OPTIMAL SENSOR LOCATION FOR NONLINEAR DYNAMIC SYSTEMS VIA EMPIRICAL GRAMIANS

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Abstract: This paper presents a new approach for determining sensor locations for nonlinear dynamic systems. The method uses empirical observability gramians for observability analysis and combines the information from this investigation with observability measures which have been previously proposed in the literature. This approach offers the advantage over other methods in that it is directly applicable to nonlinear systems without resorting to linearization of the model. The presented procedure has been applied to a binary distillation column model. Additionally, the effect of scaling of a model for sensor placement has been examined as well as the conclusions which can be drawn from different observability measures.

Keywords: Nonlinear systems, Observability, Sensor

1.0 INTRODUCTION

The chemical industry has gone through significant changes over the last few decades. Today, the emphasis is on running plants in the most optimal manner with due consideration to the safety aspects, rather than just achieving the design throughput. To meet these goals, information about a number of process variables and parameters is required. While it is possible to install additional sensors to monitor many of the aspects of a plant's operation, this can be expensive both in terms of the initial as well as the maintenance costs. In order to overcome this problem, model-based state estimation techniques can be used. By strategically measuring some key variables of the process, it is possible to compute many other variables and process parameters by using an observer. However, in order to gain the maximum benefit from this technique, the sensors have to be placed at "optimal" locations. This paper presents a method that allows computing such optimal sensor locations for plants described by nonlinear dynamic systems. Past research related to optimal sensor location has mainly been confined to linear or linearized systems (Dochain et al., 1997; Waldruff et al., 1998; Van der Berg et al., 2000, Muske, 2002). However, most chemical, petrochemical, or biochemical processes are accurately described by nonlinear dynamic systems and a linearized model may not represent the actual dynamics of the process over the entire region of operation. Due to this limitation, it is possible that different conclusions could be reached depending upon the operating point chosen for linearization. Therefore, it is desirable to use techniques, which will not have to resort to linearizing the model. The empirical gramians introduced by Lall et al. (2002) can form one piece of this investigation, because they can be used for determining observability/controllability of nonlinear systems over a specified operating region. Hahn and Edgar (2001) have shown that this new technique does provide a better representation of the input to state and state to output behavior of a nonlinear system over an operating region than the observability gramian of the linearized system. In this paper an approach for sensor placement based upon empirical observability gramians is presented. This is achieved by combining the results obtained from observability analysis via empirical gramians with measures, representing the degree of observability of a system. However, instead of proposing new measures for quantifying the degree of observability, existing measures have been used and adapted for the case where the observability analysis is based on empirical gramians. The advantage that such a technique has over other methods is that it does not resort to linearizing the system, while it can still be applied to nonlinear systems of significant complexity. Additionally, the results for different proposed measures are compared and the effect of scaling on this investigation is discussed. The methodology is applied to a distillation column separating a binary mixture.

2.0 PRELIMINARIES

2.1 Observability

Observability refers to the property of a system that allows the reconstruction of the values of the state variables given the outputs. For linear time invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

the observability gramian

$$W_o = \int_0^{\infty} e^{A^T t} C^T C e^{A t} dt \tag{2}$$

can be computed in order to determine observability of the system. If the observability gramian is a matrix of full rank then the system is observable. However, if the gramian is rank deficient then the system will not be observable and some of the states (or directions in state space) cannot be reconstructed from the output data.

2.2 Empirical observability gramian

While the observability gramian can be used for determining observability of linear systems, it may not result in sufficient information if the system is nonlinear. Extensive efforts have been undertaken in the last two decades to derive conditions for observability of nonlinear systems (Isidori, 1995). However, the results derived from these conditions are usually too complex to be interpreted for all but very simple systems. One alternative is to use the relatively new concept of empirical gramians that have been introduced by Lall et al. (2002). In order to present the definition of the empirical observability gramian the following quantities need to be defined:

$$T^n = \{T_1, \ldots\ldots, T_r; T_i \in R^{n\times n}, T_i^T T_i = I, i = 1, \ldots\ldots, r\}$$

$$M = \{c_1, \ldots\ldots\ldots, c_s; c_i \in R, c_i > 0, i = 1, \ldots\ldots\ldots, s\}$$

$$E^n = \{e_1, \ldots\ldots, e_n;\ \text{standard unit vectors in } R^n\ \}$$

where r is the number of matrices for perturbation directions, s the number of different perturbation sizes for each direction, and n the number of states of the system. The empirical observability gramian can be computed for general nonlinear systems

