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Fault detection using CUSUM based techniques with application to the Tennessee Eastman Process

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Abstract: In this paper, a cumulative sum based statistical method is used to detect faults in the Tennessee Eastman Process (TEP). The methodology is focused on three particular faults that could not be observed with other fault detection methodologies previously reported. Hotelling's-T^2 charting based on the cumulative sums of the faults' relevant variables was successful in detecting these faults, however, with significant delay. The speed of detection is further enhanced by retuning the fault's relevant controller at the expense of closed loop performance.

Keywords: Fault detection, Cumulative sum, Multivariate statistics, Control, Average run length

1. INTRODUCTION

An important aspect for the safe operation of chemical process is the rapid detection and removal of faults. Different methods have been proposed in the literature for fault detection and fault diagnosis (Venkatasubramanian *et al.*, 2003). These methods can be broadly categorized into three main classes: (1) Data driven methods; (2) Analytical methods, and (3) Knowledge based methods (Chiang *et al.*, 2001). Each of these methods has its own advantages and disadvantages depending on the problem. A number of researchers suggest combining these methods to improve detection. For examples, (Chiang and Braatz, 2001; Lee *et al.*, 2003) have observed that data driven analysis is enhanced if knowledge of the process is used to describe fundamental causal relationships among variables. Analytical methods require the use of first-principle models, thus making them less attractive for large scale systems. Therefore, they are not considered in the current work. In the chemical industry, large amount of data are measured by a variety of sensors and subsequently stored. These data generally exhibit high correlation in time and cross-correlation among variables. On the other hand, most of the data driven monitoring techniques assumed that data are uncorrelated and normally distributed. Different approaches have been proposed to mitigate the violation of these assumptions such as time series analysis and projection to latent variables methods (e.g. Principal Components Analysis) (MacGregor and Kourti, 1995).

Most monitoring data driven techniques are based on the statistical hypothesis-testing principle. Two types of errors occur when performing hypothesis testing referred to as type I and type II errors. A type I error occurs when a control chart indicates a fault in the absence of it, whereas a type II error occurs when a control chart fails to declare the existence of a fault, although it has occurred (Montgomery, 1997).

This paper proposes the application of Cumulative-Sum (CUSUM) based models for the detection of faults in the Tennessee Eastman problem (TEP) (Downs and Vogel, 1993). More specifically, the paper will investigate the application of Location CUSUM (LCS) and Scale CUSUM (SCS) based models to detect three particular faults that have been found unobservable by other algorithms previously applied to the TEP (Ding *et al*, 2009; Zhang, 2009; Chiang *et al*, 2001; Chiang and Braatz, 2001; Ku *et al*, 1995). After demonstrating the detection capability of the CUSUM based methods for each one of the three faults, a Hotelling's T^2 chart based on a cumulative sum of the observations is proposed for the individual or simultaneous detection of these three faults. Then, to quantify the fault observability, a statistical measure that is related to the speed of detection is defined. Finally, since the faults are observed from variables that are embedded within control loops, the effect of controllers' tuning parameters on the trade-offs between speed of fault detection versus process variability will be assessed. The paper is organised as follows: A description of the implemented CUSUM and Hotelling's T^2 statistics' and the metric used to gauge fault observability are given in section 2. Section 3 presents an overview of the faults considered in the Tennessee Eastman Process (TEP) and illustrates the use of the CUSUM based methods for the detection of the three abovementioned faults Then, using the statistical measure of observability presented in section 2, the tradeoffs between fault observability to process variability are investigated.

2. CUSUM, HOTELLING'S T^2 and AVERAGE RUN LENGTH (ARL)

2.1 The Cumulative sum (CUSUM) based control charts

A key disadvantage of Shewhart like control charts often used for detection is that they only use current time-interval information while not accounting for the entire time history. Hence, those charts are relatively insensitive to small shifts in the process variables especially for small signal to noise ratio. These shortcomings motivate the use of other alternatives such as the univariate or the multivariate version of the CUSUM based charts (MacGregor and Kourti, 1995). Three types of statistical charts are used in this paper. Specifically, location cumulative sum (LCS), scale cumulative sum (SCS) and the Hotelling's T^2. The current study proposes the use of a combined version of the three algorithms as described in the following section. The LCS and SCS algorithms are examples of univariate statistics while the Hotelling's T^2 is a multivariate statistics. Both the LCS and SCS are performed using the following two statistics, corresponding to a two sided hypothesis test (Hawkins and Olwell, 1998):

