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On Interval Observer Design for Time-Invariant Discrete-Time Systems

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***Abstract*—The problem of interval state observer design is addressed for time-invariant discrete-time systems. Two solutions are proposed: the first one is based on a similarity transformation synthesis, which connects a constant matrix with its nonnegative representation ensuring the observation error positivity. The second contribution shows that in the discrete-time case the estimation error dynamics always can be represented in a cooperative form without a transformation of coordinates. The corresponding observer gain can be found as a solution of the formulated LMIs. The performances of the proposed observers are demonstrated through computer simulations.**

I. INTRODUCTION

The observer design problem is very challenging and its solution is demanded in many applications [1], [2], [3]. There exist many approaches dealing with the design techniques for state observers. In some cases, due to disturbance or uncertain parameter presence the synthesis of a conventional estimator (converging in the noise-free case to the ideal value of the state) is not possible. However, an interval estimation still may be feasible. By interval (or set-membership) estimation we understand an observer that, using input-output measurements, evaluates the set of admissible values (interval) for the state at each instant of time.

There exist many interval observers proposed for continuous-time (linear and nonlinear) systems based on monotone system theory [4], [5], [6], [7]. One of the most complex assumptions for the interval observer design deals with cooperativity/monotonicity of the interval estimation error dynamics. Recently it was shown that under some mild conditions, applying *similarity* transformation, a Hurwitz matrix can be transformed to a Hurwitz and Metzler (cooperative) one [6], [8], [7]. In [6], [8] this transformation is time-varying, in [7] the transformation matrix is constant and real, it is a solution of the Sylvester equation (a constructive procedure for this solution calculation was also given in [7]). In [9] this result has been extended to the class of linear time-varying systems, when constant and time-varying similarity transformations have been proposed representing an interval matrix (a time-varying matrix) in the Metzler form.

Several set-membership state estimators have been developed for discrete-time models in the the literature using simple geometrical forms such as parallelotopes, ellipsoids, zonotopes or intervals [10], [11], [12], [13], [14], [15]. They are based on the well-known prediction/correction approach (also called open-loop observers, framers or *predictors*, then the system equations are solved starting from a set of initial conditions taking on each step the values consistent with the output measurements). The main drawback of this approach is that the convergence rate cannot be tuned since it is not based on an observer gain. As an alternative, the interval observer methodology is studied in this paper, initially developed for continuous-time systems in [4], [8], [7] and extended to uncertain discrete-time systems in [16].

In many cases the measurements are available at the discrete instants of time, then the discretized models of plants are used. It is interesting to note that a cooperative continuous-time system remains cooperative in the discrete time under (Euler) discretization. In [16], LTI discrete-time systems are considered and an interval predictor is designed in order to propagate the uncertainties on the initial state and on the additive disturbances. In the following, an interval observer is proposed for the same class of systems. Nevertheless, the design procedure consists in computing an observer gain as well as a change of coordinates. Note that a similar result has been obtained for time-varying discrete-time systems in [17]. An academic planar system and the Hénon map system are considered in this work as examples.

The paper is organized as follows. Some basic facts from the theory of interval estimation are given in Section 2. The main result is described in Section 3. Example of computer simulation is presented in Section 4.

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II. Preliminaries

The real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$ respectively, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. Euclidean norm of a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{Z} \to \mathbb{R}$ the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm:

$$||u||_{[t_0,t_1]} = \sup\{|u(t)|, t \in [t_0, t_1]\},$$

if $t_1 = +\infty$ then we will simply write $||u||$. We will denote as $\mathcal{L}_\infty$ the set of all inputs u with the property $||u|| < \infty$. Denote the sequence of integers $1, ..., k$ as $\overline{1,k}$. The symbols I_n and $E_{n\times m}$ denote the identity matrix and the matrix with all entries equal 1 respectively (with dimensions $n \times n$ and $n \times m$). For a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted as $\lambda(A)$. The relation $P \succ 0$ ($P \succeq 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is positive (nonnegative) definite.

A. Interval analysis

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{m\times n}$ define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$.