$$\dot{x}(t) = f(x(t), u(t)) \tag{3a}$$

$$y(t) = h(x(t)) \tag{3b}$$

and its definition is given by

$$W_O = \sum_{l=1}^{r} \sum_{m=1}^{s} \frac{1}{(rsc_m^2)} \int_0^{\infty} T_l \Psi^{lm}(t) T_l^T dt \tag{4}$$

where $\Psi^{lm}(t) \in R^{n\times n}$ corresponds to $\Psi_{ij}^{lm}(t) = (y^{ilm}(t) - y_{ss})^T (y^{jlm}(t) - y_{ss})$, $y^{ilm}(t)$ is the output of the system corresponding to the initial condition $x(0) = c_m T_l e_i + x_{ss}$ and y_{ss} is the steady state output of the system.

Empirical gramian can be computed for linear as well as nonlinear systems and reduces to the linear gramian (2) if the system is linear (Lall et al., 2002).

2.3 Measures for the degree of observability

While the rank of the observability gramian can be used in order to determine if a system is observable, this information is not sufficient for determining optimal sensor locations for a process. In order to address this problem, measures have been defined that quantify the degree of observability either of the entire system or of specific parts of the system. A wide variety of different measures for linear systems have been introduced over the last few decades and an overview over several of these is presented below.

Muller and Weber (1972) have outlined three candidates for measuring the degree of observability of a system based upon the linear observability gramian.

$$m_1 = l_{\min}(W_o) \tag{5a}$$

$$m_2 = \frac{n}{trace(W_o^{-1})} \tag{5b}$$

$$m_3 = [\det(W_o)]^{1/n} \tag{5c}$$

Larger values of each of these measures imply an increased degree of observability of the system. However, these measures are mainly influenced by the smallest singular values of the observability gramian.

Dochain (1997) made use of the condition number

$$C.N. = \frac{s_{\max}}{s_{\min}} \tag{6}$$

for the observability analysis of a fixed bed bioreactor. The s refer to singular values of the observability gramian matrix. A smaller value of the condition number generally implies increased observability of a system. Waldruff (1998) made use of the minimum singular value of the observability matrix

$$N.S.(W_o) = s_{\min}(W_o) \tag{7}$$

as one of the measures for determining sensor locations in tubular reactors. This method is similar to the measures based on the smallest eigenvalue given by Muller and Weber (1972). It serves as an indicator of how far the system is from being unobservable. Higher values of this criterion imply an increased degree of observability. All the above measures are strongly influenced by the minimum singular value of the system. Other approaches are based upon the maximal response which can be observed in sensor readings or upon the sum of the observability of all states. Van der Berg (2000) suggested two measures based upon these ideas. The first measure is the spectral radius

$$r(W_O) = s_{\max}(W_O) \tag{8}$$

which corresponds to the largest singular value. A large value of the measure indicates that the dominant direction in the observability gramian can be easily observed. The second measure is based on the trace of the observability gramian:

$$trace(W_O) = \sum_{i=1}^{n} s_i(W_O) \tag{9}$$

The trace can be interpreted as the sum of the singular values of the matrix. Larger values of the trace correspond to an increase in the overall observability of a system.

Summarizing, the presented measures can be put into two categories: measures which are mainly (or even exclusively) based upon the least observable direction in state space and measures which are predominantly influenced by the largest eigenvalue of the observability gramian. These findings are summarized in Table 1:

Table1: Measures for degree of observability

Smallest eigenvalue		*Largest eigenvalue*	
Measure	*Equation*	*Measure*	*Equation*
m_1	(5a)	$r(W_O)$	(8)
m_2	(5b)	$trace(W_O)$	(9)
m_3	(5c)		
$C.N.$	(6)		
$N.S.(W_O)$	(7)		

2.4 Observability analysis and sensor placement

Sensor locations are often determined in order to be able to directly measure certain states to reconstruct required information of a system. As a minimum requirement, the controlled variables need to be measured for feedback control. However, other considerations, e.g. safety or product quality may result in a need for additional measurement locations. Observability analysis can be performed in order to maximize the amount of information gained from the available measurements. Thereby, it is possible to determine sensor location in order to either

- get the most information from a certain number of measurements
- use as small a number of measurements as possible in order to obtain a required amount of information about the system.

