CORRELATION DIMENSION AND LYAPUNOV EXPONENT BASED ISOLATION OF PLANT-WIDE OSCILLATIONS

Xiaoyun Zang, John Howell

University of Glasgow, Centre for Systems and Control,
Department of Mechanical Engineering, G12 8QQ, Glasgow, UK
j.howell@mech.gla.ac.uk

Abstract: The correlation dimension and maximal Lyapunov exponent have a direct relationship with the harmonic complexity contained in non-linearity induced oscillations in chemical process plants. When combined with knowledge of the harmonic propagation of plant-wide oscillations, they can be applied to locate the source of the non-linearity. The method is demonstrated on both simulated data and real industrial data.

Keywords: correlation dimension, Lyapunov exponent, non-linearity, oscillations, process control

1. INTRODUCTION

Oscillations are a common type of plant-wide disturbance whose detection and diagnosis have generated considerable interest in recent years (Thornhill, *et al.*, 2001; Thornhill, *et al.*, 2003a; Thornhill and Hagglund 1997; Xia, *et al.*, 2003). A key issue is the determination of the root cause of a plant-wide oscillation (Qin 1998). That is an oscillation that is observed in a number of neighbouring process units and control loops. The most likely sources of oscillations are either due to valve non-linearity such as stiction or hysteresis or due to a badly tuned controller (Wallen 1997). Thornhill, *et al.*, (2001) have focused on isolating a problem non-linear element in a control loop; the most likely loop is selected by finding that measurement record that has the maximum distortion factor (D- factor), which is a measure of the harmonic content of a record. In the same paper and in (Thornhill, *et al.*, 2003b), the authors focus on an alternative measure, on what they call *non-linearity*. They recommend the application of a non-linearity statistic (the *N* measure), and hypothesize that the non-linearity in a measurement record obtained near the source will be stronger than one that it is recorded further away from the source. Both methods are based on the fact that process plants are usually 'low-pass' in nature, so filter out high frequency components in the oscillations.

In this paper, the correlation dimension and Lyapunov exponents are found to have some relationship with the harmonic content in the time series recorded from the oscillating loops. Nonlinearity induced oscillations in chemical process plants usually have asymmetric waves which contain a fundamental, plus even and odd harmonics (Zang and Howell, 2003). Hence correlation dimension and Lyapunov exponents can provide additional information towards the location of the source of nonlinearity induced, plant-wide oscillations.

The correlation dimension and Lyapunov exponents are recently developed descriptions which provide quantities for the characterisation of nonlinear, deterministic and chaotic data (Kantz and Schreiber 1997). These methods have been widely used to analyse non-linear time series in mechanical systems (Wang and Lin 2003), in medical science (Muller, *et al.*, 2003), and in chemical engineering (Guo, *et al.*, 2003). Sometimes the correlation dimension and/or the maximal Lyapunov exponent indicate non-linear or deterministic chaotic behaviour; whilst other references conclude that a larger correlation dimension or a larger maximal Lyapunov exponent corresponds to richer dynamics (Boltezar, *et al.*, 1999) or more system complexity (Burioka, *et al.*, 2001). Few have related the correlation dimension and maximal Lyapunov exponent of a time series to its harmonic content because chaotic time series, although usually periodic and oscillatory, do not contain distinct fundamentals and harmonics.

The definitions and characteristics of the correlation dimension and maximal Lyapunov exponent will be introduced first. A simple example will then be described to demonstrate their relationships with the harmonic content of an oscillating time series. A method of applying these measures to find the root cause of the oscillation will then be given. Finally three case studies will be outlined, one is simulated and two industrial.

2. TECHNIQUES

Both techniques are based on a discrete, multi-dimensional phase space representation of the data. Given the time series $x_1, x_2, x_3, \cdots, x_N$ then the multi-dimensional phase space is formed from:

$$\mathbf{X}_i = (x_i, x_{i+T}, x_{i+2T}, \cdots x_{i+(m-1)T}), \quad i = 1, 2, \cdots, N \qquad (1)$$

where T is the time interval and m is known as the embedding dimension; $\mathbf{X}_i$ is called an embedding vector (point) of m-dimension.

