FAULT DETECTION AND ISOLATION IN NON-UNIFORMLY SAMPLED SYSTEMS

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Abstract: This paper considers fault detection and isolation (FDI) in a non-uniformly sampled multirate system. By extending the Chow-Willsky scheme from single rate systems to multirate systems, one generates a primary residual vector (PRV) for fault detection. Further, by structuring the PRV to have different sensitivity/insensitivity to different faults, fault isolation is performed. The power of the proposed FDI scheme is illustrated via numerical examples.

Keywords: non-uniformly sampled multirate multivariate systems, fault detection and isolation, primary residual vector, structured residual vectors

1. INTRODUCTION

This work proposes a novel approach towards fault detection and isolation (FDI) in a non-uniformly sampled multirate multivariate system. A dynamic discrete-time (DT) system is referred to as *multirate* wherein variables are sampled with different rates. Further, the sampling is called *uniform/non-uniform* if a variable is sampled with *equally/non-equally* spaced intervals.

Multirate systems are common in the process industries. In many cases, sampling all variables in a process at a single rate is not possible, because of delays in sensors and laboratory analysis. Recently, research effort has shifted to FDI in multirate systems. Studies by Fadali and co-workers (Fadali and Liu, 1999; Fadali and Shabaik, 2002) consider processes where all variables are uniformly (or regularly) sampled with different rates. This paper considers FDI in a more general case: each variable in a continuous-time (CT) dynamic system is non-uniformly (or irregularly) sampled with a different rate. This represents a very general starting point. All other systems are subsets of this case. There is a lack of formal FDI methodology for this general case, to our best knowledge.

When sampling a CT system with different rates, the *lifting* technique can be used to convert the time-varying multirate DT system into a single rate time-invariant DT system (Khargonekar *et al.*, 1985). Consider a CT system represented by the state space model. We utilize the non-uniformly sampling technique (Sheng *et al.*, 2002) to *lift* the system. Then we extend the Chow-Willsky scheme (Chow and Willsky, 1984) to the lifted system to generate a primary residual vector (PRV) for fault detection. Furthermore, the PRV is transformed into a set of structured residual vectors (SRVs) for fault isolation. Eventually, the practicality and utility of the proposed FDI algorithms is illustrated by application to two numerical examples.

2. PROBLEM FORMULATION

Consider a dynamic system, which in the *fault-free* case is represented by the following CT state space equation

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$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\tilde{\mathbf{u}}(t) + \boldsymbol{\phi}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\tilde{\mathbf{u}}(t) \quad (1)$$

where (i) $\tilde{\mathbf{u}}(t) \in \Re^l$ and $\tilde{\mathbf{y}}(t) \in \Re^m$ are *noise-free* inputs and outputs, respectively; (ii) $\mathbf{x}(t) \in \Re^n$ is the state; (iii) $\boldsymbol{\phi}(t) \in \Re^n$ is the disturbance and can be any function of time; and (iv) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are known system matrices with appropriate dimensions.

Given a *frame period* T, throughout the paper we sample the variables in the following manners (Sheng et al., 2002):

- $\tilde{\mathbf{u}}(t)$ is sampled g times over the period $[kT, kT + T)$ at instants: $\{kT + t_1, kT + t_2, \cdots, kT + t_g\}$, where $0 = t_1 < \cdots < t_g < T$.
- $\tilde{\mathbf{y}}(t)$ is sampled p times over the same period. Within $[kT + t_i, kT + t_{i+1}]$, $\forall\ i \in [1, g]$, n_i (≥ 0) samples are taken at instants: $\{kT + t_i^1, \cdots, kT + t_i^{n_i}\}$. Similarly, $t_i \leq t_i^1 \cdots < t_i^{n_i} < t_{i+1}$, where $t_{g+1} = T$ and $p = n_1 + \cdots + n_g$.

Among the variables of the system represented by Eqn. 1, each one can be sampled differently from the others. However, for simplicity of mathematical manipulation, in the sequel we assume that (i) the inputs and the disturbances are sampled in one manner; and (ii) the m outputs are sampled in a different manner.

We consider the case of errors-in-variables (EIV). Accordingly, we denote the observed *fault-free* inputs at $kT + t_i$, $\forall\ i \in [1, g]$, by $\mathbf{u}^*(kT + t_i) = \tilde{\mathbf{u}}(kT + t_i) + \mathbf{v}(kT + t_i)$. Similarly, at $kT + t_i^j$, $\forall\ j \in [1, n_i]$, the fault-free outputs are $\mathbf{y}^*(kT + t_i^j) = \tilde{\mathbf{y}}(kT + t_i^j) + \mathbf{o}(kT + t_i^j)$. We assume $\mathbf{v}(\ )$ and $\mathbf{o}(\ )$ to be independently Gaussian distributed white noise vectors having respective covariances $\mathbf{R}_v$ and $\mathbf{R}_o$.

