{R}(\cdot)$, respectively.

Next we recall the definition of the structure at infinity given in [7]. Before we can do that, we need some terminology and one theorem. The set of all complex functions $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is denoted by $\mathbf{F}$. The set of all paths in the complex plane approaching infinity is denoted by

$$\mathbf{P} = \{p : [0,\infty) \to \mathbb{C} | \forall M > 0 : \exists \rho \geq 0 : \forall \alpha \geq \rho : |p(\alpha)| > M\}.$$

For the next definition it is convenient to define $\frac{\infty}{\infty} = 1 = \frac{0}{0}$, $\frac{a}{0} = \infty$ for $a \neq 0$ and $\frac{a}{\infty} = 0$ for $a \neq \infty$.

Definition 2.1: Let $f, g \in \mathbf{F}$ and $H \subseteq \mathbf{P}$. If

$$\forall p \in H : \exists \rho \geq 0 : \sup_{\alpha \geq \rho} \left| \frac{g(p(\alpha))}{f(p(\alpha))} \right| < \infty$$

it is said that f majorizes g along H and is denoted by $f \geq_H g$. If $f \geq_H g$ and $g \geq_H f$ notation $f =_H g$ is used.

Note, that $=_H$ is an equivalence relation. Denotation $[f]_H$ is used for the equivalence class of a function f. The following theorem is presented in [7, Theorem 3.2]. Constructing the diagonal form in the theorem is possible because $V(s)$ and $U(s)$ are allowed to be discontinuous.

Theorem 2.1: Let $H \subseteq \mathbf{P}$. A matrix $G(s) \in \mathbf{F}^{n\times m}$ can be written as

$$G(s) = V(s)\begin{bmatrix}\Lambda(s) & 0\\ 0 & 0\end{bmatrix}U(s), \tag{1}$$

where $\Lambda(s) = \operatorname{diag}(q_1(s),\ldots,q_r(s))$ is a diagonal matrix with non-zero elements, $q_1 \geq_H q_2 \geq_H \cdots \geq_H q_r$, zero blocks may be non-existent and for all $p \in P$ there exists $\rho \geq 0$ such that matrices $V(s)$ and $U(s)$ satisfy the following boundedness conditions

1) $U(p(\alpha))$ and $V(p(\alpha))$ are invertible for all $\alpha \geq \rho$,
2) $\|U(p(\alpha))\|$, $\|U^{-1}(p(\alpha))\|$, $\|V(p(\alpha))\|$ and $\|V^{-1}(p(\alpha))\|$ are uniformly bounded for $\alpha \geq \rho$.

Definition 2.2: Let $\Lambda(s) = \operatorname{diag}(q_1(s),\ldots,q_r(s))$ be as in (1). We say that $\{[q_1]_H,\ldots,[q_r]_H\}$ is the directed structure at infinity of the matrix $G(s)$ along H. The functions $q_1,q_2,\ldots,q_r$ are called structural functions along H.

The following useful characterization of bounded operators is used extensively.

Lemma 2.1: Let W and V be Hilbert spaces and let $(\phi_n)_{n\in\mathbb{Z}}$ be an orthonormal basis of W. A linear operator $T : W \to V$ is bounded if and only if $\sup_{\|h\|\leq 1}\sum_{n\in\mathbb{Z}}|\langle T\phi_n,h\rangle|^2 < \infty$.

Proof: Necessity. Assume, that T is bounded and let h_0 be an arbitrary fixed element of V satisfying $\|h_0\| \leq 1$. Set $w_N = \sum_{|n|\leq N}\overline{\langle T\phi_n,h_0\rangle}\phi_n$. By boundedness of T

$$\begin{aligned}\|T\|\|w_N\| &\geq \|Tw_N\|\\ &= \sup_{\|h\|\leq 1}|\langle Tw_N,h\rangle|\\ &\geq |\langle Tw_N,h_0\rangle|\\ &= \sum_{|n|\leq N}|\langle T\phi_n,h_0\rangle|^2.\end{aligned}$$

Since $\|w_N\|^2 = \sum_{|n|\leq N}|\langle T\phi_n,h_0\rangle|^2$ we have for all $N \in \mathbb{N}$ and $\|h_0\| \leq 1$

$$\sum_{|n|\leq N}|\langle T\phi_n,h_0\rangle|^2 \leq \|T\|^2.$$

Thus $\sup_{\|h\|\leq 1}\sum_{n\in\mathbb{Z}}|\langle T\phi_n,h\rangle|^2 < \infty$.