2.1 Correlation dimension: definition and estimation

The correlation dimension provides a tool to quantify self-similarity. A larger correlation dimension corresponds to a larger degree of complexity and less self-similarity. Stochastic signals are infinite-dimensional. The most frequently used procedure to estimate the correlation dimension was introduced by Grassberger and Procaccia (1983a, 1983b). They defined the correlation sum for a collection of points $\mathbf{X}_i$ *(i=1,2,...,N)* in some phase space to be the fraction of all possible pairs of points which are closer than a given distance ε in a particular norm:

$$C(m,\varepsilon) = \frac{2}{(N-m)(N-m-1)} \sum_{i=m}^{N} \sum_{j=i+1}^{N} \Theta(\varepsilon - \|\mathbf{X}_i - \mathbf{X}_j\|) \qquad (2)$$

where Θ is the Heaviside step function, $\Theta(x) = 0$ if $x \le 0$ and $\Theta(x) = 1$ for $x > 0$. Thus Equation 2 counts the pairs ($\mathbf{X}_i$, $\mathbf{X}_j$) whose distance is smaller than ε. To eliminate the temporal correlation, the Theiler window can be used to exclude those pairs of points that are too close in time (Kantz and Schreiber 1997).

It has been shown by Sauer & Yorke (1993) that in the limit of an infinite amount of data (i.e. $N \to \infty$) and for small ε, C scales like a power law, $C(\varepsilon) \propto \varepsilon^{D_2}$, where D_2 is known as the correlation dimension. Thus D_2 is defined by

$$D_2 = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{\partial \ln(C(m,\varepsilon))}{\partial \ln(\varepsilon)} . \qquad (3)$$

Convergence to a finite correlation dimension can be checked by plotting "effective dimensions" versus scale (ε) for various embeddings (*m*). The easiest way to proceed is to compute (numerically) the derivative of $\ln C(m,\varepsilon)$ with respect to $\ln \varepsilon$, for example by fitting straight lines to the log-log plot of $C(\varepsilon)$. If the scale is large, then self-similarity should be minimal, whereas if it is sufficiently small one observes a relationship with the embedding dimension i.e. D_2 is not totally independent. This effect is due to noise. Only on the intermediate scales can one see the desired *plateau* where the results are approximately independent of *m* and ε i.e. where D_2 is *invariant.* The region where the scale rule holds, not just the range selected for straight line fitting, is called the *scaling range*. Takens-Theiler have developed an alternative estimate of the correlation dimension:

$$D_{TT}(\varepsilon) = \frac{C(\varepsilon)}{\int_0^{\varepsilon} \frac{C(\varepsilon')}{\varepsilon'} d\varepsilon'} \qquad (4)$$

2.2 Lyapunov exponents: definition and estimation

Lyapunov exponents measure the exponential divergence (positive exponents: chaotic motion) or convergence (negative exponents: regular motion) of two initially neighbouring trajectories in a phase space. In other words, they measure the degree of unpredictability of the future. There are many different Lyapunov exponents for a dynamical system. The most important is known as the *maximal Lyapunov exponent* (λ_1) (Kantz and Schreiber 1997). Let $\mathbf{X}_i$ and $\mathbf{X}_j$ be two points in a phase space with a distance $\|\mathbf{X}_i - \mathbf{X}_j\| = \delta_0 \ll 1$ between them. Let δ_t denote the distance obtained between the two trajectories *t* units of time later i.e. $\delta_t = \|\mathbf{X}_{i+t} - \mathbf{X}_{j+t}\|$. Then λ_1 is defined by $\delta_t \simeq \delta_0 e^{\lambda_1 t}, \delta_t \ll 1$. A positive λ_1 means that there is an exponential divergence of these trajectories, i.e. chaos; a negative λ_1 implies the existence of a stable fixed point; if the motion settles down onto a limit cycle, λ_1 is zero.

