NUMERICAL METHODS FOR LARGE SCALE MOVING HORIZON ESTIMATION AND CONTROL

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Abstract: Both the linear moving horizon estimator and controller may be solved by solving a linear quadratic optimal control problem. A primal active set, a dual active set, and an interior point algorithm for solution of the linear quadratic optimal control problem are presented. The major computational effort in all these algorithms reduces to solution of certain unconstrained linear quadratic optimal control problems. A Riccati recursion procedure for efficient solution of such unconstrained problems is stated.

Keywords: Linear Model Predictive Control, Quadratic Programming, Numerical Methods

1. INTRODUCTION

Model predictive control has been established as the preferred advanced control technology in the process industries. Its success is attributed to its ability to handle hard constraints and its ability to use plant models identified by standard techniques. The implementation of model predictive control *systems* requires repeated on-line solution of a state estimation problem and an optimal control problem. Both the state estimator and the optimal controller can be formulated as constrained optimization problems (Allgöwer *et al.*, 1999; Binder *et al.*, 2001). Formulated as optimization problems and implemented in a moving horizon manner, the estimator is called a moving horizon estimator (MHE) and the controller is called a model predictive controller (MPC) or a moving horizon controller. Due to the requirement of real-time solution of both the estimation and control problem, the range and scale of processes that can be controlled by model predictive control systems depends critically on the ability to solve the constrained optimization problems efficiently and reliably.

In this paper we outline numerical methods for efficient solution of constrained moving horizon estimation and control problems for systems described by affine models and with quadratic objective functions. Both the moving horizon estimator and controller of such systems are formulated as sparse convex quadratic programs with equality and inequality constraints. This formulation is in contrast to the formulation in which the states are eliminated and the optimization problem is formulated as a dense quadratic program with inequalities only. Previously, both interior-point and active set algorithms for the solution of the quadratic program corresponding to a linear-quadratic optimal control based on a banded-matrix factorization have been proposed for solution of the model predictive control problem (Wright, 1996). The active set algorithm suggested in Wright (1996) is of the

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primal type and the linear algebra is based on updating an LU-factorization each time a constraint is added to or removed from the current working set of active constraints. The quadratic programs corresponding to the moving horizon controller and estimator, respectively, have also been solved by interior-point algorithms using Riccati iterations for the matrix factorization (Rao *et al.*, 1998; Tenny and Rawlings, 2002). The Riccati facotization is very efficient for solution of the KKT system as it exploits the specific block-diagonal structure arising in the moving horizon control and estimation problem. Alternatively, a dual active set algorithm using the Schur-complement and a general sparse matrix factorization has been suggested for solution of the model predictive control problem (Bartlett *et al.*, 2000).

We show that in the primal active-set, the dual active-set, and the interior-point algorithm for general convex quadratic programming problems, the search direction may be computed by solving a Karush-Kuhn-Tucker (KKT) system with the same structure as an equality constrained quadratic program. In quadratic programs of constrained optimal control problems, the KKT-system corresponding to the unconstrained control problem has a special structure and may be solved efficiently by a Riccati recursion. Both the constrained moving horizon estimation problem and the moving horizon control problem may be formulated as an optimal control problem and solved efficiently using the Riccati recursion.

The key contribution of this paper is a dual algorithm based on Riccati based factorization for solution of the quadratic program constituting a constrained linear-quadratic optimal control problem. It is demonstrated how the constrained linear-quadratic optimal control problem may represent a moving horizon controller and a moving horizon estimator. Furthermore, the importance and key role of the KKT-system in numerical solution of the constrained-linear quadratic optimal control problem by the primal active set algorithm as well as the dual active set algorithm and the interior-point algorithm is emphasized and explained.