$$C_i^+ = \max[0, C_{i-1}^+ + x_i - (\mu_{i.c} + k)] \quad (1)$$

$$C_i^- = \max[0, C_{i-1}^- + (\mu_{i.c} - k) - x_i] \quad (2)$$

$$C_0^+ = C_0^- = 0$$

; where k , $\mu_{i.c}$, C_i^+ and C_i^- are the slack variable, the *in control* mean , the upper and the lower CUSUM statistics, respectively. The role of the slack variables is to introduce robustness to the calculated statistics. At every new sample, the statistics' in equations (1) and (2) result in the accumulation of small deviations in the mean (LCS) or small changes in the variability (SCS). These accumulations are corrected using the slack variable and compared to zero using the (max) operation. When either one of the two statistics in equations (1) and (2) exceed a threshold H, the process is considered to be out of control. Following their respective definitions, the LCS is especially effective for detecting changes in the average whereas the SCS is suitable for detecting changes in variability. Guidelines for the selection of k and H have been reported (Hawkins and Olwell, 1998; Montgomery, 1997). Typically k is selected to be half of the expected shift in either μ or σ. H is determined so that a prespecified $\mathrm{ARL}_{o.c}$, to be defined in the following section, is achieved. It should be noticed that when using equations (1) and (2), the LCS uses the original raw data x_i, whereas the SCS uses the following standardized quantity:

$$x_i = \frac{\sqrt{|y_i|} - 0.822}{0.349} \quad (3)$$

; where y_i denotes the original raw data. A derivation of the quantities in equation (3) is given in Appendix A. Although LCS and SCS can be applied to individual measurements, there are many situations in which a pooled representative statistic for more than one variable is necessary. This is especially important when it is desired to present the operators with compact information to simplify the monitoring activities for the process. For that purpose, when the monitored variables are normally and statistically independent, the Hotelling's T^2 can be used. The Hotelling's T^2 statistics and the upper and lower control limits are given by:

$$T^2 = (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{S}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) \quad (4)$$

$$UCL = \frac{p(m+1)(m-1)}{(m^2 - mp)} . F_{\alpha,p,m-p} \quad (5)$$
$$LCL = 0$$

; where p is the number of monitored variables and m is the total number of samples. In the current work the cumulative sum statistics' are combined together into one statistic, namely, the Hotelling's T^2 as described in a later section.

2.2. The $ARL_{o.c}$ as an observability measure

Observability of a fault is referred to the ability to detect the fault from the chosen set of measurements. In the current work, the statistical measure used to gauge observability is the out-of-control Average Run Length ($\mathrm{ARL}_{o.c}$). The subscript ($o.c$) stands for *out of control*. The $\mathrm{ARL}_{o.c}$ is defined as the average number of points that must be sampled or plotted before the chart signals and it is a function of the probability of type II error (β), that is

$$ARL_{o.c} = f(\beta) \quad (6)$$

Due to their integrating nature, cumulative sum based techniques require some time before a fault can be detected, especially if the changes are very small. Accordingly, the $\mathrm{ARL}_{o.c}$ is a suitable metric to quantify this expected delay in detection.

For example, if in response to a certain fault, the calculated $\mathrm{ARL}_{o.c}$= 1, the fault would be detected, on the average, after the first sample following the onset of the fault. On the other hand, an $\mathrm{ARL}_{o.c}$= *infinity* or a very large number implies that the fault is unobservable or it takes a long time to observe it. The value of the $\mathrm{ARL}_{o.c}$ depends on the type of chart that is used for monitoring. Several analytical expressions are available for specific statistical charts (Montgomery, 1997). The above discussion showed the feasibility of using the $\mathrm{ARL}_{o.c}$ as a fault observability index. Different approaches to estimate the $\mathrm{ARL}_{o.c}$ based on the Markov chain approach appeared in the literature, (e.g. Brook and Evans, 1972) but in practice, the $\mathrm{ARL}_{o.c}$ is usually estimated from simulations conducted with random realizations of the disturbances (Woodall and Ncube, 1985). The latter approach is adopted in the current study.