Based on this understanding, a robust, consistent and unbiased estimator for the maximal Lyapunov exponent was proposed by Kantz and Schreiber (1997). They compute

$$S(\varepsilon,m,t) = \frac{1}{N} \sum_{i=1}^{N} \ln\left(\frac{1}{|U(\mathbf{X}_i)|} \sum_{\mathbf{X}_j \in U(\mathbf{X}_i)} |\mathbf{X}_{i+t} - \mathbf{X}_{j+t}|\right) \qquad (5)$$

where the reference points $\mathbf{X}_i$ are embedding vectors and $U(\mathbf{X}_i)$ is known as the neighbourhood of $\mathbf{X}_i$ with diameter ε. If $S(\varepsilon,m,t)$ exhibits a linear increase with fixed slope for all m larger than some m_0 and for a reasonable range of ε, then the slope of S is deemed to be *invariant* over this range. This slope can be taken as an estimate of the maximal exponent λ_1.

The correlation sum, correlation dimension, and the maximal Lyapunov exponent estimation in this paper are all based on the TISEAN software package (Hegger, *et al.*, 1999). Throughout the paper, $D_{TT}(\varepsilon)$ and $S(\varepsilon,m,t)$ are formed in 1-15 dimensions, D_2 and λ_1 are then estimated from the values of $D_{TT}(\varepsilon)'s$ and slopes of $S(\varepsilon,m,t)'s$ in 2-15 dimensions where the *plateau* or *linear increase* is well established, giving the mean and variance of these values.

2.3 Relationship between these invariants and harmonic content

Intuitively, a time series with high harmonic content will be more complex and unpredictable than one with a low harmonic content, and hence will have a larger correlation dimension and a larger maximal Lyapunov exponent. Here a simple example will be used to reinforce this intuition. Three 4096-sample time series were analysed to examine the effect of increasing the harmonic content:

$$\begin{aligned} &s1(k)=\sin(2\pi fk),\\ &s2(k)=\sin(2\pi fk)+0.2\sin(4\pi fk),\\ &s3(k)=\sin(2\pi fk)+0.2\sin(4\pi fk)+0.05\sin(6\pi fk),\\ &k=1,\cdots,4096 \end{aligned} \tag{6}$$

where $f=0.01$ is the fundamental frequency. The correlation dimensions and maximal Lyapunov exponents of the three time series are listed in Table 1.

It can be seen that the correlation dimension and maximal Lyapunov exponent both increase with harmonic content. This supports the claims that the correlation dimension reflects the self-similarity and the maximal Lyapunov exponent reflects the predictability. The future of time series with more harmonics should be more unpredictable and hence should have a larger invariant.

Table 1. Invariant of sinusoidal signals

Time series	Correlation dimension (D_2)	Maximal Lyapunov Exponent (λ_1)
s1(k)	1.04±0.04	2.21e-5±1.44e-5
s2(k)	1.07±0.04	2.10e-4±2.36e-5
s3(k)	1.08±0.03	8.12e-4±2.11e-4

2.4 Invariant based location of the source of the oscillations

In general the dynamic behaviour of physical processes are inherently low-pass, so time series that are recorded close to the root cause of a plant-wide oscillation will have a high harmonic content, and those recorded further away from the root cause will become more sinusoidal. However the first few harmonics of an oscillation could be amplified if the fundamental is sufficiently low to locate the first few harmonics on the positive slope of the disturbance transfer function bode plot. In this case, and since the first few harmonics dominate, the time records further away from the source could be more complex (with large signal-to-noise ratio).