Sufficiency. Assume that $M = \sup_{\|h\|\leq 1}\sum_{n\in\mathbb{Z}}|\langle T\phi_n,h\rangle| < \infty$. Let $w \in W$. Since $(\phi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis, we can write $w = \sum_{n\in\mathbb{Z}}\langle w,\phi_n\rangle\phi_n$. By Cauchy-Schwartz inequality

$$\begin{aligned}\|Tw\| &= \sup_{\|h\|\leq 1}|\langle Tw,h\rangle|\\ &\leq \sup_{\|h\|\leq 1}\sum_{n\in\mathbb{Z}}|\langle w,\phi_n\rangle||\langle T\phi_n,h\rangle|\\ &\leq \sup_{\|h\|\leq 1}\left(\sum_{n\in\mathbb{Z}}|\langle w,\phi_n\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{n\in\mathbb{Z}}|\langle T\phi_n,h\rangle|^2\right)^{\frac{1}{2}}\\ &= \sqrt{M}\|w\|\end{aligned}$$

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III. THE OUTPUT REGULATION PROBLEM

Next we present the output regulation problem (ORP). For that we need to fix a plant, an exosystem and a controller type.

A. THE PLANT

The plant is defined as follows

$$\begin{aligned}\dot{x} &= Ax + Bu + d,\ x(0) = x_0 \in W\\ y &= Cx + Du,\end{aligned}$$

where A is a linear (possibly) unbounded operator that generates a C_0-semigroup on a Hilbert space X. The input and output spaces U and Y are Hilbert spaces. The linear operators B, C and D are bounded. The disturbance term d is defined below.

B. THE EXOSYSTEM

Here we define the reference and the disturbance signals. The exosystem has an infinite number of unstable eigenvalues on the imaginary axis. With infinite-dimensional exosystems we can generate a large class of signals. Especially we can generate all the periodic signals.

Let W be a Hilbert space with an orthonormal basis $(\phi_n)_n \in \mathbb{Z}$. Furthermore, let $\omega_n \in \mathbb{R}$, $n \in \mathbb{Z}$, be such that $\inf_{n\neq m}|\omega_n - \omega_m| > 0$ and $|\omega_n| \to \infty$ as $n \to \pm\infty$. We now define a linear operator by setting

$$S = \sum_{n\in\mathbb{Z}} i\omega_n\langle\cdot,\phi_n\rangle\phi_n$$

with

$$\mathscr{D}(S) = \left\{v \in W \,\middle|\, \sum_{n\in\mathbb{Z}}\omega_n^2|\langle v,\phi_n\rangle| < \infty\right\},$$

which is a generator of a C_0-semigroup. The exosystem is defined to be

$$\begin{aligned}\dot{v} &= Sv,\ v(0) = v_0 \in W\\ y_r &= Fv,\\ d &= Ev,\end{aligned}$$

where $y_r \in U$ is the reference signal to be tracked and $d \in X$ is the disturbance signal. The operators F and E are bounded and are called the reference operator and the disturbance operator, respectively.

In Section V we consider the relation between the smoothness of the signals and the solvability of the ORP. The smoothness of the reference and disturbance signals can be controlled either by the operators F and E or by the initial state. In this paper we control the smoothness by the operators and allow the initial states to be chosen arbitrarily.

C. PROBLEM FORMULATION

The problem is to find a state-feedback control law

$$u = Kx + Lv, \tag{2}$$

such that K and L are bounded operators, $A + BK$ is a generator of an exponentially stable C_0-semigroup and the tracking error $e(t) = y(t) - y_r(t) \to 0$ as $t \to \infty$.

IV. SOLVABILITY CONDITIONS

We do the following standing assumptions:

1) The pair (A,B) is exponentially stabilizable.
2) The spectrum of S is contained in the resolvent set of A, i.e. $i\omega_n \in \rho(A)$.
3) The transfer function $P(i\omega_n) = C(i\omega_n I - A)^{-1}B + D$ is left or right invertible for all $n \in \mathbb{Z}$.