2. ALGORITHMS FOR SOLUTION OF CONVEX QPS

Consider the general convex quadratic program

$$\min_{x\in\mathbb{R}^n} \quad f(x) = \frac{1}{2}x'Gx + g'x \tag{1a}$$
$$s.t. \quad A'x = b \tag{1b}$$
$$C'x \geq d \tag{1c}$$

in which the Hessian matrix $G \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite. Assume further that the matrix $A \in \mathbb{R}^{n\times m}$ has full column rank and that the KKT-matrix

$$\begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix} \tag{2}$$

is non-singular. Further let the columns of the matrix $C \in \mathbb{R}^{n\times m_I}$ be denoted $c_i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, 2, \ldots, m_I\}$. The Lagrangian of (1) is

$$\mathcal{L}(x,\pi,\lambda) = \frac{1}{2}x'Gx + g'x - \pi'(A'x-b) - \lambda'(C'x-d) \tag{3}$$

and the necessary and sufficient conditions for optimality of (1) are

$$Gx + g - A\pi - \sum_{i\in\mathcal{A}(x)} c_i\lambda_i = 0 \tag{4a}$$
$$A'x = b \tag{4b}$$
$$C'x \geq d \tag{4c}$$
$$\lambda_i \geq 0 \qquad i \in \mathcal{A}(x) = \{i \in \mathcal{I} : c_i'x = d_i\} \tag{4d}$$
$$\lambda_i = 0 \qquad i \in \mathcal{I} \setminus \mathcal{A}(x) \tag{4e}$$

Active set algorithms for solution of (1) apply the conditions (4) in the search for the minimizer of (1). A current working set $\mathcal{W} \subset \mathcal{A}(x)$ of active constraints is recurred and the search direction is constructed by utilizing $\lambda_i = 0$ for $i \in \mathcal{I}\setminus\mathcal{W}$ and $c_i'x = d_i$ for $i \in \mathcal{W}$ and partly ignoring the remaining inequalities.

At a primal feasible point, primal active set algorithms compute the primal search direction p along which the objective function decreases (Gill and Murray, 1978). Along with the search direction p, the equality constraint Lagrange multipliers π, and the Lagrange multipliers λ associated with the current working set of active constraints are computed. (p, π, λ) are computed as the solution of

$$\begin{bmatrix} G & -A & -F \\ -A' & 0 & 0 \\ -F' & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \pi \\ \lambda \end{bmatrix} = - \begin{bmatrix} Gx+g \\ 0 \\ 0 \end{bmatrix} \quad F = [c_i]_{i\in\mathcal{W}} \tag{5}$$

The next iterate is $\bar{x} = x + \alpha p$ in which the step length α is selected to maintain primal feasibility, i.e. $c_i'\bar{x} \geq d_i$ for all $i \in \mathcal{I} \setminus \mathcal{W}$. Optimality of the current iterate is determined by the sign of λ. The current iterate is optimal if $\lambda \geq 0$ and $p = 0$. At each iteration either an inequality constraint is appended to or removed from the current working set of active constraints. This implies that a column is appended to the matrix F when an inequality constraint is appended to the current working set of active constraints. A column is removed from the matrix F when a constraint is removed from the current working set. Accordingly, the KKT-matrix at each iteration changes by a single column and row. Efficient active set algorithms exploit this simple modification of the KKT-matrix such that it is not refactorized at each iteration, but rather its factorization is updated. Initially, a primal feasible point must be provided as primal active set algorithms proceed from primal feasible points and maintains primal feasibility while improving the value of the program. Determination of the feasible point, for instance by solving a phase-I linear program, may be a substantial part of the overall computations in determining an optimal point.

Primal active set algorithms proceed by solving a sequence of equality constrained quadratic programs in which some of the inequality constraints are treated as equalities. Simultaneously, these algorithms generate the iterates such that primal feasibility is maintained. Dual active set algorithms proceed by generating iterates such that the objective function of the dual program is improved while maintaining dual feasibility (Goldfarb and Idnanini, 1983). The dual program of (1) is

$$\max_{x,\pi,\lambda} \quad \mathcal{L}(x,\pi,\lambda) \tag{6a}$$
$$s.t. \quad Gx - A\pi - C\lambda = -g \tag{6b}$$
$$\lambda \geq 0 \tag{6c}$$

Let inequality constraint r, i.e. $c_r'x \geq d_r$, be violated. In the dual active set algorithm, the search direction (p, w, v) along which the dual objective function increases is computed as the solution to the KKT system

$$\begin{bmatrix} G & -A & -F \\ -A' & 0 & 0 \\ -F' & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ w \\ v \end{bmatrix} = \begin{bmatrix} c_r \\ 0 \\ 0 \end{bmatrix} \qquad F = [c_i]_{i \in \mathcal{W}} \tag{7}$$