If the sensors are faulty, the measured outputs, $\forall\ j \in [1, n_i]$, can be represented by $\mathbf{y}(kT + t_i^j) = \mathbf{y}^*(kT + t_i^j) + \boldsymbol{\Xi}_y \mathbf{f}_y(kT + t_i^j)$, where $\boldsymbol{\Xi}_y \in \Re^{m \times d_y}$ is a matrix of fault directions; and $\mathbf{f}_y(kT + t_i^j) \in \Re^{d_y}$ are fault magnitude vectors. To represent a single sensor fault in the i^{th} output sensor, $\forall\ i \in [1, m]$ $(d_y = 1)$, $\boldsymbol{\Xi}_y \in \Re^m$ is simply the i^{th} column of the $m \times m$ identity matrix $\mathbf{I}_m$. It must be emphasized that $\mathbf{I}_x$ always refers to an $x \times x$ identity matrix throughout the paper.

The sampled inputs are similarly represented by $\mathbf{u}(kT + t_i) = \mathbf{u}^*(kT + t_i) + \boldsymbol{\Xi}_u \mathbf{f}_u(kT + t_i)$, where $\boldsymbol{\Xi}_u \in \Re^{l \times d_u}$ and $\mathbf{f}_u(kT + t_i) \in \Re^{d_u}$ resemble $\boldsymbol{\Xi}_y$ and $\mathbf{f}_y(kT + t_i^j)$, respectively.

With $\{\mathbf{u}(kT + t_i)\}$ and $\{\mathbf{y}(kT + t_i^j)\}$, $\forall\ i \in [1, g]$, $j \in [1, n_i]$, and $k = [1, \cdots)$, the goal of FDI is to indicate when $\mathbf{f}_u(kT + t_i)$ and/or $\mathbf{f}_y(kT + t_i^j)$ are non-zero; and further identify $\boldsymbol{\Xi}_u$ and/or $\boldsymbol{\Xi}_y$.

3. NON-UNIFORMLY SAMPLED SYSTEMS

Post-multiplying Eqn. 1 by a non-singular $\mathbf{e}^{-\mathbf{A}t}$, $\forall\ t \neq 0$, leads to

$$d\left[\mathbf{e}^{-\mathbf{A}t}\mathbf{x}(t)\right]/dt = \mathbf{e}^{-\mathbf{A}t}\left[\mathbf{B}\tilde{\mathbf{u}}(t) + \boldsymbol{\phi}(t)\right] \quad (2)$$

Integrating Eqn. 2 gives

$$\mathbf{x}(k+1) = \int_{kT}^{kT+T} \mathbf{e}^{\mathbf{A}(kT+T-t)}\left[\mathbf{B}\tilde{\mathbf{u}}(t) + \boldsymbol{\phi}(t)\right]dt + \underline{\mathbf{A}}\ \mathbf{x}(k) \quad (3)$$

where $\mathbf{x}(k) \equiv \mathbf{x}(t)|_{t=kT}$, $\mathbf{x}(k+1) \equiv \mathbf{x}(t)|_{t=kT+T}$, and $\underline{\mathbf{A}} \equiv \mathbf{e}^{\mathbf{A}T}$. Consequently, the following lifted state space equation can be derived,

$$\mathbf{x}(k+1) = \underline{\mathbf{A}}\ \mathbf{x}(k) + \underline{\mathbf{B}}\ \underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{E}}\ \underline{\boldsymbol{\phi}}(k) \quad (4)$$

where, $\forall\ i \in [1, g]$, $\underline{\mathbf{B}} = [\mathbf{B}_1\ \mathbf{B}_2\ \cdots\ \mathbf{B}_g]$, and $\underline{\mathbf{E}} = [\mathbf{E}_1\ \mathbf{E}_2\ \cdots\ \mathbf{E}_g]$ with

$$\mathbf{B}_i = \int_{T-t_{i+1}}^{T-t_i} \mathbf{e}^{\mathbf{A}t}\mathbf{B}dt,\ \mathbf{E}_i = \int_{T-t_{i+1}}^{T-t_i} \mathbf{e}^{\mathbf{A}t}dt,$$

$$\underline{\tilde{\mathbf{u}}}(k) = \begin{bmatrix} \tilde{\mathbf{u}}(kT + t_1) \\ \vdots \\ \tilde{\mathbf{u}}(kT + t_g) \end{bmatrix};\ \underline{\boldsymbol{\phi}}(k) = \begin{bmatrix} \boldsymbol{\phi}(kT + t_1) \\ \vdots \\ \boldsymbol{\phi}(kT + t_g) \end{bmatrix}.$$

It is arguable that $\underline{\boldsymbol{\phi}}(k)$ is not available. As will be shown later, $\underline{\boldsymbol{\phi}}(k)$ will be entirely removed from the PRV.

The integration of Eqn. 2 from $t \in [kT, kT + \tau]$, $\forall\ 0 < \tau < T$, can result in $\mathbf{x}(kT + \tau)$, whose substitution into Eqn. 1 leads to

$$\tilde{\mathbf{y}}(kT + \tau) = \mathbf{C}\int_{kT}^{kT+\tau} \mathbf{e}^{\mathbf{A}(kT+\tau-t)}\left[\mathbf{B}\tilde{\mathbf{u}}(t) + \boldsymbol{\phi}(t)\right]dt + \mathbf{C}\mathbf{e}^{\mathbf{A}\tau}\mathbf{x}(k) + \mathbf{D}\tilde{\mathbf{u}}(kT + \tau)$$

