Proceedings of the 9th International Symposium on
Dynamics and Control of Process Systems (DYCOPS 2010),
Leuven, Belgium, July 5-7, 2010
Mayuresh Kothare, Moses Tade, Alain Vande Wouwer, Ilse Smets (Eds.)

A Risk based Approach to Estimate Key Uncertainties

Farminder S. Anand*, Matthew J. Realff, Jay H. Lee*****

** School of Chemical & Biomolecular Engineering*
Georgia Institute of Technology, Atlanta, GA 30332 USA
Tel: 404-642-9151; e-mail: farminder.anand@ chbe.gatech.edu).
*** School of Chemical & Biomolecular Engineering*
Georgia Institute of Technology, Atlanta, GA 30332 USA
(e-mail: matthew.realff@chbe.gatech.edu)
**** School of Chemical & Biomolecular Engineering*
Georgia Institute of Technology, Atlanta, GA 30332 USA
(e-mail: jay.lee@chbe.gatech.edu)

Abstract: Decisions during the early stages of R&D are often made under substantial uncertainty. Evaluation of R&D alternatives under uncertainty generally does not provide a clear choice that is best under all possible scenarios. For optimal investment of R&D resources, it is important to identify the key uncertainty contributors from a decision maker's perspective. Global sensitivity analysis (GSA) is a tool that can be used to determine key uncertainties that contribute the most to the variance of the bottom-line objective. It is often the case, however, that *GSA* is not able to distinguish between the uncertainties. Motivated by this, we propose a new tool called conditional – global sensitivity analysis, which further considers the decision-maker's risk preference. The conditional sensitivity measures ($cGSA^{up}/cGSA^{down}$) quantify the contributions of different individual uncertainty factors to the upper and lower halves of the distribution function of the objective function. It is argued that the use of $cGSA^{up}$ may appeal to a risk-aversive decision maker as it leads to a lower rate of false acceptance decisions at the expense of a higher rate of false rejection decisions, whereas the use $cGSA^{down}$ does the opposite.

Keywords: GSA, c-GSA, variance, Lower Partial Moment, R&D.

1. INTRODUCTION

Evaluation of the key uncertainty contributors in the R&D process is an essential step to reduce the cost of the whole process. Global sensitivity analysis (*GSA*) and expected value of information (*EVOI*) are two tools that can be used to estimate the key uncertainty contributors. Global sensitivity analysis considers variance as a measure of risk and the importance of various uncertainties is measured as the fraction of the overall variance contributed by each uncertain factor. On the other hand, *EVOI* directly measures the value of reducing uncertainty in each factor.

The evaluation of *GSA* is significantly simpler than that of *EVOI* since the calculation of *EVOI* requires more detailed information about the effect of experimentation on the uncertainty. Though, *EVOI* provides more direct and therefore useful information than *GSA* from a decision-maker's standpoint, the computational load associated with the *EVOI* limits its usefulness. In this work we supplement the traditional *GSA* approach with a new tool called *conditional global sensitivity analysis* (*cGSA)*, which provides further insight into the selection of the key uncertainty.

For a R&D investment decision, there are two sides to the problem. First is the R&D investment cost, e.g., the experimentation cost. Second is the return or profits when the R&D project becomes successful. In order to make the overall process more profitable, one can either focus on the reduction in the R&D cost or on maximizing the probability of success of chosen projects. To address the two different strategies we consider two different risk behaviours of the decision maker:

(a). ***Decision Delayer / Just A Few Alternatives****:* This type of decision maker wants to ensure a high probability of success and is ready to accept a relatively high R&D expense. If there are just a few alternatives, one is more focused on maximizing the probability of success. In such cases, one is more concerned with not removing a potentially successful alternative than reducing the cost of higher R&D investment. Hence such a decision maker would prefer an approach that will lead to a lower FALSE REJECTION rate, even if it requires that the decision maker has to bear a higher FALSE ACCEPTANCE rate.

*(b). **Decision Maker / Many Alternatives**:* This type wants to minimize the R&D investment and is ready to accept a relative lower probability of success. If there are large numbers of alternatives, it is needed to trim down the options to just a few alternatives. Hence a decision maker in such a situation would prefer an approach that will lead to a lower FALSE ACCEPTANCE rate, even if it entails a higher FALSE REJECTION rate.