The next iterate $(\bar{x}, \bar{\pi}, \bar{\lambda})$ is computed by $\bar{x} = x + \alpha p$, $\bar{\pi} = \pi + \alpha w$, $\bar{\lambda}_{\mathcal{W}} = \lambda_{\mathcal{W}} + \alpha w$, $\bar{\lambda}_r = \lambda_r + \alpha$. $\lambda_{\mathcal{W}}$ is the Lagrange multipliers associated to the inequality constraints in the current working set of primal active constraints (and dual inactive constraints). The step length α is selected such that dual feasibility, i.e. $\lambda \geq 0$, is maintained. By construction of the search direction, the constraint (6b) is satisfied as $\lambda_i = 0$ for $i \in \mathcal{I} \setminus \mathcal{W}$. A particular advantage of the dual active set algorithm, is that an initial dual feasible point is readily available. It may be obtained as the solution to KKT-system of the equality constrained primal program (1a)-(1b), i.e. as the solution of

$$\begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \tag{8}$$

Interior-point algorithms for solution of (1) are not based on the optimality conditions (4) but on necessary and sufficient optimality conditions in the following form

$$Gx + g - A\pi - C\lambda = 0 \tag{9a}$$
$$A'x = b \tag{9b}$$
$$C'x \geq d \tag{9c}$$
$$\lambda \geq 0 \tag{9d}$$
$$\lambda_i(c_i'x - d_i) = 0 \qquad i \in \mathcal{I} \tag{9e}$$

Instead of the active set condition, $\lambda_i \geq 0$ for $i \in \mathcal{A}(x)$ and $\lambda_i = 0$ for $i \in \mathcal{I} \setminus \mathcal{A}(x)$, the formulation (9) is based on the complementarity condition (9e). Introduce slack variables, $t = C'x - d \geq 0$, and the notation

$$T = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_{m_I} \end{bmatrix} \; \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{m_I} \end{bmatrix} \; e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{10}$$

such that the conditions (9) may be expressed as

$$Gx + g - A\pi - C\lambda = 0 \tag{11a}$$
$$A'x - b = 0 \tag{11b}$$
$$C'x - d - t = 0 \tag{11c}$$
$$T\Lambda e = 0 \tag{11d}$$
$$(t, \lambda) \geq 0 \tag{11e}$$

These conditions may be regarded as a system of nonlinear equations represented as $F(x, \pi, \lambda, t) = 0$ with the requirement $(t, \lambda) \geq 0$. The Mehrotra predictor-corrector algorithm is an interior-point method which computes the search direction as a combination of a predictor and a corrector step (Rao *et al.*, 1998). At both the predictor and corrector step t and λ are maintained in the interior, i.e. $(t, \lambda) > 0$. The predictor step is a pure Newton step for $F(x, \pi, \lambda, t)$, i.e. (11a)-(11d), while the corrector step is a modified Newton step. In both cases, the structure of the equations solved in computing the search direction is

$$\begin{bmatrix} G & -A & -C & 0 \\ -A' & 0 & 0 & 0 \\ -C' & 0 & 0 & I \\ 0 & 0 & T_k & \Lambda_k \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \pi \\ \Delta \lambda \\ \Delta t \end{bmatrix} = - \begin{bmatrix} r_G \\ r_A \\ r_C \\ r_\Lambda \end{bmatrix} \tag{12}$$

3. KKT-SYSTEMS

The major computational effort in algorithms for general convex quadratic programs (1) is computation of the search direction. This corresponds to solution of (5) in the primal active set algorithm, (7) in the dual active set algorithm, and (12) in the interior point algorithm, respectively.

Consider, the equality constrained quadratic subproblem of (1)

$$\min_{x \in \mathbb{R}^n} \quad f(x) = \frac{1}{2}x'Gx + g'x \tag{13a}$$
$$s.t. \quad A'x = b \tag{13b}$$

The solution of this program is computed as the solution of the following KKT system

$$\begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \tag{14}$$

The solution of (5), (7), and (12) may essentially be reduced to solution of systems with the structure (14). This is advantageous if (14) has a special structure that facilitates its efficient computation.