Lemma 1. *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$ be a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

Proof: Note that $Ax = (A^+ - A^-)x$, that for $\underline{x} \leq x \leq \overline{x}$ gives the required estimates. ■

B. Cooperative discrete-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Schur stable if all its eigenvalues have the norm less than one, it is called nonnegative if all its elements are nonnegative. Any solution of the linear system

$$x(t+1) = Ax(t) + \omega(t),\ \omega : \mathbb{Z}_+ \to \mathbb{R}^n_+,\ t \in \mathbb{Z}_+,$$

with $x \in \mathbb{R}^n$ and a nonnegative matrix $A \in \mathbb{R}^{n\times n}_+$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ [18]. Such dynamical systems are called cooperative (monotone) [18].

Lemma 2. [7] *Given the matrices $A \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. If there is a matrix $L \in \mathbb{R}^{n\times p}$ such that the matrices $A - LC$ and R have the same eigenvalues, then there is a $S \in \mathbb{R}^{n\times n}$ such that $R = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, e_1)$ and (R, e_2) are observable for some $e_1 \in \mathbb{R}^{1\times n}$, $e_2 \in \mathbb{R}^{1\times n}$.*

This result was used in [7] to design interval observers for LTI systems with a Metzler matrix R (the matrix R is called Metzler if all its off-diagonal elements are nonnegative). The main difficulty is to prove the existence of a *real* matrix S, and to provide a constructive approach of its calculation. In [7] the matrix $S = O_R O^{-1}_{A-LC}$, where O_{A-LC} and O_R are the observability matrices of the pairs $(A - LC, e_1)$ and (R, e_2) respectively. Another (more strict) condition is that the Sylvester equation $SA - RS = QC$, $Q = SL$ has a unique solution S provided that the pair (A, C) is observable (in this case there exists a matrix L such that $\lambda(A) \neq \lambda(A - LC) = \lambda(R)$, that is equivalent to existence of a unique S [19]). In the present work we will apply this lemma to a nonnegative matrix R. Note that if the matrix $A - LC$ has only real positive eigenvalues, then R can be chosen as diagonal or Jordan representation of $A - LC$.

The application of Lemma 2 is connected with the Inverse eigenvalue problem for nonnegative matrices (i.e. the problem of existence of a nonnegative matrix R with the given set of eigenvalues $\lambda(A - LC)$), see the monograph [20] (section 11.2) for the necessary and sufficient conditions which have to be imposed on $\lambda(A - LC)$ in order that a nonnegative R exists. In [21] the fast Fourier transformation is used to design a real symmetric R with a given vector of eigenvalues.

III. Main result

Consider an LTI discrete-time system

$$x(t+1) = Ax(t) + b(t),\ y(t) = Cx(t) + v(t),\ t \in \mathbb{Z}_+, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state; $y(t) \in \mathbb{R}^p$ is the output signal available for measurements; $b : \mathbb{Z}_+ \to \mathbb{R}^n$, $b \in \mathcal{L}_\infty$ is the input; $v : \mathbb{Z}_+ \to \mathbb{R}^p$, $v \in \mathcal{L}_\infty$ is the measurement noise; A and C are real matrices of the corresponding dimensions.

In this section we will consider two approaches. The first one is based on a transformation of coordinates, which maps the estimation error dynamics to its cooperative representation. The second approach is based on the error system representation without transformation of coordinates.

A. Time-invariant transformation to cooperative form

We will need the following assumptions.

Assumption 1. *The state $x(t)$ is bounded, i.e. $x \in \mathcal{L}_\infty$.*

Assumption 2. *There exists a matrix $L \in \mathbb{R}^{n\times p}$ such that*
i) The matrix $A - LC$ is Schur stable.
ii) The matrix $A - LC$ is nonnegative.

Assumption 3. *Two functions $\underline{b}, \overline{b} : \mathbb{Z}_+ \to \mathbb{R}^n$, $\underline{b}, \overline{b} \in \mathcal{L}_\infty$ are given such that for all $t \in \mathbb{Z}_+$*

$$\underline{b}(t) \leq b(t) \leq \overline{b}(t).$$

Assumption 4. *The constant $0 \leq V < +\infty$ is given such that $||v|| < V$.*

Assumption 1 is introduced since in this work we will not consider the problem of control design (it is rather common in the literature of the observer synthesis). The first part of Assumption 2 is standard [3], [1]. The second part is crucial for the approach, it is rather restrictive and it will be relaxed later. Assumption 3 states that the input is known up to some interval error $\overline{b}(t)-\underline{b}(t)$, Assumption 4 introduces the upper bound V of the noise v amplitude.