In what follows the following two sum conditions turn out to be crucial for the solvability of the ORP:

$$\sup_{\|u\|\le 1} \sum_{n\in\mathbb{Z}} |\langle \gamma_n, u\rangle|^2 < \infty, \tag{3a}$$

$$\sup_{\|x\|\le 1} \sum_{n\in\mathbb{Z}} |\langle \pi_n, x\rangle|^2 < \infty, \tag{3b}$$

where $\gamma_n = P^{rl}(i\omega_n)(F - C(i\omega_n I - A)^{-1}E)\phi_n$ and $\pi_n = (i\omega_n I - A)^{-1}(B\gamma_n + E\phi_n)$ and P^{rl} is a left or a right inverse of $P(i\omega_k)$ which ever exists.

By Lemma 2.1 operators

$$\Gamma = \sum_{n\in\mathbb{Z}} \langle \cdot, \phi_n\rangle \gamma_n, \tag{4a}$$

$$\Pi = \sum_{n\in\mathbb{Z}} \langle \cdot, \phi_n\rangle \pi_n, \tag{4b}$$

are bounded linear operators if and only if the sum conditions (3) hold. These operators in turn satisfy the so called regulator equations. The regulator equations given for example in [6] and [2] give a necessary and sufficient condition for the solvability. Even though the following result was given only for the single output case in [6, Theorem 3.1] the proof is valid also in the current case and is therefore skipped.

Theorem 4.1: The ORP is solvable by a state-feedback control law of form (2) if and only if $A+BK$ is exponentially stable and there exists a decomposition $L = \Gamma - K\Pi$, where $\Gamma : W \to U$ and $\Pi : W \to X$ are bounded linear operators such that for all $n \in \mathbb{Z}$

$$\Pi S\phi_n - A\Pi\phi_n - B\Gamma\phi_n = E\phi_n, \tag{5a}$$

$$C\Pi\phi_n + D\Gamma\phi_n = F\phi_n. \tag{5b}$$

Before proceeding we reformulate the regulator equations so that the plant transfer function is present. This is done in the next lemma.

Lemma 4.1: The operators Γ and Π satisfy the regulator equations (5a) and (5b) if and only if they satisfy

$$P(i\omega_n)\Gamma\phi_n = F\phi_n - C(i\omega_n I - A)^{-1}E\phi_n, \tag{6a}$$

$$\Pi\phi_n = (i\omega_n I - A)^{-1}(B\Gamma + E)\phi_n. \tag{6b}$$

Proof: Necessity. Assume that the regulator equations (5a) and (5b) hold. From (5a) and noting that ϕ_n is the eigenvector of S corresponding to eigenvalue $i\omega_n$ we get

$$i\omega_n\Pi\phi_n - A\Pi\phi_n = B\Gamma\phi_n + E\phi_n$$
$$\Rightarrow \Pi\phi_n = (i\omega_n I - A)^{-1}(B\Gamma + E)\phi_n.$$

Multiplying from left by C and adding $D\Gamma\phi_n$ to both sides we get by (5b)

$$C\Pi\phi_n + D\Gamma\phi_n = P(i\omega_n)\Gamma\phi_n + C(i\omega_n I - A)^{-1}E\phi_n = F\phi_n.$$

Sufficiency. Assume that (6a) and (6b) hold. By (6b)

$$\Pi S\phi_n - A\Pi\phi_n - B\Gamma\phi_n = (i\omega_n I - A)\Pi\phi_n - B\Gamma\phi_n = E\phi_n.$$

By (6a) and (6b)

$$C\Pi\phi_n + D\Gamma\phi_n = P(i\omega_n)\Gamma_n + C(i\omega_n I - A)^{-1}E\phi_n = F\phi_n.$$

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A. THE SUM CONDITIONS

Here we show that the sum conditions (3) are crucial for the solvability of the ORP. The sum conditions are similar to the one given in [6, Corollary 4.7] for the single input single output case. However, the sum conditions given here cover general Hilbert spaces as input and output spaces and they give the solvability conditions directly in terms of the original plant instead of the closed loop plant.

It is clear that if we have less control than measurements it is practically impossible to solve the ORP. This is why the case of right invertible transfer functions is more interesting. However, for thorough analysis we consider also the case when $P(s)$ is left invertible. This is done next.