3. TENNESSEE EASTMAN PROCESS (TEP) AND THE "UNOBSERVABLE" FAULTS

The Tennessee Eastman process has been widely used as a benchmark problem to compare various monitoring solutions (Chiang *et al*., 2001; Ku *et al*., 1995; Lee *et al*., 2004; Ding *et al*., 2009). The process is open loop unstable and consists of five major unit operations, as shown in (Fig. 1): reactor, condenser, compressor, separator and stripper. The process produces two liquid products (G and H) and one by-product (F) from four gaseous reactants (A, C, D, E) and an inert (B). Based on the required product mix and production rate, the plant can be operated according to six different modes of operation. The original open loop FORTRAN code was provided by Downs and Vogel, 1993. The simulations of the plant were done with the second decentralized control structure proposed in (Lyman and Georgakis, 1995). Different monitoring techniques have been tested and reported for the TEP. These techniques have shown different capabilities in detecting the majority of the 20 faults generally assumed for the process.

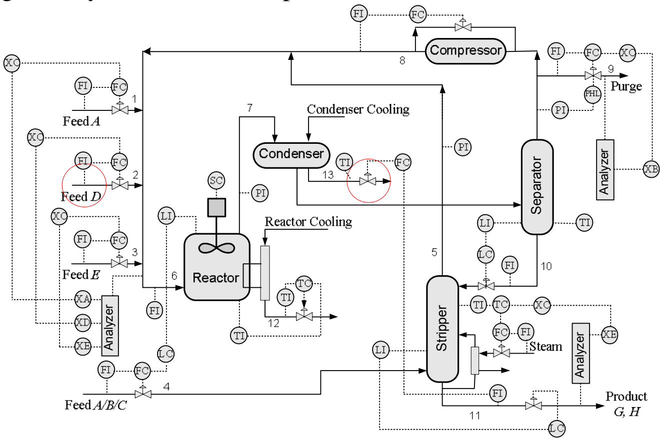

Fig. 1. Tennessee Eastman process with the second control scheme described in (Lyman and Georgakis, 1995); the circles indicate the location of the faults described in Table 1

However, all of these previously reported techniques have consistently failed in detecting the three particular faults described in Table 1.

Table.1 The "unobservable" faults of the TEP

Faults	Description	Nature
IDV(3)	D feed temp.	Step change
IDV(9)	D feed temp.	Random variation
IDV(15)	Condenser cooling water valve	Valve stiction

The resulted lack of observability when using specific techniques is attributed to the statistically insignificant changes, (i.e. changes in the process mean and/or variance), exhibited by the system when these faults occur. For fairness, it should be stressed that in most of the reported work, the detection was based on current time measurements thus the entire time histories of the measurements were not considered for detection as done in the CUSUM calculations proposed in the current study. However, the fact remains that these faults have not been detected in previous studies while they may have an important economic or operational impact. Thus it is still very relevant to attempt to detect them. Later in the paper it will be shown that CUSUM based statistics are successful in observing these three faults after a certain period of time following the occurrence of the fault. .

3.1 Previous attempts to tackle the TEP faults

Almost, all of the methods previously applied to the TEP were of multivariate nature. Among these techniques, for example, is the dynamic principal component analysis (DPCA) proposed by (Ku *el al*; 1995). Fig. 2 shows the results of the application of DPCA for the TEP using the T^2. The statistic T^2 in Fig .2 is based on the sum of squares of the scores resulting from DPCA model. The DPCA has the advantage of taking into account information along several time intervals in contrast to the conventional static PCA which is based solely on data collected at the current time. Accordingly, DPCA is more suitable for dynamical systems. The bounds of normal operation corresponding to a 95% and a 99% confidence levels i.e. no fault has occurred, are shown as dotted lines in Fig.2 and are calculated by equation (5). The meaning of these bounds is that if the T^2 is above these bounds after the occurrence of the fault, then the fault is signalled. For the plots in Fig. 2 the corresponding faults were introduced at time=160 samples. However, as shown in Fig. 2, the T^2 statistics fails to surpass the thresholds after the onset of the 3 faults, i.e. IDV(3), IDV(9) and IDV(15). Hence, these faults cannot be detected by DPCA. It should be noticed that when a PCA/DPCA are used, p is replaced with a in equation (5), where a is the number of principal components retained in the PCA/DPCA model.