Under the introduced assumptions an interval observer equations for (2) take form:

$$\begin{aligned}\underline{x}(t+1) &= A\underline{x}(t)+\underline{b}(t)+L(y(t)-C\underline{x}(t))-\overline{L}V, \\ \overline{x}(t+1) &= A\overline{x}(t)+\overline{b}(t)+L(y(t)-C\overline{x}(t))+\overline{L}V,\end{aligned} \quad (3)$$

where $\underline{x}(t)\in\mathbb{R}^n$ and $\overline{x}(t)\in\mathbb{R}^n$ are the lower and upper interval estimates for the state $x(t)$, $\overline{L}=(L^+ + L^-)E_{p\times 1}$.

Theorem 1. *Let assumptions 1–4 be satisfied. Then for all $t\in\mathbb{Z}_+$ the estimates $\underline{x}(t),\overline{x}(t)$ given by (3) are bounded and*

$$\underline{x}(t)\le x(t)\le\overline{x}(t)$$

provided that $\underline{x}(0)\le x(0)\le\overline{x}(0)$.

Proof: The equation (2) can be rewritten as follows:

$$x(t+1)=(A-LC)x(t)+L[y(t)-v(t)]+b(t).$$

Then the dynamics of the errors $\underline{e}(t)=x(t)-\underline{x}(t)$, $\overline{e}(t)=\overline{x}(t)-x(t)$ obey the equations:

$$\begin{aligned}\underline{e}(t+1) &= (A-LC)\underline{e}(t)+\underline{d}(t),\\ \overline{e}(t+1) &= (A-LC)\overline{e}(t)+\overline{d}(t),\end{aligned}$$

where $\underline{d}(t)=[\overline{L}V-Lv(t)]+[b(t)-\underline{b}(t)]$, $\overline{d}(t)=[\overline{L}V-Lv(t)]+[\overline{b}(t)-b(t)]$. According to assumptions 3, 4 we have $\underline{d},\overline{d}\in\mathcal{L}_\infty$ and $\underline{d}(t)\ge 0$, $\overline{d}(t)\ge 0$ for all $t\in\mathbb{Z}_+$. Therefore, from Assumption 2.i, the variables $\underline{e}(t)$ and $\overline{e}(t)$ are bounded, and taking in mind Assumption 1 we get the boundedness of the estimates $\underline{x}(t),\overline{x}(t)$. From Assumption 2.ii $\underline{e}(t)\ge 0$ and $\overline{e}(t)\ge 0$ for all $t\in\mathbb{Z}_+$ ($\underline{d},\overline{d}$ have the same property and $\underline{e}(0)\ge 0$, $\overline{e}(0)\ge 0$ by conditions), that implies the required order relation $\underline{x}(t)\le x(t)\le\overline{x}(t)$ for all $t\in\mathbb{Z}_+$. ■

In order to skip the part (ii) of Assumption 2, let us use Lemma 2.

Theorem 2. *Let assumptions 1–2.i, 3–4 be satisfied, there exist matrix $R\in\mathbb{R}_+^{n\times n}$ such that $\lambda(A-LC)=\lambda(R)$ and the pairs $(A-LC,e_1)$, (R,e_2) are observable for some $e_1\in\mathbb{R}^{1\times n}$, $e_2\in\mathbb{R}^{1\times n}$. Then for all $t\in\mathbb{Z}_+$ the estimates $\underline{x}(t),\overline{x}(t)$ are bounded and*

$$\underline{x}(t)\le x(t)\le\overline{x}(t)$$

provided that $\underline{x}(0)\le x(0)\le\overline{x}(0)$, where

$$\underline{x}(t)=S^+\underline{z}(t)-S^-\overline{z}(t),\ \overline{x}(t)=S^+\overline{z}(t)-S^-\underline{z}(t);$$

$$\begin{aligned}\underline{z}(t+1) &= R\underline{z}(t)+Fy(t)-\overline{F}V\\ &\quad +(S^{-1})^+\underline{b}(t)-(S^{-1})^-\overline{b}(t),\\ \overline{z}(t+1) &= R\overline{z}(t)+Fy(t)+\overline{F}V\\ &\quad +(S^{-1})^+\overline{b}(t)-(S^{-1})^-\underline{b}(t);\end{aligned} \quad (4)$$

$$\begin{aligned}\underline{z}(0) &= (S^{-1})^+\underline{x}(0)-(S^{-1})^-\overline{x}(0),\\ \overline{z}(0) &= (S^{-1})^+\overline{x}(0)-(S^{-1})^-\underline{x}(0),\end{aligned}$$

where $S=O_RO_{A-LC}^{-1}$ (O_{A-LC} and O_R are the observability matrices of the pairs $(A-LC,e_1)$, (R,e_2) respectively), $F=S^{-1}L$ and $\overline{F}=(F^+ + F^-)E_{p\times 1}$.