Based on this understanding of how harmonic content is affected as an oscillation propagates through a plant, the correlation dimension and maximal Lyapunov exponent can provide features to locate the source of plant-wide oscillations. More specifically, the time series pertaining to the root cause should have the largest correlation dimension and maximal Lyapunov exponent in cases where the plant is inherently low pass in nature over the frequency range of concern, because the harmonics have been attenuated through the propagation and also because the time series become less complex and more predictable further away from the source; and the correlation dimension and maximal Lyapunov exponent pertaining to the root cause will be the smallest in low-frequency cases where there is amplification of the first few harmonics.

3. EXAMPLES

3.1 Simulated time series

A simulated time series was generated by simulating a simple distillation column model due to Seborg, *et al.*, (1989), which was adapted by incorporating a valve model into one of its loops. Details pertaining to the revised model are given in Fig. 1; stiction in the valve (in Loop 1) was increased until the model oscillated. The static friction model that was used is that due to Horch and Isaksson (1998). Both loops oscillated. Their controlled process variables (*PVs*) were recorded, resulting in two 4096-sample time series (Fig. 2). The correlation dimension plots, D_{TT} versus ε, and the plots of $S(\varepsilon,m,t)$ versus t are given in Fig. 3 and Fig. 4 respectively.

This is a low-frequency case, because the fundamental oscillating frequency is much lower than the cut-off frequency of the disturbance transfer function bode plot of Loop2. So the correlation dimension and maximal Lyapunov exponent of the source should be smaller than the disturbed loop. The results obtained (Table 2) support this view.

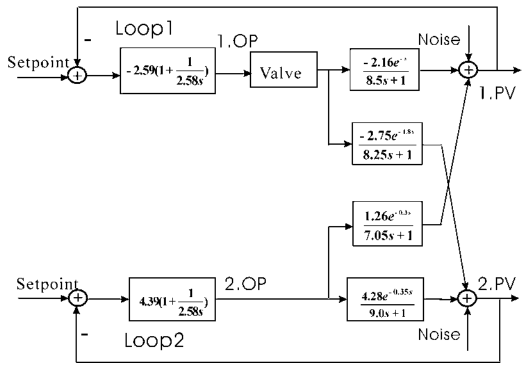

Fig. 1 The simulated plant schematic

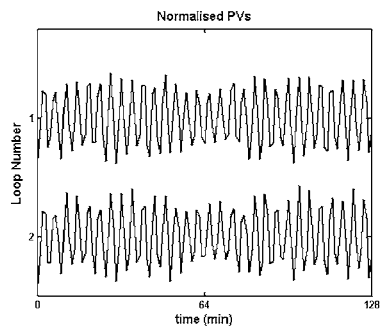

Fig. 2 Process variable time series for the two simulated loops

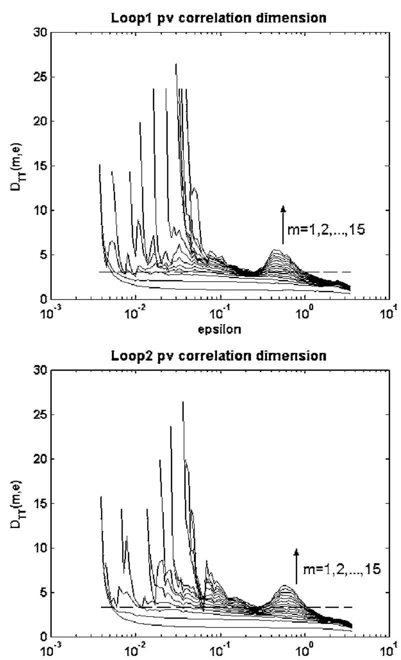

Fig. 3 Takens estimator for correlation dimension of the two loops

Table 2. Invariant estimation for the simulated case

Loop Number	Correlation Dimension (D_2)	Maximal Lyapunov exponent (λ_1)
1	3.00±0.13	0.30±0.01
2	3.28±0.13	0.31±0.01