After defining the lifted output vector:

$$\underline{\tilde{\mathbf{y}}}(k) \equiv \begin{bmatrix} \tilde{\mathbf{y}}(kT + t_1^1) \\ \vdots \\ \tilde{\mathbf{y}}(kT + t_1^{n_1}) \\ \vdots \\ \tilde{\mathbf{y}}(kT + t_g^1) \\ \vdots \\ \tilde{\mathbf{y}}(kT + t_g^{n_g}) \end{bmatrix} \in \Re^{mp},$$

one can arrive at the following lifted output equation:

$$\underline{\tilde{\mathbf{y}}}(k) = \underline{\mathbf{C}}\ \mathbf{x}(k) + \underline{\mathbf{D}}\ \underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{J}}\ \underline{\boldsymbol{\phi}}(k) \qquad (5)$$

where, $\underline{\mathbf{C}} = \left[\, \mathbf{C}_1'\ \mathbf{C}_2'\ \ldots \mathbf{C}_g' \,\right]'$ with $'$ symbolizing the transpose of the argument;

$$\underline{\mathbf{D}} = \begin{bmatrix} \mathbf{D}_{1,1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} & \mathbf{0} & \vdots \\ \vdots & & \ddots & \\ \mathbf{D}_{g,1} & \mathbf{D}_{g,2} & \mathbf{D}_{g,3} \ldots & \mathbf{D}_{g,g} \end{bmatrix};$$

and $\underline{\mathbf{J}} = \underline{\mathbf{D}}|_{\mathbf{D}_{i_1,i_2}=\mathbf{J}_{i_1,i_2}}$ with $i_1 \in [1,g]$ and $i_2 \in [1,i_1]$. Note that in the preceding matrices,

$$\mathbf{C}_i = \begin{bmatrix} \mathbf{C}e^{\mathbf{A}t_i^1} \\ \mathbf{C}e^{\mathbf{A}t_i^2} \\ \vdots \\ \mathbf{C}e^{\mathbf{A}t_i^{n_i}} \end{bmatrix},\ \mathbf{D}_{i,i} = \begin{bmatrix} \mathbf{C}\int_0^{t_i^1} e^{\mathbf{A}t}\mathbf{B}dt + \mathbf{D} \\ \vdots \\ \mathbf{C}\int_0^{t_i^{n_i}} e^{\mathbf{A}t}\mathbf{B}dt + \mathbf{D} \end{bmatrix},$$

$$\mathbf{D}_{i,j} = [\int_{t_i^1-t_{j+1}}^{t_i^1-t_j} \left(\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\right)' dt \cdots \int_{t_i^{n_i}-t_{j+1}}^{t_i^{n_i}-t_j} \left(\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\right)' dt]',$$

and $\mathbf{J}_{i,i} = \mathbf{D}_{i,i}|_{\mathbf{B}=\mathbf{I}_n,\mathbf{D}=\mathbf{0}}$, $\mathbf{J}_{i,j} = \mathbf{D}_{i,j}|_{\mathbf{B}=\mathbf{I}_n}$, $\forall j \in [1, i-1]$. The detailed derivation of Eqns. 4 and 5 can be seen in Sheng et al. (2002).

Combining Eqns. 4 and 5 eventually leads to

$$\begin{aligned} \mathbf{x}(k+1) &= \underline{\mathbf{A}}\ \mathbf{x}(k) + \underline{\mathbf{B}}\ \underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{E}}\ \underline{\boldsymbol{\phi}}(k) \\ \underline{\tilde{\mathbf{y}}}(k) &= \underline{\mathbf{C}}\ \mathbf{x}(k) + \underline{\mathbf{D}}\ \underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{J}}\ \underline{\boldsymbol{\phi}}(k) \qquad (6) \end{aligned}$$

which is the lifted model of Eqn. 1 when non-uniformly sampled.

4. FAULT DETECTION

4.1 *Description of the lifted system with faults*

We define a stacked vector:

$$\underline{\tilde{\mathbf{y}}}_s(k) \equiv [\underline{\tilde{\mathbf{y}}}'(k-s)\ \underline{\tilde{\mathbf{y}}}'(k-s+1)\ \ldots \underline{\tilde{\mathbf{y}}}'(k)]',$$

where s is the order of the parity space (Chow and Willsky, 1984). We select $s = n$ for simplicity. In the sequel, any stacked vector is defined analogously to $\underline{\tilde{\mathbf{y}}}_s(k)$. Manipulating Eqn. 6 gives

$$\underline{\tilde{\mathbf{y}}}_s(k) = \underline{\boldsymbol{\Gamma}}_s\mathbf{x}(k-s) + \underline{\mathbf{H}}_s\underline{\tilde{\mathbf{u}}}_s(k) + \underline{\mathbf{G}}_s\underline{\boldsymbol{\phi}}_s(k) \quad (7)$$

where $\underline{\tilde{\mathbf{u}}}_s(k)$ and $\underline{\boldsymbol{\phi}}_s(k)$ are also stacked vectors,

$$\underline{\boldsymbol{\Gamma}}_s = [\underline{\mathbf{C}}'\ \ \underline{\mathbf{A}}'\underline{\mathbf{C}}'\ \ \ldots (\underline{\mathbf{A}}^s)'\ \underline{\mathbf{C}}']' \in \Re^{mp(s+1)\times n};$$

$$\underline{\mathbf{H}}_s = \begin{bmatrix} \underline{\mathbf{D}} & \mathbf{0} & \ldots\ \mathbf{0} \\ \underline{\mathbf{C}}\ \underline{\mathbf{B}} & \underline{\mathbf{D}} & \vdots \\ \vdots & & \ddots \\ \underline{\mathbf{C}}\ \underline{\mathbf{A}}^{s-1}\underline{\mathbf{B}} & \underline{\mathbf{C}}\ \underline{\mathbf{A}}^{s-2}\underline{\mathbf{B}} & \ldots\ \underline{\mathbf{D}} \end{bmatrix};$$

and $\underline{\mathbf{G}}_s = \underline{\mathbf{H}}_s|_{\underline{\mathbf{B}}=\underline{\mathbf{E}},\underline{\mathbf{D}}=\underline{\mathbf{J}}}$.