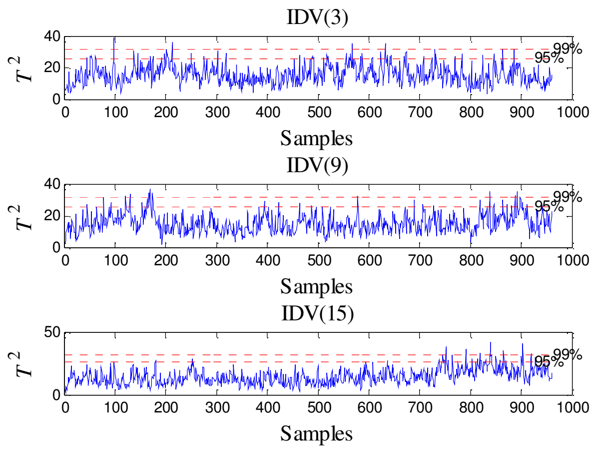

Fig. 2. T^2 based on DPCA for the three unobservable faults of TEP, i.e. IDV (3), IDV (9) and IDV (15). The horizontal dotted lines represent the 95% and 99% confidence limits.

Chang *et al*, 2001 have performed a comparative study of multivariate techniques for detection of the faults in the TEP. They compared the PCA, DPCA and the subspace Canonical Variate Analysis (CVA) algorithms. Their performance index was the misclassification rate, i.e. the number of times the fault is not detected although the fault has occurred. Their conclusion was that the CVA results in the lowest misclassification rate in particular when monitoring the residual space with respect to an identified state space model. However, the three faults in Table 1 were excluded from the overall comparison, simply, because they can not be observed and high misclassification rates were associated with them. Other methods that were applied to the TEP problem are the Dynamic Independent Component Analysis (DICA) (Lee *et al*, 2003) and recently, a new subspace identification based method proposed by (Ding *et al.*, 2009). All these methods excluded from their overall analysis the three faults given in Table 1.

3.2 The CUSUM charting approach for the TEP unobservable faults

The inability of previous techniques to detect the 3 faults given in Table 1 motivates the use of cumulative sum measures. Although Multivariate Cumulative Sum (MCUSUM) of all of the available TEP measurements could be used to detect the TEP faults (MacGregor and Kourti, 1995; Woodall and Ncube, 1985), this technique was still unable to detect these three faults. The latter has been tested by the authors; however, the results are not shown for brevity. Accordingly, it was decided to use a univariate Cusum on relevant variables as follows. Since the Cusum based statistics' are especially suitable to detect small changes in the process mean or small changes in process variability, it is important to identify the specific variables that exhibit these types of changes and apply the cusum operation on these variables. To find the variables for which the CUSUM operation should be applied, knowledge about the process was used. For example, it was observed that IDV (3) (Table 1; small constant change in feed concentration) affects the steady state in the reactor. Since the reaction is highly exothermic and to keep the level of conversion at a desired level, manipulated variable XMV [10] must change to eliminate any changes in the mean of the steady-state reactor temperature. Then, the local cumulative sum of the manipulated variable XMV[10] is expected to provide detection of the corresponding fault after sufficient errors between the new steady state mean and the old steady state mean are integrated by the CUSUM operation. Based on similar arguments it is possible to find the individual relevant variables that are most sensitive to each fault and for which the Cusum operation should be applied to detect that particular fault. The faults and the corresponding variables used for detection are given in Table 2.

Table.2 The unobservable faults/process variables pairing

Faults	Measurement*	Description
IDV(3)	XMV[10]	Reactor cooling water flow
IDV(9)	XMEAS[21]	Reactor cooling outlet temp.
IDV(15)	XMV[11]	Condenser cooling water flow

*The variable measurements as appeared in (Down and Vogel, 1993)

Also, since IDV(9) is a random disturbance around a mean and since IDV(15) results in cycling of the condenser cooling water flow due to valve stiction, the overall effect of these two faults is to increase the variance in their relevant variables as shown in Table 2. Accordingly, the location CUSUM (LCS) is applied to monitor the effect of IDV(3) since it involves a shift in mean whereas the scale CUSUM (SCS) was used to monitor the effects of both, IDV (9) and IDV(15) , since they result in changes in variance. The sampling frequency for the CUSUM charts was (1/180) Hz (3 min. time intervals). In all the following simulations, the faults are introduced after 160 samples, that is, after 8 hours of a normal operation. Fig.3 shows the application of the LCS on XMV[10] when IDV (3) occurs. In this Figure the fault was introduced at time=8 hours and was removed at time=700 hours to show whether the CUSUM statistics is able to predict both the occurrence and removal of the fault. The figure shows that the average time required for detection ($ARL_{o.c}$) is approximately 127.05 hrs. This time is calculated from the onset of the fault until the breaching of the threshold. An accurate $ARL_{o.c}$ requires averaging over a large number of noise realizations (Woodall and Ncube, 1985).