Proof: Consider the system (2) in the new coordinates $z=S^{-1}x$:

$$\begin{aligned}z(t+1) &= Rz(t)+F[y(t)-v(t)]+S^{-1}b(t),\\ y(t) &= CSz(t)+v(t).\end{aligned}$$

The dynamics of the errors $\underline{e}(t)=z(t)-\underline{z}(t)$, $\overline{e}(t)=\overline{z}(t)-z(t)$ obey the equations:

$$\begin{aligned}\underline{e}(t+1) &= R\underline{e}(t)+\underline{d}(t),\\ \overline{e}(t+1) &= R\overline{e}(t)+\overline{d}(t),\end{aligned}$$

where $\underline{d}(t)=[\overline{F}V-Fv(t)]+[S^{-1}b(t)-(S^{-1})^+\underline{b}(t)+(S^{-1})^-\overline{b}(t)]$, $\overline{d}(t)=[\overline{F}V+Fv(t)]+[(S^{-1})^+\overline{b}(t)-(S^{-1})^-\underline{b}(t)-S^{-1}b(t)]$. The matrix R is Schur stable and nonnegative, thus all arguments of Theorem 1 are valid to substantiate that $\underline{z}(t)\le z(t)\le\overline{z}(t)$ for all $t\in\mathbb{Z}_+$ (by construction $\underline{z}(0)\le z(0)\le\overline{z}(0)$). Next, using the relations (1) we get the theorem claim. ■

B. Cooperative representation in the same coordinates

There is another possibility for an interval observer construction in the case when Assumption 2.ii is not satisfied without a transformation of coordinates, but with more complex stability conditions [16].

Theorem 3. *Let assumptions 1, 3, 4 be satisfied and there exist a matrix $L\in\mathbb{R}^{n\times p}$ such that the matrix*

$$\Sigma=\begin{bmatrix}D^+ & D^-\\ D^- & D^+\end{bmatrix}$$

is Schur stable for $D=A-LC$. Then for all $t\in\mathbb{Z}_+$ the estimates $\underline{x}(t)$, $\overline{x}(t)$ are bounded and

$$\underline{x}(t)\le x(t)\le\overline{x}(t)$$

provided that $\underline{x}(0)\le x(0)\le\overline{x}(0)$, where

$$\begin{aligned}\underline{x}(t+1) &= D^+\underline{x}(t)-D^-\overline{x}(t)+\underline{b}(t)+Ly(t)-\overline{L}V,\\ \overline{x}(t+1) &= D^+\overline{x}(t)-D^-\underline{x}(t)+\overline{b}(t)+Ly(t)+\overline{L}V,\end{aligned} \quad (5)$$

and $\overline{L}=(L^+ + L^-)E_{p\times 1}$.

Proof: The system (2) can be rewritten as follows:

$$\begin{aligned}x(t+1) &= Dx(t)+L(y(t)-v(t))+b(t)\\ &= [D^+ - D^-]x(t)+L(y(t)-v(t))+b(t).\end{aligned}$$

Then the dynamics of the errors $\underline{e}(t)=x(t)-\underline{x}(t)$, $\overline{e}(t)=\overline{x}(t)-x(t)$ obeys the equations:

$$\begin{aligned}\underline{e}(t+1) &= D^+\underline{e}(t)+D^-\overline{e}(t)+\underline{d}(t),\\ \overline{e}(t+1) &= D^+\overline{e}(t)+D^-\underline{e}(t)+\overline{d}(t),\end{aligned} \quad (6)$$

where $\underline{d}(t)=[\overline{L}V-Lv(t)]+[b(t)-\underline{b}(t)]$, $\overline{d}(t)=[\overline{L}V-Lv(t)]+[\overline{b}(t)-b(t)]$. By definition, the matrices D^+, D^-