Theorem 4.2: Assume that the plant transfer function is left invertible at $i\omega_n$ for all $n \in \mathbb{Z}$. The ORP is solvable if and only if (3) hold for a choice of $P^{rl}(i\omega_n) = P^l(i\omega_n)$ and

$$(I - P(i\omega_n)P^l(i\omega_n))(F\phi_n - C(i\omega_n I - A)^{-1}E\phi_n) = 0. \tag{7}$$

Proof: Necessity. Assume that the ORP is solvable. It follows that (6a) and (6b) hold. From (6a) we get by left invertibility that $\Gamma\phi_n = \gamma_n$ and from (6b) we get $\Pi\phi_n = \pi_n$. By Lemma 2.1 the sum conditions (3) hold. Since by (6a) $F\phi_n - C(i\omega_n I - A)^{-1}E\phi_n \in \mathscr{R}(()P(i\omega_n))$ and $P(i\omega_n)P^l(i\omega_n)$ is a projection operator on $\mathscr{R}(()P(i\omega_n))$ condition (7) holds.

Sufficiency. Under the sum condition (3) the operators in (4) are bounded operators by Lemma 2.1. Under Assumption (7) it is clear that (6a) and (6b) are satisfied by the defined operators. ■

Condition (7) is natural. With non-square left invertible plants we have less control than measurements. As the reference signals live on the measurement space, it is clear that we cannot have control over all reference signals. Roughly speaking, (7) means that the part of the reference signal that is not controllable must be compensated by the error signal.

Remark 4.1: No matter how we choose the left inverse $P^l(i\omega_n)$ we have the same value for γ_n. Thus, in this case the operator Γ and Π solving the regulator equations are uniquely determined.

In the remaining part of this paper we will focus on the case where the transfer function is right invertible. The solvability condition is similar to the one given in the case of left invertible transfer functions, but no additional assumptions need to be made.

Theorem 4.3: Assume that the plant transfer function is right invertible at $i\omega_n$ for all $n \in \mathbb{Z}$. The ORP is solvable if and only if for some choice of $P^{rl}(i\omega_n) = P^r(i\omega_n)$ the sum conditions (3) hold.

Proof: Necessity. Assume that the ORP is solvable. It follows that (6a) and (6b) hold. By (6a) $\Gamma\phi_n = \gamma_n$ for some choice of right inverses $P^r(i\omega_n)$, $n\in\mathbb{Z}$. From (6b) we get $\Pi\phi_n=\pi_n$. By Lemma 2.1 the sum conditions (3) hold.

Sufficiency. Under the sum conditions (3) the operators in (4) are bounded operators by Lemma 2.1. It is a matter of trivial calculation to verify that (6a) and (6b) are satisfied by the defined operators. ■

For right invertible transfer functions the operators Γ and Π are not in general defined uniquely. However, the sum conditions (3) can be used to characterize all the regulating controllers of form (2). The next corollary follows by the proof of the above theorem and Lemma 4.1.

Corollary 4.1: If the sum conditions (3) hold for a choice of $P^{rl}(i\omega_n)=P^r(i\omega_n)$, $n\in\mathbb{Z}$, then the feedback control law (2), where K exponentially stabilizes the pair (A,B) and $L=\Gamma-K\Pi$ with operators Γ and Π in (4), solves the ORP. All the feedback control laws solving the ORP are of this form.

B. ON THE SUM CONDITIONS

Above we have seen that the solvability of the ORP is characterized by two sum conditions. Both of them are needed as shown by the following two examples.

Example 4.1: In this example the plant and the exosystem are chosen so that the sum condition (3a) fails, while the sum condition (3b) is satisfied. Let $\omega_n=n$, for $n\in\mathbb{Z}$, and $U=Y=X=\mathbb{C}$. Choose $E=0$, $D=0$, $B=C=1$, $A=-1$ and $F=\langle\cdot,w_0\rangle$, where $w_0=\sum_{n\in\mathbb{Z}}\frac{1}{n}\phi_n\in W$. Now all the assumptions in the beginning of Section IV are satisfied and

$$\gamma_n=(in+1)\langle\phi_n,w_0\rangle=\frac{in+1}{n}\text{ and }\pi_n=\langle\phi_n,w_0\rangle.$$

It is now clear that $\sup_{\|u\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\gamma_n,u\rangle|^2=\infty$ and $\sup_{\|x\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\pi_n,x\rangle|^2=\|w_0\|^2<\infty$. ▲