The following procedure is usually applied for determining optimal sensor locations for linear systems: observability gramians are computed for a variety of combinations of sensors at different locations. Scalar measures are computed from the gramians in order to compare the degree of observability of various locations for the sensor placement. A sensor configuration corresponding to the largest value of an observability measure indicates a good candidate for optimal sensor location.

3.0 OPTIMAL SENSOR LOCATION FOR NONLINEAR SYSTEMS

Since all systems in nature are nonlinear to a certain degree, a gramian of the linearized system will not always result in a good description of the state to output behavior if the operating conditions of a process can vary significantly. Due to this, it can happen that observability analysis based upon linear gramians may result in contradictory information depending upon the point of linearization of a system. In order to address these deficiencies, this work determines sensor locations by performing observability analysis of the nonlinear system via empirical gramians. This has the advantage that it can result in a more accurate description of the state to output behavior than if the gramian of the linearized system is used (Hahn and Edgar, 2002) and at the same time, it is computationally much less expensive than if Lie-algebra-based approaches (Isidori, 1995) would be used for observability analysis of nonlinear systems. While the empirical observability gramian is used for observability analysis, some of the measures that have originally been proposed for quantification of the degree of observability of linear/linearized systems can still be incorporated into this procedure: the measures need to be computed for the empirical observability gramian instead of the observability gramian of the linearized system. This is possible because the empirical gramian is an *n*-by-*n*, symmetric, and positive semi-definite matrix, similar to the gramian of a linear system. However, unlike the measures based upon linear gramians, these measures will make direct use of the nonlinear model without resorting to linearization. The procedure for computing optimal sensor locations by this approach is summarized in Figure 1.

3.1 Measuring observability of specific states

During process operations it is often required to have precise information about a particular state for process control or quality control of the product. In these cases, the values of specific states are of greater importance than the overall observability of a process. Many plants have requirements for both of these cases: certain states need to be exactly known and measured, whereas other measurements are used as an indicator of overall plant performance or safety. However, it is possible to take both of these situations into account with the proposed sensor placement framework. The observability of a specific state can be determined by analyzing the diagonal entry of the empirical observability gramian matrix corresponding to this state. The larger the magnitude of this entry, the easier it will be to reconstruct the value of this state from the available measurements. It should be pointed out that such an analysis is not always trivial. For example, if one is interested in the temperature at a specific location then this temperature should be directly measured. However, if the concentration of a product needs to be exactly known, then it is not always feasible to directly measure its value. Oftentimes, the value needs to be computed from measurements of other properties closely related to the state of interest. The presented sensor location framework can serve as an indicator which properties should be measured as well as where the measurements should be taken.

3.2 Effect of scaling on the optimal sensor location.

Most methods for determining optimal sensor locations are influenced by the magnitude of the singular values of the observability gramian. However, singular value decomposition is dependent upon scaling of the model, thereby possibly influencing sensor location decisions. A process model is generally scaled so that it reflects the sensitivity of the problem. That is, if a particular state is nearly unobservable then the scaled model should reflect poor response in the output for the perturbations in that state. The other reason to scale a model is to minimize the effect of round-off errors. Therefore, it would seem appropriate to perform scaling on all variables prior to the computation of the empirical gramian. However, scaling, though beneficial in most situations, can be a problem if the scaling factors are poorly chosen. In the past some authors have expressed their concern about bad-scaling choices. According to Paige (1981), a model should be well scaled so that it represents the actual physical picture. Waller and coworkers (1995) have stated that a bad choice of variable scaling may provide results that are not representative of the characteristics of the plant. Scaling of variables, which is essential in many cases for accurate physical interpretation and numerical stability of process model, represents an important part of

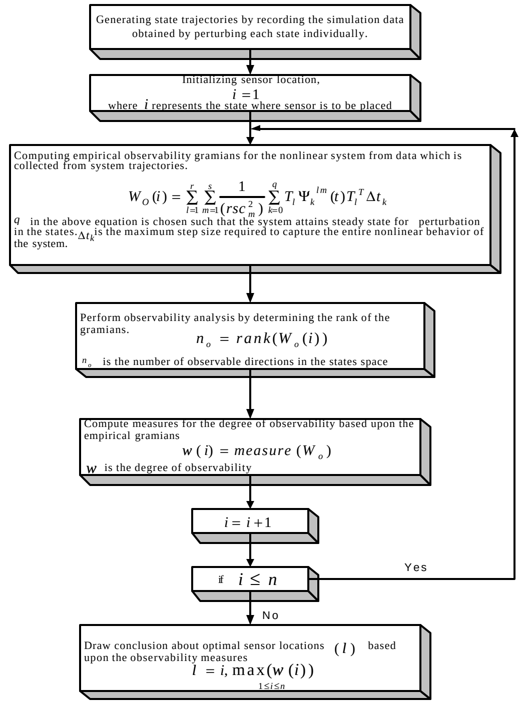

Fig. 1: Flowsheet for optimal sensor location.