When *GSA* is not able to distinguish between the uncertainties, *cGSA* (either $cGSA^{down}$ or $cGSA^{up}$) can be used for deciding the key uncertainty factor, depending on the situation and the decision maker's risk attitude.

The rest of the paper is structured as follows: Section 2 presents the background about *GSA*. Section 3 presents the new approach, *cGSA*, along with the algorithm to calculate *cGSA* and discusses the risk behaviours $cGSA^{up}$ and $cGSA^{down}$ support. Various examples to illustrate the utility of *cGSA* are presented in Section 4 and Section 5 concludes the chapter.

2. BACKGROUND

2.1 Global Sensitivity Analysis (GSA)

GSA calculates the relative importance of input variables or factors ($x_1, x_2,\ldots,x_k$) in determining the value of the output variable y. Assume a model $y = f(x_1, x_2,\ldots,x_k)$ is composed of independent random variables $x_1, x_2,\ldots,x_k$. Moreover assume that the probability density function of $x_1, x_2,\ldots,x_k$ is $p_1(x_1), p_2(x_2),\ldots,p_k(x_k)$. Given the above, the joint distribution of $x_1,x_2,\ldots,x_k$ is

$$P\left(x_1, x_2, ..., x_k\right) = \prod_{i=1}^{k} p_i\left(x_i\right) \quad (1)$$

The mean and variance of y can be calculates as:

$$E\left(y\right) = \iint\ldots\int f\left(x_1, x_2, ..., x_k\right)\prod_{i=1}^{k} p_i\left(x_i\right)dx_i \quad (2)$$

$$V\left(y\right) = \iint\ldots\int\left(f\left(x_1, x_2, ..., x_k\right) - E\left(y\right)\right)^2\prod_{i=1}^{k} p_i\left(x_i\right)dx_i$$
$$= \iint\ldots\int f^2\left(x_1, x_2, ..., x_k\right)\prod_{i=1}^{k} p_i\left(x_i\right)dx_i - E^2\left(y\right) \quad (3)$$

The first order sensitivity index is defined as:

$$S_j = \frac{V\left(E\left(y \mid x_j\right)\right)}{V\left(y\right)} = \frac{\left(U_j - E^2\left(y\right)\right)}{V\left(y\right)} \quad (4)$$

$$U_j = \int E^2\left(y \mid x_j = \tilde{x}_j\right)p_j\left(\tilde{x}_j\right)d\tilde{x}_j \quad (5)$$

Ishigami and Homma (1990) makes the calculation of the above using a single Monte Carlo loop. (Saltelli, Andres et al. 1993) suggested the following Monte Carlo based procedure for calculation of global sensitivity indexes.
Consider the two input sample matrices M_1 and M_2 as follows:

$$M_1 = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1k} \\ x_{21} & x_{22} & \ldots & x_{2k} \\ \ldots & & & \\ x_{n1} & x_{n2} & \ldots & x_{nk} \end{pmatrix}, \quad M_2 = \begin{pmatrix} x'_{11} & x'_{12} & \ldots & x'_{1k} \\ x'_{21} & x'_{22} & \ldots & x'_{2k} \\ \ldots & & & \\ x'_{n1} & x'_{n2} & \ldots & x'_{nk} \end{pmatrix} \quad (6)$$

where 'n' is the sample size used for the Monte Carlo estimation. E(y) can be calculated based on the values of y computed on matrices samples either from M_1 or M_2, whereas U_j can be calculated from values of y computed from sample from matrices M_1 and N_j.

$$N_j = \begin{pmatrix} x'_{11} & x'_{12} & \ldots & x_{1j} & \ldots & x'_{1k} \\ x'_{21} & x'_{22} & \ldots & x_{2j} & \ldots & x'_{2k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x'_{n1} & x'_{n2} & \ldots & x_{nj} & \ldots & x'_{nk} \end{pmatrix} \quad (7)$$

The approximation for U_j is as follows:

$$\hat{U}_j = \frac{1}{n-1}\sum_{r=1}^{n} f\left(x_{r1}, x_{r2}, ..., x_{rk}\right) f\left(x'_{r1}, x'_{r2}, ..., x'_{r(j-1)}, x_{rj}, x'_{r(j+1)}, ..., x'_{rk}\right) \quad (8)$$

The computational cost associated with the calculation of the all first order sensitivity indexes is S_j is $n*(k+1)$, where 'n' evaluations of '*f*' are needed to calculate *E(y)* and n sets of '*f*' for each first order sensitivity index.