Proposition 1. (Schur-complement solution). Let $G \in \mathbb{R}^{n \times n}$ be symmetric positive semi-definite. Let $\begin{bmatrix} A & F \end{bmatrix} \in \mathbb{R}^{n \times (m + m_F)}$ have full column rank. Let the KKT-matrix (2) be non-singular. Then

$$S = \begin{bmatrix} F' & 0 \end{bmatrix} \begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix}^{-1} \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{15}$$

is symmetric positive definite and has the Cholesky factorization $S = LL'$. Furthermore, the unique solution (p, s, u) of

$$\begin{bmatrix} G & -A & -F \\ -A' & 0 & 0 \\ -F' & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ s \\ u \end{bmatrix} = - \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

may be obtained by solving the following sequence of equations

$$\begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ s_0 \end{bmatrix} = - \begin{bmatrix} h \\ 0 \end{bmatrix} \tag{17a}$$

$$LL'u = -F'p_0 \tag{17b}$$

$$\begin{bmatrix} G & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta s \end{bmatrix} = - \begin{bmatrix} -Fu \\ 0 \end{bmatrix} \tag{17c}$$

$$\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} p_0 \\ s_0 \end{bmatrix} + \begin{bmatrix} \Delta p \\ \Delta s \end{bmatrix} \tag{17d}$$

Proof. See Ouellette (1981). □

Proposition 1 may be used for solution of (5) in the primal active set algorithm and solution of (7) in the dual active set algorithm. In the first case $h = Gx + g$ and in the latter case $h = -c_r$. The matrix S is not computed from scratch at each iteration. Rather, its Cholesky factorization is updated utilizing that the matrix F changes by a single column at each iteration in the active set algorithms. The Cholesky factor, L, is treated as a dense matrix. Hence, the method is most efficient when only a few inequality constraints are active at the optimal solution. The method is only efficient, when the KKT-matrix (2) has a sparse structure that can be utilized. The KKT matrix (2) used in proposition 1 remains constant and need only to be factorized once.

Proposition 2. (Interior-Point KKT System). Let $G \in \mathbb{R}^{n \times n}$ be symmetric positive semi-definite. Let (2) be non-singular. Let $(t_k, \lambda_k) > 0$. Then the unique solution $(\Delta x, \Delta\pi, \Delta\lambda, \Delta t)$ of (12) may be obtained by computation of the unique solution of

$$\begin{bmatrix} G + CT_k^{-1}\Lambda_k C' & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\pi \end{bmatrix} = - \begin{bmatrix} \tilde{r}_G \\ r_A \end{bmatrix} \tag{18a}$$

in which

$$\tilde{r}_G = -r_G + CT_k^{-1}(-r_\Lambda + \Lambda_k r_C) \tag{18b}$$

and subsequent computation of

$$\Delta t = -r_C + C'\Delta x \tag{18c}$$

$$\Delta\lambda = T_k^{-1}(-r_\Lambda - \Lambda_k \Delta t) \tag{18d}$$

Proof. Follows by simple rearrangement of (12). See Rao *et al.* (1998). □

As is evident from proposition 2, the computational burden in finding the search direction of the interior-point algorithm is solution of (18a). The structure of (18a) is identical to the structure of (14), and therefore the computation of the search direction in the interior-point algorithm is facilitated by efficient solution of (14). The KKT-matrix in (18a) changes as (λ_k, t_k) changes and must therefore be refactorized at each iteration.

4. LINEAR QUADRATIC OPTIMAL CONTROL

Proposition 1 and 2 provide methodologies for solution of inequality and equality constrained convex quadratic programming. The efficiency of these methods depends on the efficiency of the solution method for the corresponding equality constrained convex quadratic program.