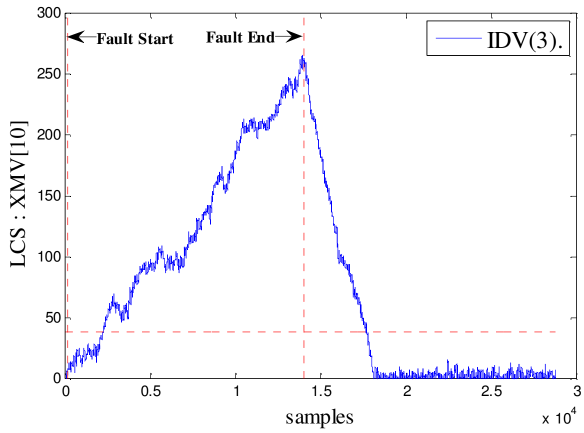

Fig. 3. The LCS for IDV (3); horizontal line represent the statistical limit. The vertical lines represent the onset (after 8 hours) and the end of the fault (after 700 hours).

It is also clear from Fig. 3 that a very long time is required for detection due to the extremely small signal to noise ratio. This explains the inability of other previously used techniques to detect this fault. Also, the algorithm is able to detect the removal of the fault after some time. It is also clear from Fig.3 that the rate of change of the LCS statistic is higher when the fault is removed than when the fault is introduced. This is expected because of the nature of the cusum algorithms given by equation (1) and (2) whereby when the rate of change of the (LSC) statistic is negative, the accumulator is reset to zero. This, in turn, accelerates the return to the statistical in control state. Different noise realizations were tested and used to calculate the average run lengths ($ARL_{o.c}$).

Fig.4 shows the detection of IDV(15) corresponding to valve stiction when using the SCS. The SCS for fault IDV(9) is not shown due to space limitation. In Fig. 4 the fault has not been removed. The figures show that the SCS and LSC were successful in observing these two faults with $ARL_{o.c}$ values given in Table 3. Thus, the CUSUM algorithms provide detection, but relatively long periods of time are required to detect the occurrence of the fault. The immediate implication is that only faults that are of longer durations than the corresponding $ARL_{o.c}$ values can be detected using the Cusum based statistics'. Although three separate control charts could be used to monitor the 3 faults (Montgomery, 1997), it is often convenient for practical purposes to monitor the process with fewer. In the current study, it is proposed to use the Hotelling-T^2 statistics to monitor the three faults with one single chart. For that purpose, the LCS algorithm is applied to XMV[10] whereas the SCS algorithm is applied , to XMEAS[21] and XMV[11], and then the corresponding cumulative sums are used to drive the Hotelling T^2 statistics defined in equation (4) ; where **x** is a vector sample composed of the 3 cumulative sums. To test for collinearity, the PCs of the covariance matrix were evaluated using a Scree plot (Chiang *et al.*, 2001). Three PCs were found for all case studies. Fig. 5, describes the Hotelling-T^2 results when only IDV(3) occurs while Fig. 6 depicts the detection of IDV(9). In addition, Fig. 7 illustrates the T^2 when both IDV(3) and IDV(15) occur simultaneously. In all cases T^2 based on the cusum statistics' were able to successfully detect the fault(s). Table.3 summarizes the relevant ARL$_{o.c}$ when the Hotelling's-T^2 charting based on the individual CUSUM statistics' was used.

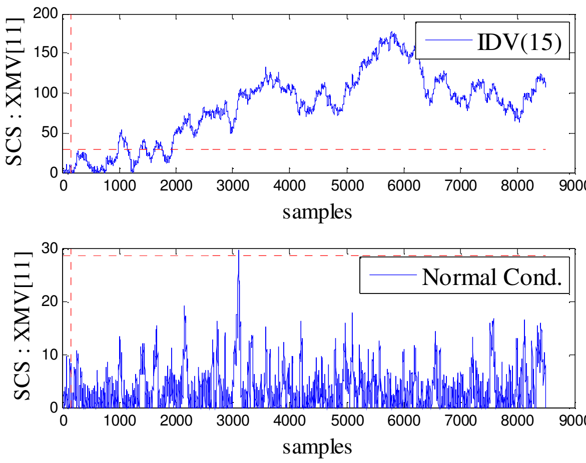

Fig. 4. The SCS for IDV (15); horizontal and vertical lines represent the statistical limit and the fault onset, respectively.