It follows from $\mathbf{u}(k)$ and $\mathbf{y}(k)$ that the lifted vectors of measured inputs and outputs are

$$\begin{aligned} \underline{\mathbf{u}}(k) &= \underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{v}}(k) + \boldsymbol{\Xi}_u^g\underline{\mathbf{f}}_u(k) \\ \underline{\mathbf{y}}(k) &= \underline{\tilde{\mathbf{y}}}(k) + \underline{\mathbf{o}}(k) + \boldsymbol{\Xi}_y^p\underline{\mathbf{f}}_y(k) \qquad (8) \end{aligned}$$

where, $\underline{\mathbf{v}}(k) = \underline{\tilde{\mathbf{u}}}(k)|_{\tilde{\mathbf{u}}()=\mathbf{v}()}$, $\underline{\mathbf{f}}_u(k) = \underline{\tilde{\mathbf{u}}}(k)|_{\tilde{\mathbf{u}}()=\mathbf{f}_u()}$, $\underline{\mathbf{o}}(k) = \underline{\tilde{\mathbf{y}}}(k)|_{\tilde{\mathbf{y}}()=\mathbf{o}()}$, and $\underline{\mathbf{f}}_y(k) = \underline{\tilde{\mathbf{u}}}(k)|_{\tilde{\mathbf{u}}()=\mathbf{f}_y()}$. In addition, $\boldsymbol{\Xi}_u^g \equiv \mathbf{I}_g \otimes \boldsymbol{\Xi}_u$; $\boldsymbol{\Xi}_y^p \equiv \mathbf{I}_p \otimes \boldsymbol{\Xi}_y$; and $\otimes$ is the Kronecker tensor product.

Stacking Eqn. 8 facilitates the relationship between the stacked vectors:

$$\begin{aligned} \underline{\mathbf{u}}_s(k) &= \underline{\tilde{\mathbf{u}}}_s(k) + \underline{\mathbf{v}}_s(k) + \boldsymbol{\Xi}_{s,u}^g\underline{\mathbf{f}}_{s,u}(k) \\ \underline{\mathbf{y}}_s(k) &= \underline{\tilde{\mathbf{y}}}_s(k) + \underline{\mathbf{o}}_s(k) + \boldsymbol{\Xi}_{s,y}^p\underline{\mathbf{f}}_{s,y}(k) \qquad (9) \end{aligned}$$

where $\boldsymbol{\Xi}_{s,u}^g = \mathbf{I}_{s+1} \otimes \boldsymbol{\Xi}_u^g$, $\boldsymbol{\Xi}_{s,y}^p = \mathbf{I}_{s+1} \otimes \boldsymbol{\Xi}_y^p$. Moreover, using Eqn. 9 can rewrite Eqn. 7 as

$$\begin{aligned} \underline{\mathbf{y}}_s(k) - \underline{\mathbf{H}}_s\underline{\mathbf{u}}_s(k) &= \underline{\boldsymbol{\Gamma}}_s\mathbf{x}(k-s) - \underline{\mathbf{H}}_s\underline{\mathbf{v}}_s(k) + \underline{\mathbf{o}}_s(k) \\ &- \underline{\mathbf{H}}_s\boldsymbol{\Xi}_{s,u}^g\underline{\mathbf{f}}_{s,u}(k) + \boldsymbol{\Xi}_{s,y}^p\underline{\mathbf{f}}_{s,y}(k) + \underline{\mathbf{G}}_s\underline{\boldsymbol{\phi}}_s(k) \qquad (10) \end{aligned}$$

4.2 *Design of the PRV for fault detection*

We select a matrix $\mathbf{W}_0$ from the *left null space* (LNS) of $\underline{\boldsymbol{\Gamma}}_s^G \equiv [\underline{\boldsymbol{\Gamma}}_s\ \underline{\mathbf{G}}_s]$. By extending the Chow-Willsky scheme (Chow and Willsky, 1984), pre-multiplying both sides of Eqn. 10 by $\mathbf{W}_0$, we obtain the PRV as follows,

$$\begin{aligned} \mathbf{e}_s(k) &\equiv \mathbf{W}_0\left[\underline{\mathbf{y}}_s(k) - \underline{\mathbf{H}}_s\underline{\mathbf{u}}_s(k)\right] \qquad (11) \\ &= \mathbf{W}_0\left[\boldsymbol{\Xi}_{s,y}^p\underline{\mathbf{f}}_{s,y}(k) - \underline{\mathbf{H}}_s\boldsymbol{\Xi}_{s,u}^g\underline{\mathbf{f}}_{s,u}(k)\right] + \mathbf{e}_s^*(k) \end{aligned}$$

where the unknown state vector $\mathbf{x}(k-s)$ and the lifted disturbance $\underline{\boldsymbol{\phi}}_s(k)$ have been completely decoupled. Besides, $\mathbf{e}_s^*(k) = \mathbf{W}_0\left[-\underline{\mathbf{H}}_s\underline{\mathbf{v}}_s(k) + \underline{\mathbf{o}}_s(k)\right]$, which is a zero-mean Gaussian distributed random vector with covariance $\mathbf{R}_{s,e}$ (Johnson and Wichern, 1998), i.e. $\mathbf{e}_s(k) \sim \aleph(\mathbf{0}, \mathbf{R}_{s,e})$.