are nonnegative. According to assumptions 3, 4 the inputs $\underline{d}(t)$, $\overline{d}(t)$ are also nonnegative, thus the estimation error dynamics is cooperative and starting from $\underline{e}(0) \geq 0$, $\overline{e}(0) \geq 0$ we get that $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t \in \mathbb{Z}_+$. Since the matrix Σ is Schur stable and the inputs $\underline{d}(t)$, $\overline{d}(t)$ are bounded, the variables $\underline{e}(t)$, $\overline{e}(t)$ possess the same property, that in combination with Assumption 1 implies boundedness of $\underline{x}(t), \overline{x}(t)$. ■

It is worth doing some comments about stability verification of the matrix Σ. According to [22] a nonnegative matrix Σ is Schur stable if there exists a diagonal positive definite matrix $P_\Sigma \in \mathbb{R}^{2n\times 2n}$ such that $\Sigma^{\mathrm{T}} P_\Sigma \Sigma - P_\Sigma \prec 0$, or this condition can be replaced with existence of a vector $p_\Sigma \in \mathbb{R}_+^{2n}$, $p_\Sigma > 0$ such that $p_\Sigma^{\mathrm{T}}(\Sigma - I_{2n}) < 0$. Due to block symmetric structure of the matrix Σ and its nonnegativeness, the stability of Σ is equivalent to stability of the matrix $\overline{D} = D^+ + D^-$. Indeed consider the copositive Lyapunov function $V(t) = \underline{p}^{\mathrm{T}}\underline{e}(t) + \overline{p}^{\mathrm{T}}\overline{e}(t)$ for some vectors $\underline{p} > 0$, $\overline{p} > 0$ for the system (6), since $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t \in \mathbb{Z}$, then this function is positive definite. Existence of such a Lyapunov function under the condition $\Delta V(t) = V(t+1) - V(t) < 0$, $\forall t \in \mathbb{Z}$, satisfied for any initial conditions $\underline{e}(0)$, $\overline{e}(0) \in \mathbb{R}_+^n$ for the unperturbed system (6) (the system (6) with $\underline{d}(t) = \overline{d}(t) = 0$), is equivalent to its Schur stability. Thus $\Delta V(t) = [\underline{p}^{\mathrm{T}} D^+ + \overline{p}^{\mathrm{T}} D^- - \underline{p}^{\mathrm{T}}]\underline{e}(t) + [\underline{p}^{\mathrm{T}} D^- + \overline{p}^{\mathrm{T}} D^+ - \overline{p}^{\mathrm{T}}]\overline{e}(t)$ and a reasonable choice is $\underline{p} = \overline{p}$, then $\Delta V(t) = \underline{p}^{\mathrm{T}}[\{D^+ + D^-\} - I_{2n}][\underline{e}(t) + \overline{e}(t)]$, that is equivalent to $\underline{p}^{\mathrm{T}}\{\overline{D} - I_n\} < 0$ or the Schur stability of $\overline{D}$.

Verification of stability of $\overline{D}$ with simultaneous computations of L and $\underline{p}$ can be performed via the following LMIs. By its construction, $D = A - LC$, then $D^+ = A^+ - L^+ C$ and $D^- = A^- + L^- C$, where L^+, L^- are the parts of the matrix $L = L^+ - L^-$ with sign indefinite entries which enter into matrices D^+ or D^- respectively. Thus

$$\begin{aligned}\overline{D} &= D^+ + D^- = A^+ - L^+ C + A^- + L^- C \\ &= \overline{A} - (L^+ - L^-)C = \overline{A} - LC.\end{aligned}$$

Now the nonnegative matrix $\overline{D}$ is Schur stable if there exists a diagonal positive definite matrix P such that $\overline{D}^{\mathrm{T}} P \overline{D} - P \prec 0$. Since P is diagonal, then the property $P \succ 0$ implies that all elements of P are positive. By applying Schur complement, the matrix $\overline{D}$ stability follows the facts that there exist a diagonal matrix $P \succ 0$ and a matrix $Y \in \mathbb{R}^{n\times p}$ such that

$$\begin{bmatrix} P & P\overline{A} - YC \\ (P\overline{A} - YC)^{\mathrm{T}} & P \end{bmatrix} \succ 0 \qquad (7)$$

under the constraint

$$P\overline{A} - YC \geq 0, \qquad (8)$$

where $L = P^{-1}Y$. Since the matrix P is elementwise positive the constraint (8) implies that the matrix $\overline{D} = \overline{A} - LC$ is nonnegative, while the LMI (7) ensures the Schur stability of $\overline{D}$. Therefore the following corollary can be formulated.

