ROBUST CONTROL OF A SOLIDIFICATION PROCESS WITH PARAMETRIC UNCERTAINTY

Beathe Furenes*, Bernt Lie*

* Telemark University College, P.O. Box 203, N-3901 Porsgrunn, Norway

Abstract: Robust control of a solidification process is considered. In solidification processes, the properties of the solidified material are significantly determined by the solidification rate, defined as the time derivative of the solid/liquid interface position. Most often the position can not be measured directly, and thus the interface position must be estimated in order to employ feedback control. The solidification model has a variable structure where the state equations depend on the interface position. An observer based on a linearized model is developed. A gain-scheduled PI-controller is implemented, and the robustness properties are tested for parametric uncertainty in the model.

Copyright ©2007 IFAC.

Keywords: Robust control, state estimation, phase transition

1. INTRODUCTION

Robustness of a system refers to a system that functions adequately for all admissible uncertainties. Sources for system uncertainty may be poor plant knowledge, uncertain or slowly varying parameters, nonlinearities (e.g. hysteresis or friction), errors or unknown initial conditions, reduced-order models, additive unknown internal or external noise, and environmental influence (Weinmann, 1991).

In some systems, there are no measurements of the output to be controlled. Instead, other output variables are measured. Then, if the system is observable, the controlled output can be estimated from the other measured outputs. The separation principle states that for linear systems the state feedback control and an observer can be designed independently (Friedland, 1986). However, a critical assumption for the separation principle is that the observer includes an exact dynamic model of the plant, which is almost never valid in reality since the precise dynamic model is rarely known. Thus, it is of importance to check the robustness properties of the system when the controller and observer are designed through the use of the separation principle.

In solidification processes, the solidification rate significantly determines the properties of the solidified material. Some processes require certain solidification rates in order to achieve well-defined microstructures, and the solidification rate may also affect the capturing of impurities in the solid phase. By controlling the solidification rate, the product quality can often be improved.

In order to employ feedback control of the solidification rate, a measurement or reliable online estimate of the interface position must be available. An example of direct measurement of the interface position is given in Drevermann et al. (2004b). The movement of the interface is measured by an ultrasonic pulse-echo technique which can be used for rod-like sample geometries.
2. A GENERAL CONTROL CONFIGURATION WITH UNCERTAINTY

Many linear control problems including model uncertainty can be formulated using the block diagram in figure 1 (Skogestad and Postlethwaite, 1996). $P$ is the generalized plant model, $K$ is the controller, and the perturbation $\Delta$, assumed to be bounded, represents the uncertainty. The control signal $u$ is generated from the controller, $K$, based on information in the sensed outputs $v$ (e.g. commands, measured plant outputs, measured disturbances, etc.). The exogenous inputs $w$ may be commands, disturbances, and noise. The exogenous outputs, $y$, are error signals to be minimized. $u_\Delta$ is the perturbation input (the output of the feedback perturbation $\Delta$), whereas $y_\Delta$ is the perturbation output (the input to the feedback perturbation $\Delta$).
Fig. 3. MΔ-structure for robust closed-loop stability analysis. M = \( N_{y\Delta u\Delta} \) is the transfer function from \( u\Delta \) to \( y\Delta \).

2.1 Robust stability for unstructured uncertainty

Unstructured uncertainty is often referred to as full-block complex perturbation uncertainty. \( \Delta \) is then allowed to be any (full) complex transfer function matrix satisfying \( \|\Delta\|_\infty \leq 1 \).

If \( M(s) \) is nominally stable and the perturbations \( \Delta(s) \) are stable, then the \( M\Delta \)-structure in figure 3 is stable for all perturbations \( \Delta \) satisfying \( \|\Delta\|_\infty \leq 1 \) if and only if (Skogestad and Postlethwaite, 1996)

\[
\sigma[M(j\omega)] < 1, \quad \forall \omega \Leftrightarrow \|M\|_\infty < 1. \tag{3}
\]

2.2 Robust stability for structured uncertainty

If the uncertainty in the model is structured, i.e. \( \Delta \) is block-diagonal, the above stability condition (3) is often too conservative. It is then desirable to obtain a tighter robust stability condition.