- In the ideal case, $\mathbf{e}_s(k) = \mathbf{0}$, because $\underline{\mathbf{o}}_s(k) = \mathbf{0}$, $\underline{\mathbf{v}}_s(k) = \mathbf{0}$, $\underline{\mathbf{f}}_{s,u}(k) = \mathbf{0}$, and $\underline{\mathbf{f}}_{s,y}(k) = \mathbf{0}$.
- In the fault-free case, $\mathbf{e}_s(k) = \mathbf{e}_s^*(k)$.
- In the presence of any sensor faults,

$$\mathbf{e}_s(k) = \mathbf{e}_s^*(k) + \mathbf{e}_s^f(k) \qquad (12)$$

where

$$\mathbf{e}_s^f(k) = \mathbf{W}_0 \left[\boldsymbol{\Xi}_{s,y}^p \underline{\mathbf{f}}_{s,y}(k) - \underline{\mathbf{H}}_s \boldsymbol{\Xi}_{s,u}^g \underline{\mathbf{f}}_{s,u}(k)\right]$$

is the fault contribution. In this case, $\mathbf{e}_s(k) \sim \aleph(\mathbf{e}_s^f(k), \mathbf{R}_{s,e})$.

Therefore, fault detection is equivalent to checking if $\mathbf{e}_s(k)$ is *zero-mean*. This can be done by testing if $\eta_s(k) = \mathbf{e}_s'(k)\mathbf{R}_{s,e}^{-1}\mathbf{e}_s(k)$ follows a central chi-square distribution with degrees of freedom equal to Rank($\mathbf{W}_0$) (Johnson and Wichern, 1998). With a pre-selected level of significance α, while $\eta_s(k) < \chi_\alpha^2[\text{Rank}(\mathbf{W}_0)]$ indicates that all sensors function normally, $\eta_s(k) \geq \chi_\alpha^2[\text{Rank}(\mathbf{W}_0)]$ triggers alarming of any faulty sensors.

Finally, we investigate the calculation of $\mathbf{W}_0$. We define Rank($\mathbf{W}_0$) $\equiv (mp - ng)(s+1) - n$ and will use this quantity in the sequel. It can be proved that $\mathbf{W}_0$ has Rank($\mathbf{W}_0$) independent rows and Rank($\mathbf{W}_0$) > 0 if $mp > ng$ and $s = n$. Therefore, a non-trivial solution to $\mathbf{W}_0$ can exist.

Denote $\tilde{\underline{\mathbf{H}}}_s \equiv [\mathbf{I}_{mps+mp} \mid -\underline{\mathbf{H}}_s]$. Then, it follows from Eqn. 11 that $\mathbf{e}_s(k) = \mathbf{W}_0\tilde{\underline{\mathbf{H}}}_s \left[\,\underline{\mathbf{y}}_s'(k) \;\; \underline{\mathbf{u}}_s'(k)\,\right]'$.

$\mathbf{e}_s(k)$ should have maximized sensitivity with any sensors. In line with this criterion, it comes from (Li and Shah, 2002) that

$$\mathbf{W}_0' = \text{the largest eigenvectors of } \underline{\boldsymbol{\Gamma}}_s^{G,\perp}\tilde{\underline{\mathbf{H}}}_s\tilde{\underline{\mathbf{H}}}_s' \text{ related to non-zero eigenvalues,}$$

where $\underline{\boldsymbol{\Gamma}}_s^{G,\perp} = \mathbf{I}_{mps+mp} - \underline{\boldsymbol{\Gamma}}_s^G \left[\left(\underline{\boldsymbol{\Gamma}}_s^G\right)' \underline{\boldsymbol{\Gamma}}_s^G\right]^{-1} \left(\underline{\boldsymbol{\Gamma}}_s^G\right)'$.

We consider the isolation of a *single* faulty sensor at each time. Since the considered system has l inputs and m outputs, the goal of isolation is achievable by generating $(m+l)$ SRVs, where the i^{th} SRV is made insensitive to the i^{th} sensor fault but most sensitive to other sensor faults, $\forall\ i \in [1, m+l]$.

Mathematically, since the i^{th} SRV is

$$\mathbf{r}_{s,i}(k) = \mathbf{W}_i\mathbf{e}_s(k) \tag{13}$$

designing a set of SRVs is equivalent to computing $(m+l)$ transformation matrices $\mathbf{W}_i$.