Corollary 1. *Let assumptions 1, 3, 4 be satisfied and there exist a diagonal matrix $P \succ 0$ and a matrix $Y \in \mathbb{R}^{n\times p}$ such that (7), (8) be true. Then the result of Theorem 3 is valid.*

The main advance of this corollary is that it allows us to use the numerical routines for the matrix L selection, the YALMIP toolbox of Matlab in particular can be used to find a solution of such a constrained problem [23].

Remark 1. It is worth to stress that b could be a function of the state x provided that there exist known bounded signals $\overline{b}, \underline{b}$ satisfying Assumption 3. Therefore, the presented interval observers (3) and (4) can be applied to nonlinear systems in the output canonical form, for instance. A mild reformulation of theorems 1, 2 for this case is skipped for brevity of presentation. Application of these theorems to nonlinear systems is illustrated on example in Section 4.

In all cases, the width of the estimated interval (after some transients) is proportional to the system uncertainty (i.e., the bounds $\underline{b}$,$\overline{b}$ and V). Improvement of interval estimation accuracy can be achieved optimizing the value of the observer gain L in H_∞ sense, for example.

IV. EXAMPLES

In this section we consider two examples. The first one is an academic planar system for which we will compare the interval observers proposed in theorems 2 and 3. The second example is the Hénon chaotic system.

A. Comparison of observers (4) and (5)

Consider the following system

$$\begin{aligned}x(t+1) &= Ax(t) + b(t), \\ y(t) &= Cx(t) + v(t), \\ b(t) &= b_0(t) + \delta b(t, x(t)),\end{aligned}$$

where $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$, $v(t) \in \mathbb{R}$ are respectively the state, the output and the measurement noise ($||v|| \leq V = 0.1$, for simulation we used $v(t) = V\sin(t)$). The signals

$$\begin{aligned}b_0(t) &= [\sin(0.1t)\ \cos(0.2t)]^{\mathrm{T}}, \\ \delta b(t, x(t)) &= \delta[\sin(0.5t\, x_2(t))\ \sin(0.3t)]^{\mathrm{T}}\end{aligned}$$

are the known portion of b and its uncertain deviation δb from the nominal b_0 from which we know that it is bounded by $\delta = 0.5$. Thus $\underline{b}(t) = b_0(t) - \delta$, $\overline{b}(t) = b_0(t) + \delta$ and assumptions 3, 4 are satisfied. Finally,

$$A = \begin{bmatrix} 0.3 & -0.7 \\ 0.6 & -0.5 \end{bmatrix},\ C = [1\ 0],$$

and there is no observer gain $L \in \mathbb{R}^2$ making the matrix $D = A - LC$ nonnegative. The matrix A is Schur stable (it has complex eigenvalues), but the matrix $\overline{A} = A^+ + A^-$ is unstable. Applying Matlab YALMIP toolbox we obtain

$$L = [0.3\ 0.6]^{\mathrm{T}},$$

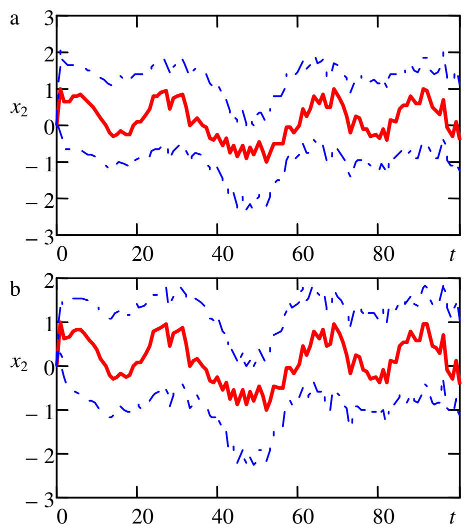

Figure 1. Comparison of interval estimation performed by the observer (4) (part (a)) and (5) (part (b))

then $\lambda(D) = [0 \ \ -0.5]^{\mathrm{T}}$ and $\lambda(\overline{D}) = [0 \ 0.5]^{\mathrm{T}}$, therefore all conditions of Theorem 3 are satisfied for this choice of L. The matrices