3. CONDITIONAL – GLOBAL SENSITIVITY ANALYSIS

As explained in section 2.1, *GSA* provides information about the contribution by individual factors to the overall uncertainty, where uncertainty is measured in terms of variance. However, there are situations where *GSA* is unable to distinguish among several uncertain factors in terms of their importance, i.e. two or more factors have equal or nearly equal global sensitivity index (*GSI*). In such a case, it is difficult to differentiate between the factors and decide in which factor to reduce uncertainty through experimentation. In order to accommodate such a situation, a new tool called *cGSA* is presented, which may be able to differentiate between the factors having equal *GSI* based on the situation at hand and the decision maker's risk preference.

3.1 Definition

Conditional global sensitivity analysis is the measure of the contribution of variance by individual factors to the conditional objective function. Here conditional distribution refers to the part of the distribution of model output *y* above or below its mean $E(y)$, for $cGSA^{up}$ or $cGSA^{down}$, respectively. The conditional (upside or downside) global sensitivity index ($cGSI^{up}$ or $cGSI^{down}$) can be calculated using (9) or (10) compared to GSI which is calculated using (4).

$$cGSI_j^{up} = \frac{V\left(E\left(y \mid x_j, y > E\left(y\right)\right)\right)}{V\left(y \mid y > E\left(y\right)\right)} \tag{9}$$

$$cGSI_j^{down} = \frac{V\left(E\left(y \mid x_j, y < E\left(y\right)\right)\right)}{V\left(y \mid y < E\left(y\right)\right)} \tag{10}$$

3.2 Algorithm

Calculation of $cGSI^{up}/cGSI^{down}$ follows the same approach as *GSI* with the difference being that the random samples for the various factors conform to the conditionality constraint, i.e. the factors transformed by the model to the objective function value should lie above/below the mean of the model output *E*(*y*). So, in essence the problem is how to sample from the distribution of the factors so that the objective function values belong to the respective side of the distribution. This problem can be addressed by developing classification functions, which make sure that the model output will lie on the desired side of the mean of the model output distribution.

It is important to note, that the sampling from the factor distributions introduces correlation between the factors. Global sensitivity analysis problem for correlated factor distributions has been addressed in the literature (McKay 1995), (Saltelli 2002),(Jacques, Lavergne et al. 2006), (Xu and Gertner 2008). The generic idea is to follow the same approach for unconditional *GSA* but introduce correlation between the parameters by re-ordering the samples from the factor distributions by following the ordering scheme introduced in (Iman and Conover 1982). Moreover, the measure of uncertainty contribution for each factor is considered the same as for the traditional *GSA* approach, i.e., $V(E(Y|X_j))$ (Saltelli 2002).

The correlation introduced in *cGSA* cannot be calculated by using the re-ordering scheme of (Iman and Conover 1982), since the conditionality induces a relationship between the input factors that cannot be easily represented by either correlation or rank correlation. Hence in order to maintain the original distribution properties of the factors and account for the natural relationship introduced by the 'conditionality', we first sample the factors from their original distribution functions and then accept or reject the sample based on the result of the classification function separating the objective function values below and above the cut-off. (Note: this is very similar to the Acceptance-Rejection algorithm). The importance of using the classification function instead of the model is that the model can be computationally intensive and a high rejection rate can significantly increase the computational load.

Moreover, since the classification function would be an approximation, we would consider an enveloping classification function i.e. a classification function with high priority for acceptance of all possible valid combinations of factor values at the cost of relatively low priority for rejection of invalid combination of factor values. Though this would require additionally checking the validity of the sample point it would make the *cGSI* calculation more accurate. The complete procedure to calculate the *cGSI* given below assumes that *GSI* has been performed already and the user wants to perform *cGSI* for follow-up analysis:

3.3. *Theoretical Interpretation*

To understand the theoretical interpretation of the results of reducing uncertainty in the key $cGSA^{up}$ and $cGSA^{down}$ uncertainty and its application consider the following model output *y* given by (11).

$$y = x_1 + x_2 \tag{11}$$

$$x_1 = \mathrm{beta}(2,10) \tag{12}$$

$$x_2 = \mathrm{beta}(10,2) \tag{13}$$

The input uncertainties (x_1 and x_2) are given by (12) and (13). The distributions of x_1 and x_2 are positively and negatively skewed respectively as shown in figure 1. The distribution of model output y is shown in figure 2. Properties of the input uncertainties along with the model output are shown in Table 1.