Introduce a new notation $\underline{\mathbf{P}}_s \equiv \mathbf{W}_0\tilde{\underline{\mathbf{H}}}_s$. Consequently, $\mathbf{e}_s(k) = \underline{\mathbf{P}}_s \left[\,\underline{\mathbf{y}}_s'(k)\; \underline{\mathbf{u}}_s'(k)\,\right]'$, and it follows from Eqn. 13 that $\mathbf{r}_{s,i}(k) = \mathbf{W}_i\underline{\mathbf{P}}_s \left[\underline{\mathbf{y}}_s'(k)\; \underline{\mathbf{u}}_s'(k)\right]'$. Note that Rank$\left(\underline{\mathbf{P}}_s\right)$ = Rank$\left(\mathbf{W}_0\right)$.

Denote, $\forall\ i \in [1, m]$ and $j \in [1, l]$,

$$\underline{\mathbf{y}}_s(k,i) \equiv \begin{bmatrix} \underline{\mathbf{y}}(k-s,i) \\ \vdots \\ \underline{\mathbf{y}}(k,i) \end{bmatrix}, \underline{\mathbf{u}}_s(k,j) \equiv \begin{bmatrix} \underline{\mathbf{u}}(k-s,j) \\ \vdots \\ \underline{\mathbf{u}}(k,j) \end{bmatrix},$$

where

$$\underline{\mathbf{y}}(k,i) = \begin{bmatrix} y(kT+t_1^1,i) \\ \vdots \\ y(kT+t_1^{n_1},i) \\ \vdots \\ y(kT+t_g^1,i) \\ \vdots \\ y(kT+t_g^{n_g},i) \end{bmatrix} \in \Re^p$$

and $\underline{\mathbf{u}}(k,j) = [u(kT+t_1,j)\ \cdots\ u(kT+t_g,j)]'$. Note that $y(kT+\tau,i)$ and $u(kT+\mu,j)$ represent the i^{th} element of $\mathbf{y}(kT+\tau)$ for $\tau \in [t_1^1, \cdots, t_1^{n_1}, \cdots, t_g^1, \cdots, t_g^{n_g}]$, and the j^{th} element of $\mathbf{u}(kT+\mu)$ for $\mu \in [t_1, \cdots, t_g]$, respectively.

We re-organize $\underline{\mathbf{y}}_s(k)$ and $\underline{\mathbf{u}}_s(k)$ as follows,

$$\underline{\mathbf{y}}_s(k) = \begin{bmatrix} \underline{\mathbf{y}}_s(k,1) \\ \vdots \\ \underline{\mathbf{y}}_s(k,m) \end{bmatrix}, \quad \underline{\mathbf{u}}_s(k) = \begin{bmatrix} \underline{\mathbf{u}}_s(k,1) \\ \vdots \\ \underline{\mathbf{u}}_s(k,l) \end{bmatrix}.$$

As a consequence, columns in $\underline{\mathbf{P}}_s$ must be regrouped such that the first $ps+p$ columns correspond to $\underline{\mathbf{y}}_s(k,1)$, the second $ps+p$ columns to $\underline{\mathbf{y}}_s(k,2)$, and so forth.

We use $\underline{\mathbf{P}}_{s,i}$ to represent those columns in $\underline{\mathbf{P}}_s$ associated with $\underline{\mathbf{y}}_s(k,i)$ if $i \in [1,m]$; or $\underline{\mathbf{u}}_s(k, i-m)$ if $i \in [m+1, m+l]$. Since $\mathbf{r}_{s,i}(k)$ is designed to be insensitive to $\underline{\mathbf{y}}_s(k,i)$, $\mathbf{W}_i\underline{\mathbf{P}}_{s,i} = \mathbf{0}$ must be ensured.

Moreover, each $\mathbf{W}_i$ must have a maximum covariance with $\underline{\mathbf{P}}_s$. The algorithms of calculating $\mathbf{W}_0$ can be directly applied to calculating $\mathbf{W}_i$, if replacing $\tilde{\underline{\mathbf{H}}}_s\tilde{\underline{\mathbf{H}}}_s'$ by $\underline{\mathbf{P}}_s\underline{\mathbf{P}}_s'$ and $\underline{\boldsymbol{\Gamma}}_s^G$ by $\underline{\mathbf{P}}_{s,i}$.

It follows from Eqns. 12 and 13 that

$$\mathbf{r}_{s,i}(k) = \mathbf{r}_{s,i}^*(k) + \mathbf{r}_{s,i}^f(k) \tag{14}$$

where $\mathbf{r}_{s,i}^f(k) = \mathbf{W}_i\mathbf{e}_s^f(k)$, and $\mathbf{r}_{s,i}^*(k) = \mathbf{W}_i\mathbf{e}_s^*(k)$. From the distribution of $\mathbf{e}_s^*(k)$, it is known that $\mathbf{r}_{s,i}^*(k) \sim \aleph\left(\mathbf{0}, \mathbf{R}_{s,i}\right)$, where $\mathbf{R}_{s,i} = \mathbf{W}_i\mathbf{R}_{s,e}\mathbf{W}_i'$. In addition, $\mathbf{r}_{s,i}(k) \sim \aleph\left(\mathbf{0}, \mathbf{R}_{s,i}\right)$ if the i^{th} sensor is faulty according to the selected isolation logic; and $\mathbf{r}_{s,i}(k) \sim \aleph\left(\mathbf{r}_{s,i}^f(k), \mathbf{R}_{s,i}\right)$ once the j^{th} $(j \neq i)$ sensor is faulty. Therefore, if $\mathbf{r}_{s,i}(k)$ is zero-mean, $\forall\ i \in [1, m+l]$, but $\mathbf{r}_{s,j}(k)$ is non-zero mean $\forall$ $\{j \neq i\} \bigcap \{j \in [1, m+l]\}$, we conclude that the i^{th} sensor has failed.