The structured singular value \( \mu(M) \) depends on both \( M \) and \( \Delta \), and thus takes advantage of the structure in the uncertainty. \( \mu(M) \) is defined as (Skogestad and Postlethwaite, 1996)

\[
\mu(M)^{-1} \triangleq \min_{\Delta} \{ \sigma(\Delta) | \det(I - M\Delta) = 0 \} . \tag{4}
\]

A general feedback system is internally stable for all possible block-diagonal perturbations with \( \sigma[M(j\omega)] < 1, \forall \omega \), if and only if the nominal closed-loop system is internally stable and

\[
\mu[M(j\omega)] \leq 1, \quad \forall \omega. \tag{5}
\]

3. SYSTEM DEFINITION

The solidification process studied in this paper is sketched in figure 4. A mould with height \( L \), assumed to be be covered by a thick layer of insulation on the vertical surfaces, is situated in a solidification chamber and is initially filled with melt. Heat is extracted from the melt by cooling the bottom mould wall at a rate \( Q_0 \). At the top of the mould, the melt is heated at a rate \( Q_L \) by imposing electrical current on a resistance heater. As the solid begins to form, the interface position, \( h \), and the temperature profiles \( T_s(t, z) \) and \( T_l(t, z) \), will change with time; subscripts \( s \) and \( l \) refer to solid and liquid phase, respectively. For simplicity \( Q_0 \) is assumed to be constant during solidification, whereas \( Q_L \) is used to control the solidification rate. The number of available (temperature) measurements in the melt are \( n_y \), and the estimate of the interface position is based on the measured temperatures.

The mathematical model of this process is based on the heat diffusion equation. There is one fixed boundary condition and one moving boundary condition for both phases. The independent variables are time \( t \) and position \( z \), and the resulting model is a system of two PDEs and one ODE. The spatial domain is split into two subdomains by the interface, one for each phase.

The discretization method is based on a fixed-grid finite-difference method, and is mainly taken from Chun and Park (2000). Details about the implementation of the model are found in Furenes and Lie (2006a).

The state vector with dimension \( n_x = N + 1 \) is given as

\[
x = (x_1 \cdots x_N x_{N+1})^T = (T_1 \cdots T_N h)^T \tag{7}
\]

where \( T_1, \ldots, T_N \) are the temperatures in the spatial grid cells and \( N \) is the number of grid cells. \( h \) is the interface position. The input variables are

\[
u = (u_1 u_2)^T = (\dot{Q}_0 \dot{Q}_L)^T \tag{8}
\]

where, for simplicity, \( \dot{Q}_0 \) is assumed to be constant. Temperature measurements are available at \( n_y \) positions in the melt

\[
y_m = (y_1 \cdots y_{n_y})^T, \tag{9}
\]

where \( n_y \leq N \).
Table 1. State definitions for the solidification model.

<table>
<thead>
<tr>
<th>State definitions for the temperatures</th>
<th>Node no. (n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dT_n}{dt} = \alpha_n \frac{T_{n+1} - T_n}{(\Delta z)^2} - \frac{Q_0}{Ab_n \Delta z}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{dT_n}{dt} = \alpha_n \frac{T_{n+1} - 2T_n + T_{n-1}}{(\Delta z)^2}$</td>
<td>(2, i - 1), (i + 1, N - 1)</td>
</tr>
<tr>
<td>$\frac{dT_n}{dt} = \alpha_n \frac{T_{n+1} - 2T_n + \frac{k_x}{\rho} T_{n-1} + T_m (1 - \frac{\Delta z}{\rho})}{(\Delta z)^2}$</td>
<td>i</td>
</tr>
<tr>
<td>$\frac{dT_n}{dt} = \alpha_n \frac{T_{n+1} - 2T_n + \frac{k_x}{\rho} T_{n-1} + T_m (1 - \frac{\Delta z}{\rho})}{(\Delta z)^2}$ + $\frac{\alpha_n (\Delta H_f) dh}{k_t \Delta z}$</td>
<td>i + 1</td>
</tr>
<tr>
<td>$\frac{dT_n}{dt} = 2\alpha_n \frac{Q_i}{Ab_n \Delta z} - \frac{T_n - T_{n-1}}{(\Delta z)^2}$</td>
<td>N</td>
</tr>
</tbody>
</table>

State definition for the interface position

$\frac{dh}{dt} = \frac{k_x}{\rho} \left( 1 - \frac{\Delta z}{\rho} \right)$

$\frac{dh}{dt} = \frac{k_l}{\Delta H_f} \left( 1 - \frac{\Delta z}{\rho} \right)$

$\frac{dh}{dt} = \frac{k_l}{\Delta H_f} \left( 1 - \frac{\Delta z}{\rho} \right)$

$\frac{dh}{dt} = \frac{k_l}{\Delta H_f} \left( 1 - \frac{\Delta z}{\rho} \right)$

The controlled output is $y_c = h$.

The definition of the state equations for the temperatures and the position vary with the nodal index $i$, where the interface position is located between node indices $i$ and $i + 1$. The model thus has a variable structure of the system of the form

$dx_k = f_k(x, u, i) dt$ for $k = 1, ..., N + 1$,

where $i$ can be found by comparing the nodal temperatures with the melting temperature, $T_m$:

$i = \max \left( T_k \leq T_m \right) , \quad k = 1, ..., N$.