computing measures for observability analysis. In order to address the scaling issue, the results from sensor location for the original model and for a scaled version of the model are compared in this work. Each variable, in the scaled model, is normalized by dividing it by its steady state value.

4.0 CASE STUDY

The presented sensor selection procedure will be applied to a specific example in this section. Since several different aspects will be investigated, the case study is limited to determining the optimal location if a single sensor is used. However, if the influence of placing a sensor on one part of the model has only a minor effect on the observability of other parts of the model, then conclusions can be drawn about subsequent placement of several sensors. The reader is referred to Hahn and Edgar (2002) for details on the calculation of the empirical gramians.

4.1 Distillation column model

For the analysis, a distillation column model with 30 trays for the separation of a binary mixture is considered (Hahn and Edgar, 2002). The column has 32 states and is assumed to have a constant relative volatility of 1.6. The feed is introduced in the middle compartment (17th tray) as a saturated liquid. The feed stream has a composition of $x_f = 0.5$, distillate and bottom product purities are $x_d = 0.935$ and $x_b = 0.065$, respectively. The reflux ratio is held at a constant value of 3.0.

4.2 Measuring product observability

In practice, composition sensors are generally placed at the top or bottom of the distillation column. The rationale behind this is to place the sensor close to the product for optimal product observability (Luyben, 1992). For the distillation column, the top and bottom products are given by the first and the last state of the model. Similarly, the first and the last diagonal entries of the empirical observability gramian correspond to the variance of the measurements that is caused by changes in these states. In order to gain as much information as possible about the product concentrations, it is desirable to determine sensor locations that maximize these two diagonal entries. Figure 2 shows a plot of the diagonal entries as the sensor location is varied along the height of the column for the original model and for the scaled system. It can be concluded from the figure that the most information about the bottom product can be obtained by directly measuring this product. However, contradictory conclusions could be drawn for the top product when comparing the results obtained for the scaled and the original model. The original model indicates that measuring the state corresponding to the top product will give the most information about this product. Additionally, the further the measurement is moved away from the top the harder it becomes to observe the top product. While the scaled model also indicates that measuring the top product is the best method for determining its value, the trend of decreasing "observability" of the top product as the measurement is moved away from this location does not hold for the scaled model. Instead, the scaled model could lead to the conclusion that measuring the bottom product may give more information about the top product than some states near the top product. These results are surprising at first glance, because they clearly indicate that scaling a model can have a negative effect on sensor selection by leading to conclusions that have no physical interpretation. The reason for this behavior is that all states in this model correspond to concentrations of the same component.

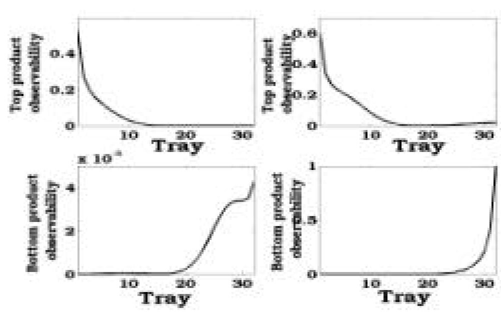

Fig. 2: Top and bottom product observability with sensor location at different trays for the original model (top left for top product and bottom left for bottom product) and scaled model (top right for top product and bottom right for bottom product)

Thereby, if a scaled model is used then a slight increase in the absolute value of the molar flow in one of the product streams can lead to a significant change of this value on a percentage basis. Due to this, the observability of the scaled model is skewed towards the states with lower concentrations (in this case towards the bottom of the column). While scaling in general may be an important part of this analysis, it can be concluded that at least for this specific case, it would result in misleading information.