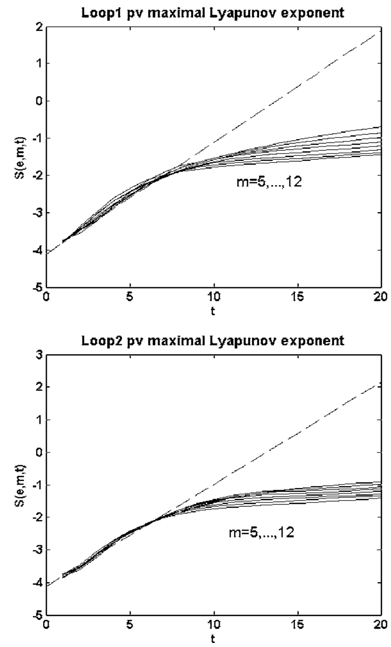

Fig. 4 Maximal Lyapunov exponents of the two loops

3.2 Case study 1 (low-frequency case)

Case study 1 pertains to industrial data, provided by the Eastman Chemical Company, which was recorded whilst a plant was oscillating. A plant assessment found a sticking valve in the loop with Tag22 (Thornhill, *et al.*, 2002; Xia, *et al.*, 2003). Figure 5 gives the 4096-sample time series for each of the six dominant loops that showed these plant-wide oscillations. The correlation dimensions and maximal Lyapunov exponents of each time series are plotted and estimated. The results are listed in Table 3.

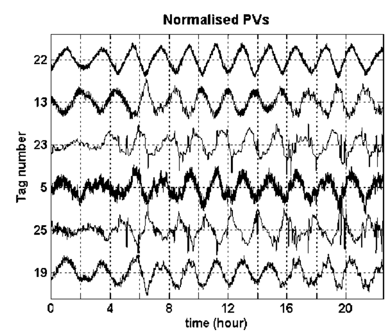

Fig. 5 Time series plots of the six loops of interest

Table 3. Invariant estimation for Easterman case study

Tag Number	Correlation Dimension (D_2)	Maximal Lyapunov exponent (λ_1)
22	1.52±0.03	0.004±0.0001
13	2.33±0.01	0.022±0.004
23	-	-
5	-	-
25	3.86±0.01	0.029±0.007
19	2.29±0.05	0.007±0.0004

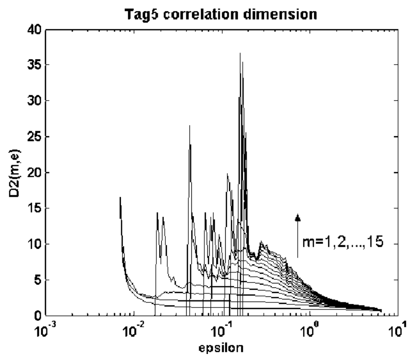

Fig. 6 (a) No typical correlation dimension found in Tag5

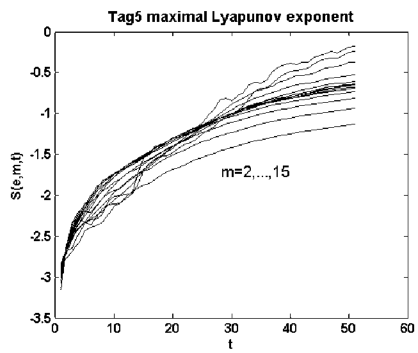

Fig. 6 (b) No exponential divergence of Tag5

Once again, this is a low-frequency case study. Tag22 has the smallest correlation dimension and maximal Lyapunov exponent inferring that it should be the nearest to the root cause. Interestingly Fig. 6 (a) shows the correlation dimension plot of Tag5, where no typical correlation dimensions can be found. All curves from different embedding dimensions behave different and there is no common behaviour. Fig. 6 (b) shows that the time series pertaining to Tag5 exhibit no linear increase, reflecting the lack of exponential divergence of nearby trajectories. And the time series pertaining to Tag23 also has no typical dimension and maximal Lypunov exponent. A possible explanation for this is that the stochastic content (for example, noise) in the time series dominates. These observations also match the propagation schematic provided by (Thornhill, *et al.*, 2002), where Tag23 and Tag5 are both relatively far away from Tag22. Although Tag19 is farther away than Tag13, it has a smaller correlation dimension and maximal Lyapunov exponent. This is because the intrinsic low-pass filtering attenuates the higher harmonics and simplifies the time series pertaining to Tag19.