The state space model for the temperatures and the interface position are given in Table 1. The dimensionless distance from the interface position $h$ to node $i$ is given by

$\zeta = \zeta(h) = \frac{h - i \Delta z}{\Delta z}$.

The grid size is $\Delta z = L/\rho$ and $\bar{k} = k/\rho$.

Figure 5 shows the scaled temperature distribution and the interface position as functions of time at constant heating and cooling conditions.

4. STATE ESTIMATION

4.1 Linearization of the model

Since the solidification process is a batch process, there exists no single operating point around which the system can be linearized (Bouvin, 1998). The solidification model undergoes structural changes in the definition of the state variables each time the interface position crosses a grid line (when the nodal index $i$ changes), and hence the system is linearized around a trajectory given by the interface position. The $A$ and $B$ matrices will change with the nodal index $i$, and hence the $A$ and $B$ matrices need to be calculated for every index $i \in 1, ..., N$, whereas the $C$-matrix is independent of the index. The procedure of the switching between the different systems is handled by the events function, and is described in (Furenes and Lie, 2006b).

4.2 Calculation of steady state Kalman filter gains

The different $A$ matrices were used as inputs to the steady state Kalman filter gain algorithm (see e.g. (Lewis, 1986) for more information on Kalman filter theory), and thus the Kalman gain $K_f$ values also depends on the interface position. The switching between the different filter gains follows the model switching.

5. CONTROLLER DESIGN AND NOMINAL STABILITY

A gain-scheduled PI controller is used. The stability margins for 3 operating points are shown in Table 2, where $S = (I + L)^{-1}$, $T = (I + L)^{-1} L$, $L = GK_c$, and $G = E (sI - A)^{-1} B$. It should be noted that even though the linearized system is stable, this does not guarantee the overall nonlinear system to be stable.
The signals for the generalized plant are identified in Table 3.

### Table 3. Definition of the signals in the uncertain system.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensed outputs:</td>
<td>$v = (e_c e_f)^T = (r - y_n - y_m)$</td>
</tr>
<tr>
<td>Control signals:</td>
<td>$u = (u_c u_f)^T$</td>
</tr>
<tr>
<td>Exogenous inputs:</td>
<td>$w = (r d n)^T$</td>
</tr>
<tr>
<td>Exogenous outputs:</td>
<td>$z = (e_c e_f)^T$</td>
</tr>
<tr>
<td>Uncertain inputs:</td>
<td>$u_{\Delta} = (u_{\Delta}^c u_{\Delta}^f)^T$</td>
</tr>
<tr>
<td>Uncertain outputs:</td>
<td>$y_{\Delta} = (y_{\Delta}^c y_{\Delta}^f)^T$</td>
</tr>
</tbody>
</table>

6.2 Robust stability tests

The $M_{\Delta}$-structure in figure 3 is used for robust stability analysis. The upper bound of $\mu(M)$ is calculated using the Matlab Robust Control toolbox command `mu`. Figure 7 shows $\mu(M)$ and $\hat{\sigma}(M)$ for two different levels of uncertainty in the $A$ and $B$-matrices (10% and 40% in both) at one operating point ($h = 0.7$). The figure shows that the condition $\hat{\sigma}(M) \leq 1$ is too conservative when
Δ is block-diagonal. According to the robust stability condition in (5), the controller gain should be detuned in order to achieve robust stability for uncertainties larger than 40% at operating point $h = 0.7$. The test for robust stability should be performed at all operating points. There is no guarantee that the tests of the linearized subsystems holds for the total nonlinear system, but such robust stability will give a hint about the robustness properties for the nonlinear system.

7. CONCLUSIONS

The goal of the reported work is to study the robustness properties of a solidification system with a Kalman filter and a gain-scheduled PI-controller. Parametric uncertainty is introduced to the state space model, and by QR-factorization a simple method for pulling out the uncertainty matrix is illustrated. The test cases show that the singular value criteria is too conservative when the uncertainty is represented by a block-diagonal matrix.

In this study, the radiation and conduction dynamics for the control devices at the boundaries have been neglected. Also, in the simulated cases, temperature measurements are assumed to be available in the melt. However, in most practical industrial processes, only temperature measurements outside the mould wall are available. The method proposed in this paper, can be utilized for more realistic instrumentation settings. Then the model should be extended to include state variables of the temperatures outside the mould walls, and more complex boundary conditions must be modeled. The model extensions as well as validation with plant data is ongoing and will be included in future work.

REFERENCES