4.3 Determining optimal sensor locations

Product measurement sensitivity is not the only factor when sensor locations are investigated. It is just as important to place sensors at locations where the most information about the state of the process can be gained. When analyzing the results obtained for sensor location by using the spectral norm and the trace of the empirical observability gramian, both measures return the same optimal location. However, when comparing the results for the scaled model and the original model (Figure 3), major difference between these two approaches can again be seen. For the original model the 6^{th} state seems to be the best location and the 25^{th} state would be the second best option, if one is interested in placing an additional sensor in the stripping section. For the scaled model, the bottom-most state in the stripping section (32^{nd} state) would seem to be the best option. However, the sensitivity at the very top and bottom is at a minimum for most columns as the driving force for mass transfer is small compared to rest of the column (Luyben, 1992). Hence, the best sensor location should be at a certain distance from either end of the column. The results for the original model are in line with this physical knowledge, but the results for the scaled model would lead to contradicting conclusions. This investigation can serve as another indicator that special attention has to be paid when considering the use of scaling for sensor location. At least for this type of investigation, scaling should be avoided. Another conclusion that can be drawn from the results is that the feed tray is not a good choice for placing a sensor. This conclusion is based upon observations using both types of measures for the scaled and the original model. This seems physically

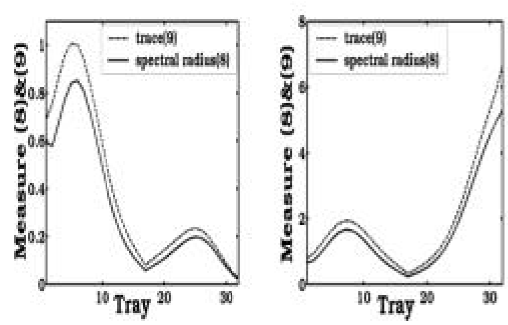

Fig. 3: Sensor location with trace/norm measure for original model (left) and scaled model (right)

justified as the feed tray is affected by the feed composition and is least sensitive to perturbations around its nominal point. In order to corroborate these findings, the location of the feed tray has been varied. Figure 4 shows the measures for a feed located at the 10^{th} and the 25^{th} tray. In both cases, the feed tray seems to be the worst location for placing a sensor. However, it should be pointed out that measuring the feed concentration, but not the concentration on the feed tray, can have beneficial effects if feedforward control strategies are to be applied. Another observation from this case study is that the measures based on the smallest singular value do not provide any relevant results for sensor placement.

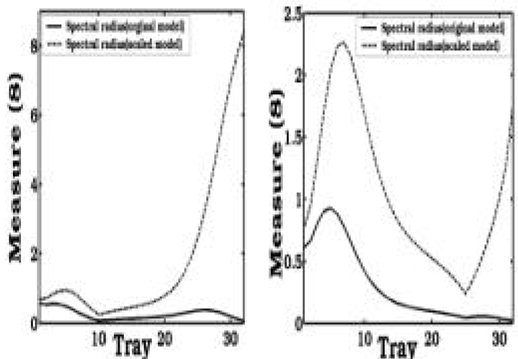

Fig. 4: Sensor location with 10^{th} tray as feed tray (left) and 25^{th} tray as feed tray (right)

5. CONCLUSION

This paper investigated a technique for computing optimal sensor locations for nonlinear dynamic systems. This has been achieved by first computing the empirical observability gramian for the nonlinear system over a pre-specified operating region and then determining observability measures based upon the empirical gramian. This method has the advantage over other techniques in that it does not resort to linearization of the model while at the same time it is computationally tractable. In addition it is possible to automate the presented procedure enabling computation of optimal sensor locations by formulating and solving an optimization problem.

When analyzing the various observability measures used for sensor location it can be concluded that for systems where the number of states far exceeds the number of measurements, as in the presented case study, methods based on maximal response energy provide better results than methods based upon maximizing observability of the least observable direction as the magnitude of the smallest singular value is often very close to zero. Spectral norm and trace of the empirical gramian are measures that can result in more accurate information, especially if some of the singular values are close to zero. In the presented case study, the measures indicate that there is one location in the stripping and one location in the rectifying section which are optimal for sensor placement.

Another important aspect of this work was investigation of the effect of scaling on the sensor placement. While scaling would seem to be a natural choice when computing gramians, it can lead to false conclusions in this investigation. The results indicate that scaling may not result in appropriate information if all the variables are representing the same type of physical property, i.e. concentrations for this case. The effect that scaling has on the computation of gramians when different types of variables are present, e.g. temperature and concentration variables, will be investigated in future work.

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