We can also use a scalar statistic $\eta_{s,i}(k) = \mathbf{r}_{s,i}'(k)\mathbf{R}_{s,i}^{-1}\mathbf{r}_{s,i}(k)$ as a fault isolation index. We can conclude that the i^{th} sensor is faulty, if $\eta_{s,i}(k)$ is less but $\eta_{s,j}(k)$ is larger than a pre-determined confidence limit.

5. NUMERICAL EVALUATION

A quadruple tank system (Ge and Fang, 1988), where tanks with the same height and same cross section are serially connected by outlets that have an identical cross section. In the system, the input is the water flowing into Tank 1, and the controlled variables are the levels $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)]'$ in Tanks 1 up to 4, respectively.

The dynamics of the system can be described by the following CT state space model :

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\tilde{u}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) \qquad (15)$$

where besides $\mathbf{x}(t)$, $\tilde{\mathbf{y}}(t) \in \Re^4$ is the output vector,

$$\mathbf{A} = \begin{bmatrix} -0.0457 & 0.0457 & 0 & 0 \\ 0.0457 & -0.0914 & 0.0457 & 0 \\ 0 & 0.0457 & -0.0914 & 0.0457 \\ 0 & 0 & 0.0457 & -0.0914 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0.0020 & 0 & 0 & 0 \end{bmatrix}', \text{ and } \mathbf{C} = \mathbf{I}_4.$$

In accordance with Ge and Fang (1988), the noise-free and fault-free input to the tank system is simulated by $\tilde{u}(t) = 1 + 0.36 sin(t)$.

With such an input, we use the function 'ode45' in MATLAB^{TM} to simulate Eqn. 15, generating the CT signals $\{\tilde{\mathbf{y}}(t), \tilde{u}(t)\}$.

Select a frame period $T = 0.5$ minute. For $k = [0, 1, \cdots)$, within the period $[kT, kT+T]$ we sample $\tilde{u}(t)$ at $t = kT$ and $t = kT+0.2$; and $\tilde{\mathbf{y}}(t)$ at $t = kT$ and $t = kT + 0.3$, respectively; resulting in

$$\underline{\tilde{\mathbf{u}}}(k) = \begin{bmatrix} \tilde{u}(kT) \\ \tilde{u}(kT+0.2) \end{bmatrix}, \ \underline{\tilde{\mathbf{y}}}(k) = \begin{bmatrix} \tilde{\mathbf{y}}(kT) \\ \tilde{\mathbf{y}}(kT+0.3) \end{bmatrix}.$$

Applying $\underline{\tilde{\mathbf{u}}}(k)$ and $\underline{\tilde{\mathbf{y}}}(k)$ to Eqn. 6 gives us the lifted system model, $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$, $\underline{\mathbf{D}}$, $\underline{\mathbf{E}}$, and $\underline{\mathbf{J}}$. Since there are 5 sensors in total, we generate 5 SRVs for fault isolation. We calculate the models for the PRV and 5 SRVs, respectively.

We introduce Gaussian distributed white noise to $\underline{\tilde{\mathbf{u}}}(k)$ and $\underline{\tilde{\mathbf{y}}}(k)$ to produce 1000 samples of training data that are used to estimate $\mathbf{R}_{s,e}$ and $\mathbf{R}_{s,i}$. Given $\alpha = 0.01$, the confidence limit of $\eta_s(k)$ is $\chi^2_{0.01}(36) = 58.5713$. In addition, the confidence limit for each $\eta^i_s(k)$ is $\chi^2_{0.01}(26) = 45.6420$, $\forall\ i \in [1, 5]$. We define the scaled FDI indices, $\bar{\eta}_s(k) \equiv \eta_s(k)/58.5713$ and $\bar{\eta}^i_s(k) \equiv \eta^i_s(k)/45.6420$, all of which have a confidence limit, 1, and will be plotted to show FDI results.

A drift fault simulated by $(t - t_f)/500$, where t_f is the time at which the fault begins to occur, is introduced to one sensor at $t = t_f = 501 * T = 205.5$ minutes. The FDI results are depicted in Figure 1, where in the x-axis each sample represents 0.5 minute. In addition, Fd represents $\bar{\eta}_s(k)$, and $\{Fi_1, \cdots, Fi_5\}$ represents $\{\ \bar{\eta}^1_s(k), \cdots,\ \bar{\eta}^5_s(k)\}$, respectively.

Fd is beyond 1 after the occurrence of the fault, indicating the success of fault detection. Moreover, since Fi_1 (< 1) is unaffected by the fault, while $\{Fi_2, Fi_3, Fi_4, Fi_5\}$ (> 1) have been affected by the fault, one infers that the first output sensor has a fault. Notice that there is a delay in fault detection, because a drift fault evolves with time very slowly.

5.1 *Comparison with single rate FDI schemes*

We use another quadruple tank system (Johansson, 2000) as a benchmark to conduct comparative study among the newly proposed multirate FDI scheme and the standard single rate one. Detailed description of the tank system can be found in the reference mentioned above.