Table 1: Statistical properties of the input uncertainty and the model output

	Mean	*Standard Deviation*	*Skewness*
x_1	*0.167*	*0.104*	*0.92*
x_2	*0.833*	*0.104*	*-0.92*
Y	*1.0*	*0.147*	*0*

Consider that the true decision criterion (14) is based on a *cutoff*, i.e. if the correct value of the model output lie above the *cutoff* the decision maker would keep the R&D alternative and if the model output lie below the *cutoff* the decision maker would reject the R&D alternative. For simplicity, assume the *cutoff* to be at the mean of the model output y (= 1). This means that the correct decision is to accept the technology 50% of the sampled cases and reject it in the other 50% of the cases. Consider the decision criterion under uncertainty (15) is driven by just the mean value, i.e., if the posterior mean ($\mu^i_{y,posterior}$, for realization i), after the uncertainty reduction, is below the *cutoff*, the alternative is rejected, and if the posterior mean ($\mu^i_{y,posterior}$) is above the *cutoff*, then the alternative is accepted. Furthermore, assume that if the decision maker tries to reduce uncertainty in any input factor, then the correct value of that factor would be known exactly, i.e. the posterior value of that input factor would be a point estimate, though the actual point estimate that results would be a sample from the prior distribution.

$$\text{True Decision} = \begin{cases} \text{Accept}, & \text{if} \quad y > cutoff \\ \text{Reject}, & \text{if} \quad y < cutoff \\ \text{No Decision}, & \text{otherwise} \end{cases} \tag{14}$$

$$\text{Decision Criterion} = \begin{cases} \text{Accept}, & \text{if} \quad \mu^i_{y,posterior} > cutoff \\ \text{Reject}, & \text{if} \quad \mu^i_{y,posterior} < cutoff \\ \text{No Decision}, & \text{otherwise} \end{cases} \tag{15}$$

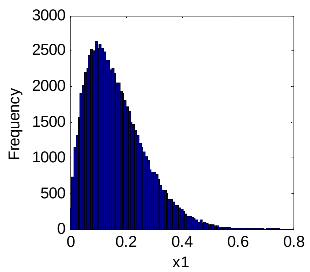

Figure 1: (a) Frequency distribution of random variable x_1, (b) Frequency distribution of random variable x_2.

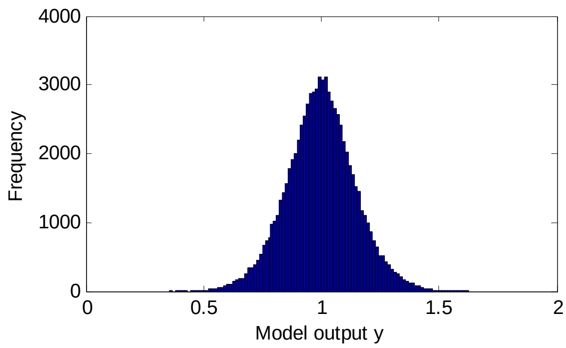

Figure 2: Frequency distribution of model output y.

$$y^i_{posterior} = \hat{x}^i_1 + x_2 \tag{16}$$

$$\mu^i_{y,posterior} = \hat{x}^i_1 + \mu_2 \tag{17}$$

$$E\left(\mu_{y,posterior}\right) = \mu_1 + \mu_2 \tag{18}$$

Since the two input uncertainties have similar distributions and the model structure is simply the addition of the two uncertainties, it is obvious that both the uncertainties will equally contribute to the uncertainty in the model output, which means they would have an equal GSI value. Additionally, x_1 will be identified as the key unceratainty if $cGSA^{up}$ is used, whereas x_2 will be if $cGSA^{down}$ is used. If the key $cGSA^{up}$ uncertainty is reduced, we will obtain a negatively skewed posterior distribution ($y^i_{posterior}$) for model output y for any particular realization i, but the mean of these posterior distributions ($\mu_{y,posterior}$) obtained over all possible realizations would be positively distributed. To clarify, if uncertainty x_1 is reduced then a posterior point estimate ($\hat{x}_1$) of x_1 is obtained, which is sampled from the positively skewed prior distribution of x_1 (12). Then, as in (16), the shape of the posterior distribution is mainly driven by the input uncertainty x_2, and hence $y^i_{posterior}$ is a negatively skewed distribution. The mean of the posterior distribution ($\mu^i_{y,posterior}$) for any realization i is given by (17), the distribution of which overall all possible realizations is positively skewed as can be seen from (17). The expected value of the posterior mean of y (i.e. $\mu_{y,posterior}$) is constant (18), where μ_1 and μ_2 are the mean of the prior distributions of x_1 and x_2, respectively.