Table.3 The estimated ARL$_{o.c}$ for the LSC, SCS and T^2

Fault	Statistics	*ARL$_{o.c}$ (hr)
IDV(3)	LCS	127.05
IDV(9)	SCS	8.20
IDV(15)	SCS	41.00
IDV(3)	T^2	102.40
IDV(9)	T^2	276.05
IDV(15)	T^2	89.65
IDV(3) & IDV(15)	T^2	41.30

*All ARL$_{o.c}$ are calculated from after onset of the faults (i.e. after 8 hours)

4. CONTROL DETECTION INTERACTION

Since the CUSUM operations presented in the previous section are applied to variables that are used either for manipulation or as feedback within closed loop control schemes, there is a possibility to speed up the detection of the faults by re-tuning the fault's relevant controllers. The potential tradeoffs between control design and fault detection do not receive much attention except for a few attempts (e.g. Tyler and Morari, 1994). These tradeoffs generally arise from the fact that faster detection requires higher variability in the variables used for detection whereas higher variability generally translates into lower product uniformity or higher wear of actuators. As noted from Table 3, although the CUSUM based statistics' were successful in detecting the 3 faults under consideration, the resulting ARL$_{o.c}$ values were relative large. To shorten the time for detection given by the ARL$_{o.c}$ it is proposed to re-tune the controllers and to check the impact of this tuning operation on the ARL$_{o.c}$ and on the variability. By way of illustration, IDV(15), and its corresponding controller are considered, that is, the condenser cooling water valve. Since a cascade control scheme is implemented for this loop, the master PI controller was re-tuned.

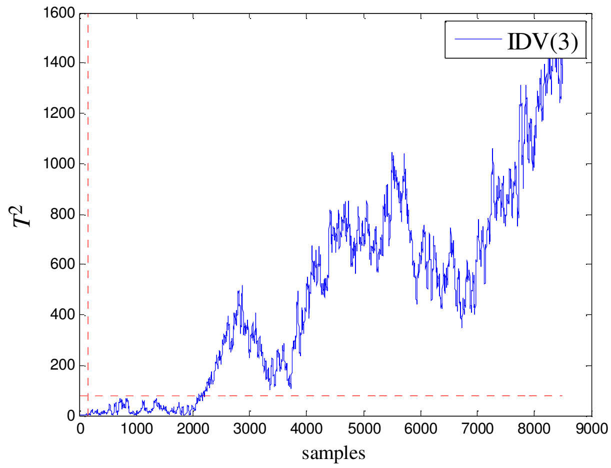

Fig. 5. The Hotelling'T^2 for IDV (3); horizontal and vertical lines represent the statistical limit and the fault onset, respectively.

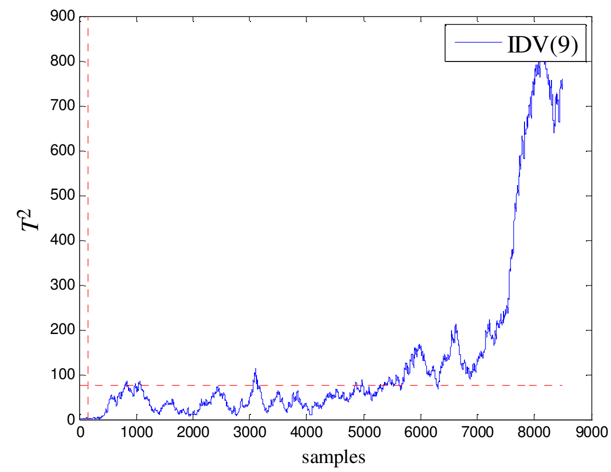

Fig. 6. The Hotelling'T^2 for IDV (9); horizontal and vertical lines represent the statistical limit and the fault onset, respectively.