Example 4.2: In this example the plant and the exosystem are chosen so that the sum condition (3b) fails, while the sum condition (3a) is satisfied. Let $\omega_n=n$, for $n\in\mathbb{Z}$, and $U=Y=X=Z$. Choose $B=D=E=F=I$,

$$A=\sum_{n\in\mathbb{Z}}\left(in-\frac{1}{|n|+1}\right)\langle\cdot,\phi_n\rangle\,\phi_n$$

and

$$C=\sum_{n\in\mathbb{Z}}\frac{1}{|n|+1}\langle\cdot,\phi_n\rangle\,\phi_n.$$

Elementary calculations show that

$$P^{-1}(s)=\sum_{n\in\mathbb{Z}}\frac{(|n|+1)(s-in)+1}{(|n|+1)(s-in)+2}\langle\cdot,\phi_n\rangle\,\phi_n.$$

Now all the standing assumptions are satisfied and

$$\gamma_n=-\frac{1}{2}\phi_n\text{ and }\pi_n=\frac{|n|+1}{2}\phi_n.$$

It is now clear that $\sup_{\|u\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\gamma_n,u\rangle|^2=\frac{1}{2}<\infty$. Since $\sum_{n\in\mathbb{Z}}|\langle\pi_n,\phi_n\rangle|^2=\frac{(|n|+1)^2}{4}$, the sum condition (3b) fails. ▲

Since the supremum over an infinite number of series is hard to find in practice, we seek some additional conditions that simplify the situation. This is our aim for the rest of this section.

Lemma 4.2: Let B be a finite rank operator and let $\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty$. The sum conditions (3) hold if and only if

$$\sup_{\|x\|\le 1}\sum_{n\in\mathbb{Z}}\left|\left\langle(i\omega_nI-A)^{-1}E\phi_n,x\right\rangle\right|^2<\infty. \tag{8}$$

Proof: It is easy to see that under the assumption made

$$\sup_{\|u\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\gamma_n,u\rangle|^2\le\sum_{n\in\mathbb{Z}}\sup_{\|u\|\le 1}|\langle\gamma_n,u\rangle|^2=\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty.$$

From the exponential stabilizability of (A,B) it follows that X can be decomposed as $X=X_+\oplus X_-$ so that X_+ is finite-dimensional and A has the corresponding decomposition $A=\begin{bmatrix}A_+ & 0\\ 0 & A_-\end{bmatrix}$, where A_+ is a generator of C_0-semigroup and A_- is a generator of an exponentially stable C_0-semigroup [3, Theorem 5.2.6]. By [3, Lemma 2.1.11] $\|(sI-A_-)^{-1}\|$ is uniformly bounded in some right half-plane including the imaginary axis. Since A_+ is an operator on a finite-dimensional space $\|(i\omega I-A_+)^{-1}\|\to 0$ as $|\omega|\to\infty$. It follows that there exists $M>0$ such that $\|(i\omega_nI-A)^{-1}\|<M$ for all $n\in\mathbb{Z}$, because $i\omega_n$ do not cluster at finite points. By assumption and the boundedness of B

$$\begin{aligned}\sup_{\|u\|\le 1}\sum_{n\in\mathbb{Z}}|\left\langle(i\omega_nI-A)^{-1}B\gamma_n,u\right\rangle|^2&\le\sum_{n\in\mathbb{Z}}\|(i\omega_nI-A)^{-1}B\gamma_n\|^2\\&\le M\|B\|\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty.\end{aligned}$$

It follows, that $\sup_{\|u\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\pi_n,u\rangle|^2<\infty$ if and only if (8) holds. ■

Theorem 4.4: If the output space Y is finite-dimensional and B is a finite-rank operator, then the ORP is solvable if and only if

$$\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty \tag{9a}$$

and

$$\sup_{\|x\|\le 1}\sum_{n\in\mathbb{Z}}\left|\left\langle(i\omega_nI-A)^{-1}E\phi_n,x\right\rangle\right|^2<\infty. \tag{9b}$$

Proof: By Lemma 4.2 it is sufficient to show that $\sup_{\|x\|\le 1}\sum_{n\in\mathbb{Z}}|\langle\gamma_n,x\rangle|^2<\infty$ if and only if $\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty$, which is true by Riesz representation theorem. ■

It is obvious that the latter sum condition is satisfied, if the disturbance operator is of finite-rank. We get the following corollary.

Corollary 4.2: Let the output space Y be finite-dimensional and the operators B and E be of finite-rank. The ORP is solvable if and only if

$$\sum_{n\in\mathbb{Z}}\|\gamma_n\|^2<\infty.$$

V. ON THE REQUIRED SMOOTHNESS PROPERTIES

In this section we characterize the required smoothness properties of the reference signals for the ORP to be solvable in terms of the structure at infinity. We concentrate on the reference signals and assume the disturbance signals to be smooth enough. We are able to characterize all the reference operators for which the ORP is solvable.