Similarly, if uncertainty in key $cGSA^{down}$ uncertainty is reduced, we will obtain a positively skewed posterior distribution for model output y. The frequency distribution of the mean of the posterior model output for scenarios reducing uncertainty in key $cGSA^{up}$ and key $cGSA^{down}$ uncertainty are shown in figure 3 and figure 4.

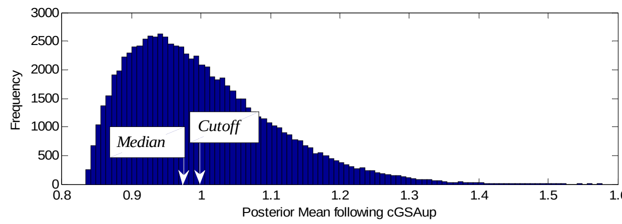

Figure 3: Frequency distribution of posterior mean of model output *y* when key $cGSA^{up}$ uncertainty (i.e. x_1) is reduced

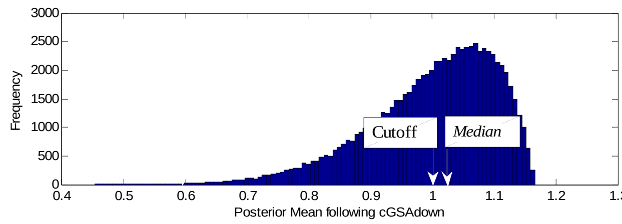

Figure 4: Frequency distribution of posterior mean of model output *y* when key $cGSA^{down}$ uncertainty (i.e. x_2) is reduced.

In order to evaluate the pros and cons of the $cGSA^{up}$ and $cGSA^{down}$, false acceptance and false rejection is defined by (29) and (30) respectively, where N symbolizes the frequency. False acceptance is defined as the percentage of reject decisions which are falsely concluded to be acceptance decision. Similarly false rejection is defined as the percentage of the correct acceptance decisions which are falsely rejected.

$$\text{False Acceptance} = 100 / N(\text{Correct Decision = Reject}) \bullet N(\text{Correct Decision = Reject, Predicted Decision = Accept}) \quad (19)$$

$$\text{False Rejection} = 100 / N(\text{Correct Decision = Accept}) \bullet N(\text{Correct Decision = Accept, Predicted Decision = Reject}) \quad (20)$$

Consider the scenario, when cGSA^{up} is used to determine which factor's uncertainty should be reduced. Posterior mean that one obtains in this case would be positively skewed, as shown in Figure 3. For a positively skewed distribution the median lies below the mean, as indicated in the same figure. Hence, the mean-based decision criterion shown in (15) would lead to a higher percentage of rejection instances. This is confirmed in the numerical results shown in Table 2. On the other hand, if cGSA^{down} is used instead, the resulting posterior mean would be negatively skewed as shown in Figure 4. Hence, it leads to a higher percentage of acceptance decisions, which is confirmed in the numerical results in Table 2.

In addition, the results presented in Table 2 indicate that the reduction in uncertainty in the factor identified by cGSA^{up} not only leads to a higher percentage of rejection decisions but also a higher rate of false rejection decisions. Again, the opposite is true if cGSA^{up} is used instead.