The unconstrained linear quadratic optimal control problem is the equality constrained convex quadratic program

$$\min_{\{x_{k+1}, u_k\}_{k=0}^{N-1}} \quad \phi = \sum_{k=0}^{N-1} l_k(x_k, u_k) + l_N(x_N) \tag{19a}$$

$$s.t. \qquad x_{k+1} = A_k' x_k + B_k' u_k + b_k \tag{19b}$$

in which the stage costs of the objective function are

$$l_k(x_k, u_k) = \frac{1}{2}\left(x_k' Q_k x_k + 2x_k' M_k u_k + u_k' R_k u_k\right) + q_k' x_k + r_k' u_k + f_k \tag{20a}$$

$$l_N(x_N) = \frac{1}{2} x_N' P_N x_N + p_N' x_N + \gamma_N \tag{20b}$$

The matrices $\begin{bmatrix} Q_k & M_k \\ M_k' & R_k \end{bmatrix}$ and P_N are assumed to be symmetric positive semi-definite and the KKT-matrix of the problem is assumed to be non-singular.

The necessary and sufficient optimality conditions for (19) are

$$Q_k x_k + M_k u_k + q_k - \pi_{k-1} + A_k \pi_k = 0 \tag{21a}$$

$$M_k' x_k + R_k u_k + r_k + B_k \pi_k = 0 \tag{21b}$$

$$P_N x_N + p_N - \pi_{N-1} = 0 \tag{21c}$$

$$x_{k+1} = A_k' x_k + B_k' u_k + b_k \tag{21d}$$

In the case $N = 2$, the necessary and sufficient optimality conditions may be expressed as the KKT-system

$$\left[\begin{array}{cccc|cc} R_0 & & & & B_0 & \\ & Q_1 & M_1 & & -I & A_1 \\ & M_1' & R_1 & & & B_1 \\ & & & P_2 & & -I \\ \hline B_0' & -I & & & & \\ & A_1' & B_1' & -I & & \end{array}\right] \left[\begin{array}{c} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \hline \pi_0 \\ \pi_1 \end{array}\right] = - \left[\begin{array}{c} r_0 + M_0' x_0 \\ q_1 \\ r_1 \\ p_2 \\ \hline b_0 + A_0' x_0 \\ b_1 \end{array}\right]$$

which may be rearranged to

$$\begin{bmatrix} R_0 & B_0 & & & & \\ B_0' & 0 & -I & & & \\ & -I & Q_1 & M_1 & A_1 & \\ & & M_1' & R_1 & B_1 & \\ & & A_1' & B_1' & 0 & -I \\ & & & & -I & P_2 \end{bmatrix} \begin{bmatrix} u_0 \\ \pi_0 \\ x_1 \\ u_1 \\ \pi_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} r_0 + M_0' x_0 \\ b_0 + A_0' x_0 \\ q_1 \\ r_1 \\ b_1 \\ p_2 \end{bmatrix}$$

The necessary and sufficient conditions (21) for optimality of (19) are sparse and highly structured. The following proposition prescribes a Riccati iteration procedure for solution of (21).

Proposition 3. (LQ Optimal Control Solution). Let x_0, $\{A_k, B_k, b_k, Q_k, M_k, R_k, q_k, r_k\}_{k=0}^{N-1}$, and $\{P_N, p_N\}$ be given. Then the solution $\{u_k, \pi_k, x_{k+1}\}_{k=0}^{N-1}$ of (21) may be obtained by the following procedure

(1) Compute

$$R_{e,k} = R_k + B_k P_{k+1} B_k' \tag{22a}$$
$$K_k = -R_{e,k}^{-1}(M_k' + B_k P_{k+1} A_k') \tag{22b}$$
$$a_k = -R_{e,k}^{-1}(r_k + B_k(P_{k+1} b_k + p_{k+1})) \tag{22c}$$
$$P_k = Q_k + A_k P_{k+1} A_k' - K_k' R_{e,k} K_k \tag{22d}$$
$$\begin{aligned} p_k = &(A_k + K_k' B_k)(P_{k+1} b_k + p_{k+1}) \\ &+ q_k + K_k' r_k \end{aligned} \tag{22e}$$

for $k = N-1, N-2, \ldots, 0$.

(2) Compute the primal solution $\{u_k, x_{k+1}\}_{k=0}^{N-1}$ for $k = 0, 1, \ldots, N-1$ by

$$u_k = K_k x_k + a_k \tag{23a}$$
$$x_{k+1} = A_k' x_k + B_k' u_k + b_k \tag{23b}$$

(3) Obtain the dual solution $\{\pi_k\}_{k=0}^{N-1}$ by computing

$$\pi_{N-1} = P_N x_N + p_N \tag{24a}$$
$$\pi_{k-1} = A_k \pi_k + Q_k x_k + M_k u_k + q_k \tag{24b}$$

for $k = N-1, N-2, \ldots, 1$.