Since the structure at infinity is only defined for plants with finite-dimensional input and output spaces we choose $U = \mathbb{C}^m$ and $V = \mathbb{C}^k$. All the eigenvalues of the exosystem are purely imaginary so the structure at infinity is considered along the imaginary axis. This is why in Theorem 2.1 we choose

$$H = \{p_1(\alpha), p_2(\alpha)\}, \tag{10}$$

where $p_1(\alpha) = i\alpha$ and $p_2(\alpha) = -i\alpha$.

In the case of right invertible transfer function, we saw that the operators Γ and Π are not unique. A question rises whether there is a systematic way of choosing Γ and Π in such a manner that the ORP is solvable if and only if the chosen operators satisfy the regulator equations. The next theorem gives a positive answer to this question.

Theorem 5.1: Write $P(s)$ as in Theorem 2.1, where H is given in (10). The ORP is solvable if and only if the sum conditions (9) hold with the choice $P^{rl}(i\omega_n) = V^{-1}(i\omega_n)\begin{bmatrix}\Lambda^{-1}(i\omega_n)\\ 0\end{bmatrix}U^{-1}(i\omega_n)$.

Proof: Sufficiency is obvious, so only the necessity remains to be proved. By Corollary 4.4 there exists such a choice $P^r(i\omega_n)$ for all $n \in \mathbb{Z}$ that (9) holds. Now we need to show that (9) holds if we replace $P^r(i\omega_n)$ by $P_+(i\omega_n) = U^{-1}(i\omega_n)\begin{bmatrix}\Lambda^{-1}(i\omega_n)\\ 0\end{bmatrix}V^{-1}(i\omega_n)$. Every right inverse of $P(s)$ can be written in the form $P^r(s) = U^{-1}(s)\begin{bmatrix}\Lambda^{-1}(s)\\ J(s)\end{bmatrix}V^{-1}(s)$. It is sufficient to show that there exists $m > 0$ such that $\|P^r(s)y\| \geq m\|P_+(s)y\|$ for any $y \in Y$.

Let y be an arbitrary element of Y and set $\begin{bmatrix}y_1(s)\\ y_2(s)\end{bmatrix} = \begin{bmatrix}\Lambda^{-1}(s)\\ J(s)\end{bmatrix}V^{-1}(s)y$. Now $\begin{bmatrix}\Lambda^{-1}(s)\\ 0\end{bmatrix}V^{-1}(s)y = \begin{bmatrix}y_1(s)\\ 0\end{bmatrix}$ so we have reduced the problem to showing that there exists $m > 0$ such that $\left\|U(i\omega_n)\begin{bmatrix}y_1(i\omega_n)\\ y_2(i\omega_n)\end{bmatrix}\right\| \geq m\left\|U(i\omega_n)\begin{bmatrix}y_1(i\omega_n)\\ 0\end{bmatrix}\right\|$.

For any invertible matrix U we have

$$\begin{aligned}\frac{1}{\|U\|}\left\|U\begin{bmatrix}y_1\\ 0\end{bmatrix}\right\| \leq \left\|\begin{bmatrix}y_1\\ 0\end{bmatrix}\right\| &\leq \left\|\begin{bmatrix}y_1\\ y_2\end{bmatrix}\right\| \\ &= \left\|U^{-1}U\begin{bmatrix}y_1\\ y_2\end{bmatrix}\right\| \\ &\leq \|U^{-1}\|\left\|U\begin{bmatrix}y_1\\ y_2\end{bmatrix}\right\|.\end{aligned}$$

Since $U(s)$ and $U^{-1}(s)$ are uniformly bounded on the imaginary axis, there exists $m > 0$ such that $\frac{1}{\|U(i\omega_n)\|\|U^{-1}(i\omega_n)\|} > m$ and the claim follows by the above inequality. ■

The previous theorem shows that the solvability of the ORP is closely related to the structure at infinity. We are now able to show that if the disturbance signal is smooth enough, the solvability can be characterized by using the smoothness of the reference signal. To guarantee the required smoothness for the disturbance signal we assum, that

$$\sum_{n\in\mathbb{Z}} \left\|P_+(i\omega_n)C(i\omega_n I - A)^{-1}E\phi_n\right\|^2 < \infty \tag{11}$$

where $P_+(i\omega_n) = U^{-1}(i\omega_n)\begin{bmatrix}\Lambda^{-1}(i\omega_n)\\ 0\end{bmatrix}V^{-1}(i\omega_n)$ and U, Λ and V are from Theorem 2.1.