4. TEST CASE STUDIES

To further demonstrate the utility of cGSA, we consider a generic model *y* composed of two skewed model inputs (x_1 and x_2) and two symmetric distributed model inputs (h_1 and h_2) with parameters (y_0, *a*, *b*, *c*, *d*, *e*, *f*). The values and prior distributions of the various underlying parameters ($T_1, T_2, \theta_1, \theta_2$) of x_1, x_2, h_1 and h_2 are given by (29) – (33). The frequency distributions of the two model input factors x_1 and x_2 are shown in Figure 5 and those for model inputs h_1 and h_2 are shown in Figure 6. For reducing uncertainty of parameters θ_1 and θ_2, experiments are conducted to collect measurements for g_1 and g_2 given by (26) and (27) respectively. These measurements are corrupted by Gaussian noises of (34) and (35) with their variances given by (48).

$$y = y_0 + a.x_1 + b.x_2 + c.x_1.x_2 + d.h_1 + e.h_2 + f.h_1.h_2 \quad (21)$$

$$x_1 = \left(z_1^T \theta_1 - T_1\right)^2 \quad (22)$$

$$x_2 = \left(z_2^T \theta_2 - T_2\right)^2 \quad (23)$$

$$h_1 = z_1^T \theta_1 \quad (24)$$

$$h_2 = z_2^T \theta_2 \quad (25)$$

$$g_1 = z_1^T \theta_1 + \varepsilon_1 \quad (26)$$

$$g_2 = z_2^T \theta_2 + \varepsilon_2 \quad (27)$$

$$z_1 = z_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \quad (28)$$

$$\theta_1 \sim N\left(\mu_{\theta_1}, \Sigma_{\theta_1}\right) \quad (29)$$

$$\theta_2 \sim N\left(\mu_{\theta_2}, \Sigma_{\theta_2}\right) \quad (30)$$

$$\mu_{\theta_1} = \mu_{\theta_2} = \begin{bmatrix} -5 & 10 & 5 \end{bmatrix}^T \quad (31)$$

$$\Sigma_{\theta_1} = \Sigma_{\theta_2} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \quad (32)$$

$$T_1 = T_2 = 5 \quad (33)$$

$$\varepsilon_1 \sim N\left(0, \sigma_1^2\right) \quad (34)$$

$$\varepsilon_2 \sim N\left(0, \sigma_2^2\right) \quad (35)$$

$$\sigma_1^2 = \sigma_2^2 = 0.05 \quad (36)$$

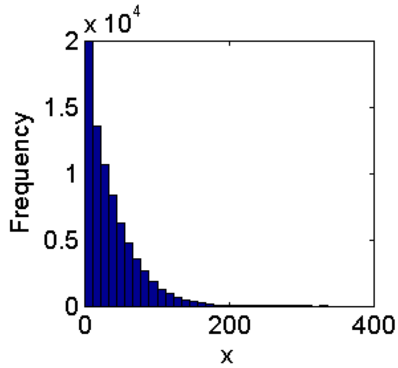

Figure 5: Frequency distribution of input uncertainties (for both x_1 and x_2)

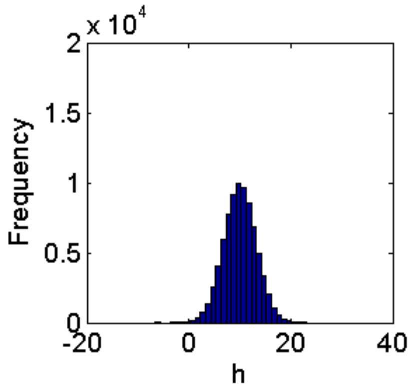

Figure 6: Frequency distribution of input uncertainties (for both h_1 and h_2)

To evaluate the performance of the cGSA, both cGSI^{up} (9) and cGSI^{down} (10) are calculated, along with it the rates of false acceptance (19) and false rejection (20).

Procedure to calculate the probabilities are explained as follows:

Consider the scenario when cGSA (either cGSA^{up} or cGSA^{down}) identifies x_1 as the key uncertainty. This means more information should be obtained to reduce the uncertainty in x_1. In order to acquire more information, one conducts an experiment, with z_1 and z_2 obtained through D-optimal design (denoted by $\eta_1^{\exp}$) on the model g_1 (see (26)), and after obtaining the experimental measurement ($g_{1,\exp}^i$) of g_1, the posterior distribution, $\theta_{1,posterior}^i \sim N\left(\mu_{\theta_{1,posterior}^i}, \Sigma_{\theta_{1,posterior}^i}\right)$, of θ_1 is obtained.