Proof. See Rao *et al.* (1998) or Ravn (1999). □

Let $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then this method has complexity $O(N(n^3 + m^3))$ while a dense method on the same KKT-system has complexity $O(N^3(n+m)^3)$. Corresponding dense quadratic programs obtained by elimination of the states has complexity $O(N^3 m^3)$. The Riccati based factorization is thus two order of magnitudes faster than the dense based approach when the horizon N length is much larger than the state dimension n.

The constrained linear quadratic optimal control problem is (19) with the additional constraints

$$C_k' x_k + D_k' u_k + c_k \geq d_k \quad k = 0, 1, \ldots, N-1 \tag{25a}$$
$$C_N' x_N + c_N \geq d_N \tag{25b}$$

The search direction in algorithms for solution of this problem is computed efficiently by combination of proposition 3 and either proposition 1 or 2.

The model predictive controller and the moving horizon estimator are particular instances of the optimal control problem, i.e. (19) and (25). For simplicity and to focus on the essentials, this is demonstrated for the model predictive controller and moving horizon estimator without inequality constraints. These formulations are easily extended to the inequality constrained cases.

5. MODEL PREDICTIVE CONTROL

The unconstrained model predictive controller for linear systems may be expressed as

$$\min_{\{y_k, w_{k+1}, u_k\}_{k=0}^{\infty}} \quad \phi = \sum_{k=0}^{\infty} \tilde{l}_k(y_k, \Delta u_k) \tag{26a}$$
$$s.t. \quad w_{k+1} = \tilde{A}_k' w_k + \tilde{B}_k' u_k + \tilde{b}_k \tag{26b}$$
$$y_k = \tilde{C}_k' w_k + \tilde{c}_k \tag{26c}$$

in which the stage cost is

$$\tilde{l}_k(y_k, \Delta u_k) = \frac{1}{2}||y_k - z_k||_{\tilde{Q}_k}^2 + \frac{1}{2}||\Delta u_k||_{\tilde{S}_k}^2 \tag{27}$$

The goal stated by this cost, is to keep the systems output $\{y_k\}_{k=0}^{\infty}$ close to some prescribed trajectory $\{z_k\}_{k=0}^{\infty}$ while simultenously limiting the actuator variation $\Delta u_k = u_k - u_{k-1}$. Let the data at stage k of this controller be $\mathcal{M}_k = \left\{\tilde{A}_k, \tilde{B}_k, \tilde{b}_k, \tilde{C}_k, \tilde{c}_k, \tilde{Q}_k, \tilde{S}_k\right\}$ and let $\mathcal{M} = \left\{\tilde{A}, \tilde{B}, \tilde{b}, \tilde{C}, \tilde{c}, \tilde{Q}, \tilde{S}\right\}$. Assume that the controller (26) is parameterized as

$$\mathcal{C} = \{\mathcal{M}_k\}_{k=0}^{\infty} = \{\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{N-1}, \mathcal{M}, \mathcal{M}, \ldots\} \tag{28}$$

and that the reference trajectory has the parameterization

$$\{z_k\}_{k=0}^{\infty} = \{z_0, z_1, \ldots, z_{N-1}, z, z, \ldots\} \tag{29}$$

The optimal steady state consistent with the controller model (26) is obtained by solution of the quadratic program

$$\min_{u,w,y} \quad \frac{1}{2}||y - z||_{\tilde{Q}}^2 + \frac{1}{2}||u - u_s||_{R_s}^2 \tag{30a}$$
$$s.t. \quad w = \tilde{A}' w + \tilde{B}' u + \tilde{b} \tag{30b}$$
$$y = \tilde{C}' w + \tilde{c} \tag{30c}$$

in which u_s is a target of the input if there are degrees of freedom in excess and R_s is computed by the procedure (Muske, 1997)

$$\tilde{N} = \text{Null}\left(\begin{bmatrix} I - \tilde{A}' & -\tilde{B}' \end{bmatrix}\right) = \begin{bmatrix} \tilde{N}_x \\ \tilde{N}_u \end{bmatrix} \tag{31a}$$
$$\alpha = \text{Null}\left(\tilde{N}_x' \tilde{C} \tilde{C}' \tilde{N}_x\right) \tag{31b}$$
$$R_s = \tilde{R} \tilde{N}_u \alpha \alpha' \tilde{N}_u' \tilde{R} \tag{31c}$$