$$R = \begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix}, \; S = \begin{bmatrix} 0.609 & 0.814 \\ -1.162 & 0.581 \end{bmatrix}$$

satisfy to all conditions of Theorem 2. Therefore now we can apply both observers (4) and (5) in this examples. The results of simulation for the interval estimation of the unmeasured coordinate x_2 are shown in Fig. 1. As we can conclude in this examples the observers demonstrate a similar performance. An advantage of (5) is that we can use the LMI techniques to calculate L, however performance in this case is critically dependent on ability to increase the stability margin of $\overline{D}$ by L, which is a harder problem than stabilization of D required for (4). From another side, performance of the observer (4) is influenced by additional transformation of coordinates S, which also decreases the accuracy of estimation.

B. Hénon chaotic system

Consider a variant of the Hénon model:

$$\begin{aligned} x(t+1) &= Ax(t) + r[1 - a(t)x_1^2(t) + d(t)], \\ y(t) &= x_1(t) + v(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^2$ is the state, $y(t) \in \mathbb{R}$ is the output, $v(t) \in \mathbb{R}$ is the measurement noise ($||v|| \le V = 0.1$ and Assumption 4 is satisfied) and $d(t) \in \mathbb{R}$ is the disturbance ($||d|| \le \delta = 0.015$),

$$A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}, \; r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \underline{a} \le a(t) \le \overline{a},$$
$$\underline{a} = 1, \; \overline{a} = 1.4.$$

The system has an uncertain parameter $a(t)$ with the values from an interval. If $a(t) = \overline{a}$, then the system equations

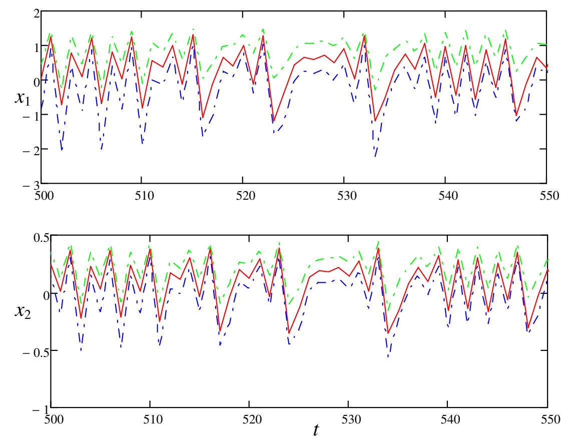

Figure 2. The results of simulation for Hénon model

become identical to the chaotic Hénon model. Thus assume that $||x|| < +\infty$ (Assumption 1 holds). Let us rewrite the system as follows:

$$x(t+1) = Ax(t) + b(t), \, b(t) = r[1 - a(t)[y(t) - v(t)]^2 + d(t)]$$

then clearly it is in the form (2) and Assumption 3 is valid for

$$\begin{aligned} \underline{b}(t) &= r[1 - \overline{a}y^2(t) - 2\overline{a}|y(t)|V - \overline{a}V^2 - \delta], \\ \overline{b}(t) &= r[1 - \underline{a}y^2(t) + 2\overline{a}|y(t)|V - \underline{a}V^2 + \delta], \end{aligned}$$

that justifies Remark 1. Finally, Assumption 2 is verified for $L = [-0.1 \ 0.1]^{\mathrm{T}}$ and $C = [1 \ 0]$ (the matrix $A - LC$ is Schur stable and nonnegative). Therefore, all conditions of Theorem 1 are satisfied and the interval observer (3) solves the problem of interval state estimation. The results of simulation are shown in Fig 2.

V. Conclusion

The paper is devoted to interval observer design for the LTI discrete-time systems. Two techniques have been proposed. The first one is based on a static transformation of coordinates, which connects a stable LTI discrete-time system with its nonnegative representation. The second technique uses a nonlinear transformation of the system in a nonnegative form, the observer gain can be calculated as a solution of LMIs. The efficiency is shown on example of computer simulation for a chaotic system.

In comparison with continuous-time systems, the discrete-time interval observers admit a relaxation of some applicability conditions (there are more results on design of a nonnegative matrix rather than a Metzler one [20], [21]) and always there exists an interval observer (5) with a cooperative estimation error dynamics.

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