In order to systematically compare the cGSA^{up} and cGSA^{down}, N (= 8000) realizations of θ_1, θ_2, ε_1 and ε_2 are generated and stored in θ_1^i, θ_2^i, ε_1^i and ε_2^i (where i = 1, 2.., N). To generate the experimental realization $g_{1,\exp}^i$, a sample θ_1^i from the prior distribution of θ_1 and a sample noise, ε_1^i from the noise distribution ε_1 are used. The posterior distribution for θ_1 can be calculated using (37) and (38).

$$\mu_{\theta_1^{posterior}} = \left(\left(\eta_1^{\exp}\right)^T \left(\eta_1^{\exp}\right) + \left(\Sigma_{\theta_1} / \sigma_1^2\right)^{-1}\right) \bullet \left(\left(\eta_1^{\exp}\right)^T \left(g_1^{\exp}\right) + \left(\Sigma_{\theta_1} / \sigma_1^2\right)^{-1} \mu_{\theta_1}\right) \quad (37)$$

$$\Sigma_{\theta_1^{posterior}} = \sigma_1^2 \left(\left(\eta_1^{\exp}\right)^T \left(\eta_1^{\exp}\right) + \left(\Sigma_{\theta_1} / \sigma_1^2\right)^{-1}\right) \quad (38)$$

Now given this posterior distribution ($\theta_{1,posterior}$) of θ_1, the posterior distribution of x_1 is easily obtained using (22). Next using the posterior distribution of x_1 and prior distribution of x_2 the updated distribution of y is obtained. The required properties of the posterior distribution are calculated and the decision is predicted based on the decision criterion of (15).

In the above, one sample of experimental realization $g_{1,\exp}^i$ is considered and it leads to one updated distribution of y and then one decision (either selection or rejection) is made; this constitutes of one scenario. Similarly N (N = 8000) scenarios are generated and using the definition of false acceptance (19) and false rejection (20) the values are reported.

Consider the model structure in (39). Given the input uncertainties the distribution of the model output y is symmetric. The GSA results show that both the uncertainties are equally important. Further cGSA^{up} suggests that x_1 is the main uncertainty while cGSA^{down} suggests that x_2 is the key uncertainty. Results for two different *cutoff* values of 12 and 88 are shown in Tables 4 and 5 respectively. Results support the argument that the uncertainty reduction in the key uncertainty indicated by cGSA^{up} leads to a higher rate of false rejection and a lower rate of false acceptance, and vice versa.

$$y = 50 + x_1 - x_2 \quad (51)$$

Table 4: Comparison of cGSA^{up} and cGSA^{down} for the *cutoff* value of 12

	x_1	x_2
GSA	0.5	0.5
cGSA^{up}	0.88	0.00
cGSA^{down}	0.01	0.76
False Rejection (%)	0	0
False Acceptance (%)	7	41

Table 5: Comparison of cGSA^{up} and cGSA^{down} for the *cutoff* value of 88

	x_1	x_2
GSA	0.5	0.5
cGSA^{up}	0.88	0.00
cGSA^{down}	0.01	0.76
False Rejection (%)	41	7
False Acceptance (%)	0	0

We have conducted many more numerical studies with different parameter values and decision criterion but these are not presented here due to the space limitation.

REFERENCES

Iman, R. L. and W. J. Conover (1982). "A distribution-free approach to inducing rank correlation among input variables." *Communications in Statistics - Simulation and Computation* 11(3), 311 - 334.

Ishigami, T. and T. Homma (1990). An importance quantification technique in uncertainty analysis for computer models *Proceedings of the ISUMA'90, First International Symposium on Uncertianty Modelling and Analysis* University of Maryland

Jacques, J., C. Lavergne, et al. (2006). "Sensitivity analysis in presence of model uncertainty and correlated inputs." *Reliability Engineering & System Safety* 91(10-11), 1126-1134.

McKay, M. D. (1995). Evaluating Prediction Uncertainty, U.S. Nuclear Regulatory Commission and Los Alamos National Laboratory. Report NUREG/CR-6311.

Saltelli, A. (2002). "Making best use of model evaluations to compute sensitvity indices " *Computer Physics Communications* 145, 280-297.

Saltelli, A., T. H. Andres, et al. (1993). "Some new techniques in sensitivity analysis of model output " *Computational Statistics & Data Analysis* 15, 211-238.

Xu, C. and G. Z. Gertner (2008). "Uncertainty and sensitivity analysis for models with correlated parameters." *Reliability Engineering and Systems Safety* 93, 1563-1573.