3.3 Case Study 2 (high-frequency case)

A set of refinery data (courtesy of a SE Asian refinery) was examined. Previously Thornhill, *et al.*, (2001) have performed spectral PCA on the data and suggested that 12 loops were associated with a plant-wide oscillation observed in the data. They then determined the distortion factor and N- measure of each of these 12 loops and concluded that the source of non-linearity was located in the loop associated with one of Tags 13, 33 or 34. Figure 7 shows the time trends of the 12 loops. The estimated correlation dimensions and maximal Lyapunov exponents for the 12 loops are listed in Table 4.

This is a high-frequency case study. The source of such plant-wide oscillations should have the largest correlation dimension and maximal Lyapunov exponent. From Table 4, it can be seen that the time series pertaining to Tag13 and Tag33 have larger correlation dimensions and maximal Lyapunov exponents than those pertaining to other loops, which agrees with the conclusions made by Thornhill, *et al.*, (2001). Note that the time series pertaining to Tag11 also has a large correlation dimension and maximal Lyapunov exponent. It is suspected that this loop contained harmonics which were not multiples of the fundamental pertaining to the oscillation (0.06 min^{-1}). These harmonics may contribute to the large correlation dimension and maximal Lyapunov exponent. Note that the time series pertaining to Tags 24, 25, 4 and 19 have no typical correlation dimensions and maximal Lyapunov exponents, because these loops are contaminated by noise.

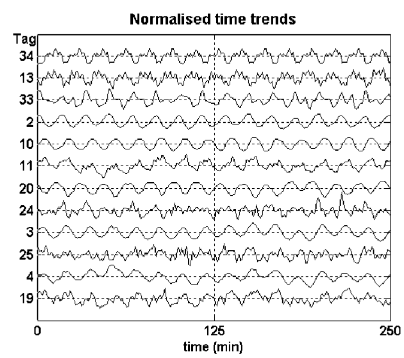

Fig. 7 Time series plots of the twelve loops of interest

Table 4. Invariant estimation for SE Asian case study

Tag Number	Correlation Dimension (D_2)	Maximal Lyapunov exponent (λ_1)
34	1.24±0.01	0.011±0.0003
13	2.78±0.06	0.057±0.0008
33	2.16±0.12	0.030±0.004
2	1.88±0.08	0.010±0.0002
10	1.79±0.10	0.011±0.001
11	2.87±0.04	0.027±0.0007
20	1.87±0.05	0.019±0.001
24	-	-
3	2.05±0.02	0.029±0.001
25	-	-
4	-	-
19	-	-

4. CONCLUSION

The correlation dimension and maximal Lyapunov exponent of a time series are demonstrated to be associated with its harmonic content. Based on the assessment of the propagation of harmonic content of plant-wide oscillations, it is possible to find the source of non-linearity by analysing invariants of the time series pertaining to the oscillating loops. Compared to the N-measure proposed by Thornhill, *et al.*, (2001), these methods do not involve surrogate data, and are hence more time efficient.

Second-order Volterra models that are capable of exhibiting asymmetric responses to symmetric input changes (Doyle *et al*., 1995) might provide additional information.

ACKNOWLEDGEMENTS

Xiaoyun Zang & John Howell gratefully acknowledge John Cox & Michael Paulonis from the *Eastman Chemical Company, Kingsport, TN, USA*, for the supply of real plant data.