Lemma 5.1: Write $P(s)$ as in Theorem 2.1, where H is given in (10). Set $P^{rl}(i\omega_n) = P_+(i\omega_n) = U^{-1}(i\omega_n)\begin{bmatrix}\Lambda^{-1}(i\omega_n)\\ 0\end{bmatrix}V^{-1}(i\omega_n)$. Assume, that the sum conditions (8) and (11) hold. The ORP is solvable if and only if $\sum_{n\in\mathbb{Z}}\|P_+(i\omega_n)F\phi_n\|^2 < \infty$.

Proof: Sufficiency. Assume that $\sum_{n\in\mathbb{Z}}\|P_+(i\omega_n)F\phi_n\|^2 < \infty$. By the triangle inequality

$$\|\gamma_n\|^2 \leq 2\left(\|P^r(i\omega_n)F\phi_n\|^2 + \left\|P_+(i\omega_n)C(i\omega_n I - A)^{-1}E\phi_n\right\|^2\right).$$

By Lemma 4.4 sufficiency follows.

Necessity. Assume that the ORP is solvable. By the reverse triangle inequality

$$\sum_{n\in\mathbb{Z}}\left|\|P_+(i\omega_n)F\phi_n\| - \left\|P_+(i\omega_n)C(i\omega_n I - A)^{-1}E\phi_n\right\|\right|^2 \leq \sum_{n\in\mathbb{Z}}\|\gamma_n\|^2 < \infty.$$

Since $\sum_{n\in\mathbb{Z}}\left\|P_+(i\omega_n)C(i\omega_n I - A)^{-1}E\phi_n\right\|^2 < \infty$ the claim follows.

■

We are now able to characterize all the reference signals that can be regulated by a controller (2) under the assumption that the disturbance signals are smooth or non-existent. For this we write a bounded operator $F : W \to U$ in the form

$$F = \sum_{n\in\mathbb{Z}}\langle\cdot, \phi_n\rangle V(i\omega_n)\begin{bmatrix}\langle f_1, \phi_n\rangle\\ \vdots\\ \langle f_m, \phi_n\rangle\end{bmatrix}. \tag{12}$$

That every bounded operator from W to U can be presented in this form follows from Riesz representation theorem, boundedness properties of $V(i\omega_n)$ and Lemma 2.1.

Theorem 5.2: Write $P(s)$ as in Theorem 2.1, where H is given in (10) and F in (12). Assume that the sum conditions (8) and (11) hold. The ORP is solvable if and only if $\left(\frac{\langle f_j, \phi_n\rangle}{q_j(i\omega_n)}\right)_{n\in\mathbb{Z}} \in \ell^2$ for all $j = 1, \ldots, m$.

Proof: By Lemma 5.1 the ORP is solvable if and only if $\sum_{n\in\mathbb{Z}}\|P_+(i\omega_n)F\phi_n\|^2 < \infty$, where $P_+(i\omega_n) = U^{-1}(i\omega_n)\begin{bmatrix}\Lambda^{-1}(i\omega_n)\\ 0\end{bmatrix}V^{-1}(i\omega_n)$. The result follows immediately by writing F as in (12) and noting that $\|U^{-1}(i\omega_n)\| < \infty$ and $\|U(i\omega_n) < \infty$. ■

The following example illustrates the importance of the above theorem. It shows that for certain type of signals the required smoothness properties are far stricter than for others.

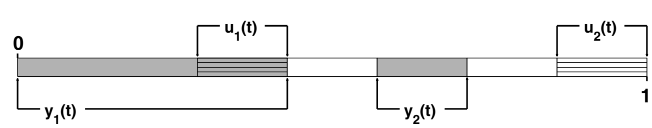

Fig. 1. A heated metal bar.