The dynamics of the tank system around an operating point can be represented by the following CT state space equations (Johansson, 2000):

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.016 & 0 & 0.042 & 0 \\ 0 & -0.011 & 0 & 0.033 \\ 0 & 0 & -0.042 & 0 \\ 0 & 0 & 0 & -0.033 \end{bmatrix} \mathbf{x}(t)$$
$$+ \begin{bmatrix} 0.083 & 0 \\ 0 & 0.063 \\ 0 & 0.048 \\ 0.031 & 0 \end{bmatrix} \mathbf{u}(t), \ \mathbf{y}(t) = 0.5\mathbf{I}_4\mathbf{x}(t) \qquad (16)$$

where in $\mathbf{x}(t)$, its i^{th} element, $x_i(t)$ for $i \in [1, 4]$, represents the variation of the level in the i^{th} tank. Similarly, in $\mathbf{y}(t)$, its i^{th} element $y_i(t)$ represents the measured level variation in the i^{th} tank. The inputs to the system are voltages applied to two pumps which provide water to the four tanks. We simulate the inputs by pseudo random binary signals with small magnitude. The frequency bands for the frequency contents of the inputs are chosen to be [0,0.03], and [0,0.05], expressed in fractions of the Nyquist frequencies.

In the second tank system, there are 6 sensors in total. Accordingly, 1 PRV and 6 SRVs are generated for FDI. Furthermore, from the PRV and SRVs, the related FDI indices are calculated and scaled to have unit confidence limit. Similarly, the i^{th} fault isolation index is made insensitive to fault in the i^{th} sensor but most sensitive to faults in any other sensors.

Comparative studies are conducted among (a) FDI using multirate data; and (b) FDI using a low single rate data, where different types of faults in different sensors are simulated.

A complete failure was introduced to one sensor. We sampled the two inputs with a sampling interval of 1 minute and the four outputs with a sampling interval of 2 minutes. The corresponding FDI results are shown in Figure 2, where Fd and $\{Fi_1, \cdots, Fi_6\}$ are similar to those in Figure 1. Figure 2 shows that the faulty sensor has been successfully detected and isolated.

Furthermore, we sample the inputs and outputs with an identical sampling interval of 2 minutes, and then perform FDI on the data collected with a slower rate. The relevant FDI results are demonstrated in Figure 3. Apparently, the FDI performance in this case is much worse compared with that illustrated in Figure 2. This would be expected since we are now downsampling the data and thus ignoring potentially useful information.

6. CONCLUSION

A novel approach to detection and isolation of sensor faults in non-uniformly sampled multirate dynamic systems has been proposed. This approach has been applied to a simulation example, where different types of faults including bias, drift, and precision degradation, are introduced to sensors, respectively, and are successfully detected and isolated.

Comparative studies have also been carried out via another quadruple tank system. It has been verified that multirate data-based FDI outperforms the slow single rate data-based FDI.

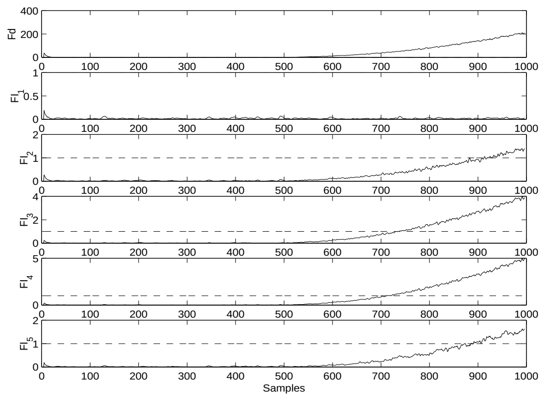

Fig. 1. A fault in the 1^{st} tank system is successfully detected and isolated with multirate data.

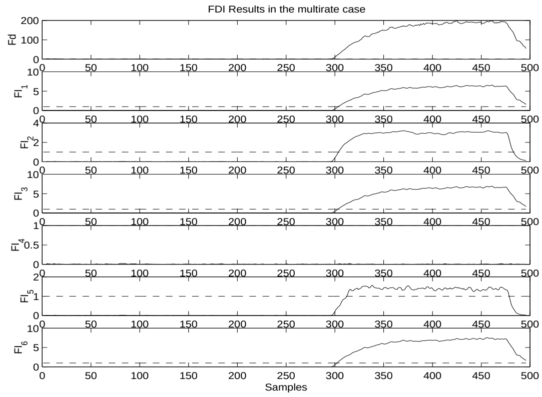

Fig. 2. A fault in the second tank system is successfully detected and isolated with multirate data.

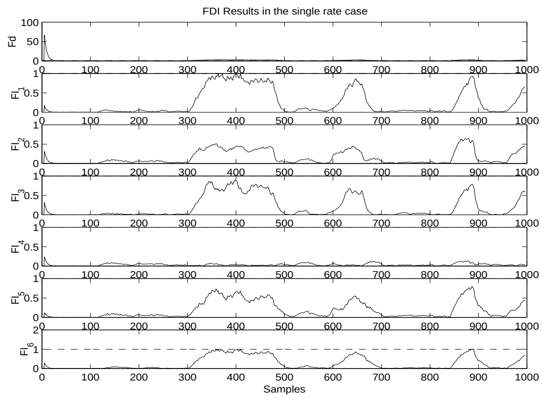

Fig. 3. Using slow single rate data fails to detect and isolate the fault in the second tank system.

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