REFERENCES

Boltezar, M., N. Jaksic, I. Simonovski and A. Kuhelj (1999). Dynamical behaviour of the planar non-linear mechanical system -- PART II: Experiment. *Journal of Sound and Vibration*, **226**(5), 941-953.

Doyle, F. J., B. A. Ogunnaike and R. K. Pearson (1995). Nonlinear model-based control using second-order Volterra models. *Automatica*, **31**(5), 697-714.

Grassberger, P. and I. Procaccia (1983a). Characterisation of strange attractors. *Physical Review Letters,* **50**, 346-349.

Grassberger, P. and I. Procaccia (1983b). Measuring the strangeness of strange attractors. *Physica D: Nonlinear Phenomena*, **9**(1-2), 189-208.

Guo, Q., G. Yue, T. Suda and J. Sato (2003). Flow characteristics in a bubbling fluidized bed at elevated temperature. *Chemical Engineering and Processing*, **42**(6), 439-447.

Hegger, R., H. Kantz and T. Schreiber (1999). Practical implementation of nonlinear time series methods: the TISEAN package. *Chaos*, **9**, 413.

Horch, A. and A. J. Isaksson (1998). A method for detection of stiction in control valves. In: *Proceedings of the IFAC Workshop on on-line fault detection and supervision in the chemical process industry*. Session 4B. Lyon, France.

Kantz, H. and T. Schreiber (1997). *Nonlinear time series analysis*. Cambridge University Press, Cambridge.

Muller, V., W. Lutzenberger, H. Preil, F. Pulvermuller and N. Birbaumer (2003). Complexity of visual stimuli and non-linear EEG dynamics in humans. *Cognitive Brain Research*, **16**(1), 104-110.

Qin, S. J. (1998). Control performance monitoring -- a review and assessment. *Computers & Chemical Engineering*, **23**(2), 173-186.

Sauer, T. and J. Yorke (1993). How many delay coordinates do you need? *Int.J.Bifurcation and Chaos*, **3**, 737.

Seborg, D. E., T. F. Edgar and D. A. Mellichamp (1989). *Process Dynamics and Control*. John Wiley & Sons, Inc.

Thornhill, N. F. and T. Hagglund (1997). Detection and diagnosis of oscillation in control loops. *Control Engineering Practice*, **5**(10), 1343-1354.

Thornhill, N. F., S. L. Shah and B. Huang (2001). Detection of distributed oscillations and root-cause diagnosis. In: *Proceedings of the CHEMFAS-4*. Cheju, Korea. pp. 167-172.

Thornhill, N. F., C. Xia, J. Howell, J. Cox and M. Paulonis (2002). Analysis of plant-wide disturbances through data-driven techniques and process understanding. In: *Proceedings of the IFAC World Congress*. Barcelona, Spain.

Thornhill, N. F., B. Huang and H. Zhang (2003a). Detection of multiple oscillations in control loops. *Journal of Process Control*, **13**(1), 91-100.

Thornhill, N. F., J. W. Cox and M. A. Paulonis (2003b). Diagnosis of plant-wide oscillation through data-driven analysis and process understanding. *Control Engineering Practice,* **11**(12), 1481-1490.

Wallén, A. (1997). Valve diagnositics and automatic tuning. In: *Proceedings of 1997 American Control Conference*. Albuquerque, New Mexico. pp. 2930-2934.

Wang, W. J. and R. M. Lin (2003). The application of pseudo-phase portrait in machine condition monitoring. *Journal of Sound and Vibration*, **259**(1), 1-16.

Xia, C. and J. Howell (2003). Isolating multiple sources of plant-wide oscillations via independent component analysis. In: *Proceedings of the 5th IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes (SAFEPROCESS'03)*. Washington DC, USA.

Zang, X. and J. Howell (2003). Discrimination between bad tuning and non-linearity induced oscillations through bispectral analysis. In: *SICE Annual Conference 2003 in Fukui proceedings*. Fukui, Japan.