When a target (u, w) has been computed the model predictive controller algorithm solves the dynamic quadratic program

$$\min \quad \phi = \sum_{k=0}^{N-1} l_k(x_k, u_k) + l_N(x_N) \tag{32a}$$
$$s.t. \quad x_{k+1} = A_k' x_k + B_k' u_k + b_k \tag{32b}$$

in which

$$x_k = \begin{bmatrix} w_k \\ u_{k-1} \end{bmatrix} \; A_k' = \begin{bmatrix} \tilde{A}_k' & 0 \\ 0 & 0 \end{bmatrix} \; B_k' = \begin{bmatrix} \tilde{B}_k' \\ I \end{bmatrix} \; b_k = \begin{bmatrix} \tilde{b}_k \\ 0 \end{bmatrix} \tag{33}$$

and $l_k(x_k, u_k)$ is of the form (20a) with the parameters

$$Q_k = \begin{bmatrix} \tilde{C}_k \tilde{Q}_k \tilde{C}_k' & 0 \\ 0 & \tilde{S}_k \end{bmatrix} \quad M_k = \begin{bmatrix} 0 \\ -\tilde{S}_k \end{bmatrix} \quad R_k = \tilde{S}_k \tag{34a}$$

$$q_k = \begin{bmatrix} \tilde{C}_k \tilde{Q}_k (\tilde{c}_k - z_k) \\ 0 \end{bmatrix} \qquad r_k = 0 \tag{34b}$$

$$f_k = \frac{1}{2}(\tilde{c}_k - z_k)' \tilde{Q}_k (\tilde{c}_k - z_k) \tag{34c}$$

Let

$$x = \begin{bmatrix} w \\ u \end{bmatrix} \quad Q = \begin{bmatrix} \tilde{C}\tilde{Q}\tilde{C}' & 0 \\ 0 & \tilde{S} \end{bmatrix} \quad M = \begin{bmatrix} 0 \\ -\tilde{S} \end{bmatrix} \quad R = \tilde{S} \tag{35}$$

and compute P from the Riccati equation

$$\begin{aligned} P &= Q + APA' \\ &\quad - (M + APB')(R + BPB')^{-1}(M + APB')' \end{aligned} \tag{36}$$

Then the selected cost-to-go function in (32)

$$l_N(x_N) = \frac{1}{2}(x_N - x)' P (x_N - x) \tag{37}$$

is identical to (20b) with the parameters

$$P_N = P \quad p_N = -Px \quad \gamma_N = \frac{1}{2} x' P x \tag{38}$$

Consequently, the dynamic quadratic program of model predictive control is an instance of a linear quadratic optimal control problem.

6. CONCLUSION

Model predictive control and moving horizon estimation with affine models are instances of the linear quadratic optimal control problem. By utilizing the structure of the linear quadratic optimal control problem, efficient methods for solution of the linear quadratic optimal control problem have been outlined. These methods are linear in the control horizon. They are therefore particularly attractive for predictive control with long horizons. Long horizons in predictive control are often preferable, because nominal stability can be guaranteed provided the control horizon is sufficiently large.

The search-direction of active-set algorithms for solution of the constrained linear quadratic optimal control problems, (19) and the additional constraints (25), may be efficiently computed using the decomposition stated in proposition 1 and the Riccati iteration stated in proposition 3. When the model parameterization is fixed a priori, as for instance for linear time invariant systems, the factorization of the unconstrained linear quadratic optimal control problem may be done off-line, and the method increases even further in efficiency.

The search direction of the Mehrotra predictor-corrector interior-point algorithm for solution of the constrained linear quadratic optimal control problem may be efficiently computed by combination of proposition 2 specialized to (19,25) and proposition 3.

The algorithms outlined for solution of the linear quadratic optimal control problem can be applied as the QP-solver for solution of the nonlinear optimal control problem by SQP algorithms.

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