Example 5.1: A metal bar is heated with two heaters and its temperature is measured along two intervals as shown in Figure 1. The resulting system is

$$\frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) - x(z,t) + 7\cdot \mathbf{1}_{[\frac{2}{7},\frac{3}{7}]}(z)u_1(t) + 7\cdot \mathbf{1}_{[\frac{6}{7},1]}(z)u_2(t),$$
$$y_1(t) = \int_0^{\frac{3}{7}} x(z,t)\mathrm{d}z,$$
$$y_2(t) = \int_{\frac{4}{7}}^{\frac{5}{7}} x(z,t)\mathrm{d}z,$$
$$x(z,0) = x_0(z),$$
$$\frac{\partial x}{\partial z}(0,t) = 0 = \frac{\partial x}{\partial z}(1,t),$$

where $\mathbf{1}_{[a,b]}(z)$ is the unit step function on the interval $[a,b]$.

Calculations similar to those in [3, Example 4.3.11] show that the transfer function is

$$P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix},$$

where

$$p_{11}(s) = \frac{1}{s+1} + 7\frac{\left\{\begin{array}{c} -\cosh\left(\sqrt{s+1}\right) + \cosh\left(\frac{6}{7}\sqrt{s+1}\right) \\ -\cosh\left(\frac{2}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{1}{7}\sqrt{s+1}\right) \end{array}\right\}}{2(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)},$$
$$p_{12}(s) = \frac{7}{(s+1)^{3/2}}\frac{\sinh\left(\frac{1}{7}\sqrt{s+1}\right)\sinh\left(\frac{3}{7}\sqrt{s+1}\right)}{\sinh\left(\sqrt{s+1}\right)},$$
$$p_{21}(s) = 7\frac{\left\{\begin{array}{c} -\cosh\left(\frac{6}{7}\sqrt{s+1}\right) - 2\cosh\left(\frac{5}{7}\sqrt{s+1}\right) \\ +\cosh\left(\frac{4}{7}\sqrt{s+1}\right) \\ +2\cosh\left(\frac{1}{7}\sqrt{s+1}\right) - 4 \end{array}\right\}}{2(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)},$$
$$p_{22}(s) = \frac{7\sinh\left(\frac{1}{7}\sqrt{s+1}\right)\left(\sinh\left(\frac{5}{7}\sqrt{s+1}\right) - \sinh\left(\frac{4}{7}\sqrt{s+1}\right)\right)}{(s+1)^{3/2}\sinh\left(\sqrt{s+1}\right)}.$$

The structure at infinity of $P(s)$ along the imaginary axis is $\left\{\frac{1}{s+1}, \frac{1}{(s+1)^{3/2}}e^{-\frac{4}{7}\sqrt{s+1}}\right\}$.

Consider the ORP with $\omega_n = n$ and $E = 0$. Define the following two reference operators:

$$F_1 = \sum_{n\in\mathbb{Z}} \langle \cdot, \phi_n \rangle V(in) \begin{bmatrix} \langle f_1, \phi_n \rangle \\ 0 \end{bmatrix}$$

and

$$F_2 = \sum_{n\in\mathbb{Z}} \langle \cdot, \phi_n \rangle V(in) \begin{bmatrix} 0 \\ \langle f_2, \phi_n \rangle \end{bmatrix},$$

where $V(\cdot)$ is the matrix valued function in (1). Because of the boundedness properties of $V(in)$, the smoothness of the reference signals generated by using the reference operators F_1 and F_2 are essentially the same, if $f_1 = f_2$. By Theorem 5.2 the ORP with the reference operator F_1 is solvable if $(\langle f_1, \phi_n \rangle (in+1))_{n\in\mathbb{Z}} \in \ell^2$. Similarly, the ORP with the reference operator F_2 is solvable if $\left(\langle f_2, \phi_n \rangle (in+1)^{3/2} e^{\sqrt{in+1}/7}\right)_{n\in\mathbb{Z}} \in \ell^2$. The required smoothness of the reference signals generated by using F_2 is of exponential type, while using F_1 only polynomial smoothness is required. ▲

VI. CONCLUSIONS

In this paper we have considered the output regulation problem with a feedforward controller and an infinite-dimensional exosystem. We have given a necessary and sufficient condition for the solvability in terms of two sums. For a fixed plant we were able to characterize all the reference operators for which the problem is solvable, provided that the disturbance signal is smooth enough.

In this note a fairly complete answer for the solvability was given. However, a detailed analysis for the required smoothness properties of the disturbance signals was not made. We are planning to do a similar analysis for the smoothness properties required in the case of the robust regulation problem with an infinite-dimensional exosystem.

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