The ARL$_{o.c}$ of IDV (15) and the variability results as a function of the controller proportional gain (K) are shown in Fig.8. As can be seen from Fig .8, there is a significant interaction between the control and the detection scheme. The re-tuning of the controller significantly reduces the ARL$_{o.c}$ that would be required to observe IDV (15), but at the expense of significant degradation in performance as shown by the increased variability in the manipulated variable value. This variability may translate into significant wear of the corresponding valve. Thus, there is a motivation to seek for a trade-off between detection speed and closed loop performance provided that the related costs are available. The formulation of such optimization problem using the CUSUM based detection techniques is currently under investigation.

CONCLUSION

A CUSUM based statistic combined with the Hotelling's T^2 charting is proposed. This method was successful in detecting three faults in the Tennessee Eastman problem that were impossible to observe with other previously applied methods. The 3 univariate CUSUMs were combined into one control chart by using Hotelling'T^2 statistics. Potential enhancements to the speed in detecting these faults, gauged by the $ARL_{o.c}$, can be achieved by formulating an optimization problem that explicitly considers the tradeoffs between detection and control performance.

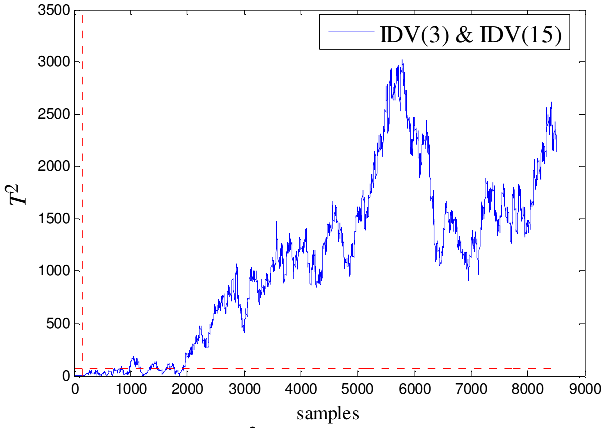

Fig. 7. The Hotelling'T^2 for the simultaneous occurrence of IDV(3) and IDV(15); horizontal and vertical lines represent the statistical limit and the fault onset, respectively.

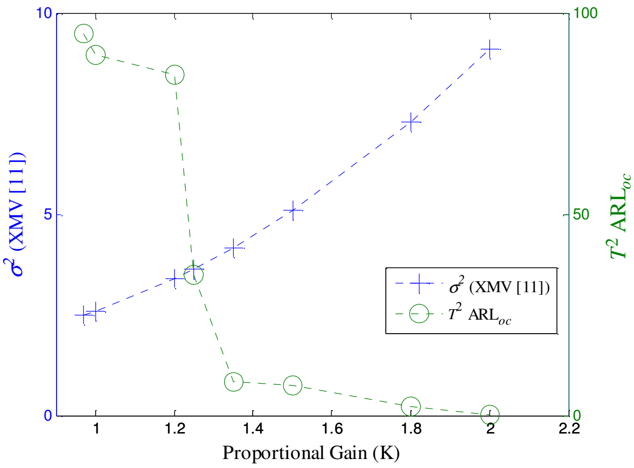

Fig. 8. The change in variability and the T^2-$ARL_{o.c}$ as a function of the master controller's gain (XMV[11] is considered in the case).

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Appendix A. SCALE CUSUM PARAMETERS

The parameters of the Scale CUSUM have been derived as follows. Let, $x_i \sim N(0,\sigma^2)$, $i=1,2,\ldots,n$ and $y_i=|x_i/\sigma|^\lambda$. The characteristic of the distribution of y_i are easily worked out from those of standard normal distribution. That is

$$\Pr[y_i < c] = 2\Phi(c^{1/\lambda}) - 1 \tag{A1}$$

where Φ () denote the standard normal function. Furthermore, the k^{th} moment of y_i is as follows:

$$E[y_i^k] = \mu = 2^{0.5.\lambda.k}.\Gamma[0.5(\lambda.k+1)]/\sqrt{\pi} \tag{A2}$$

; where Γ() is the gamma function. With λ=0.5, the transformed variate y_i has a distribution which is very close to normal. In particular, using (A2), the first and second moments are given as followings:

$$E[y_i^k] = E[y_i^1] = \mu = 2^{0.25}.\Gamma(3/4)/\sqrt{\pi} = 0.82218 \tag{A3}$$

$$V(y_i) = \sqrt{(2/\pi)}.\Gamma(1) - \mu^2 = (0.34915)^2 \tag{A4}$$