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Conditions for Identifiability Using Routine Operating Data for a First-Order ARX Process Regulated by a Lead-Lag Controller

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Abstract: The conditions for closed-loop identifiability using routine operating data are largely unknown. In this paper, the closed-loop identifiability conditions for a first-order autoregressive process with exogenous input (ARX) that is regulated using a 3-parameter lead-lag controller and that has no external excitation will be examined using an analytical approach. Despite the convoluted nature of the intermediate results, the final conditions for absolute identification of the stable region can be concisely stated. These results suggest that the class of internal model controllers (IMCs) can, despite their aggressive behaviour, successfully identify an ARX model without any external excitation. As well, Monte Carlo simulations performed using MATLAB confirmed the analytical results that were obtained. Future work in this area can focus on extending the results to other model structures, to other types of controllers, and to higher order processes.

Keywords: ***Closed-loop identification, identifiability, internal model control, autoregressive models***

1. INTRODUCTION

In control engineering, the identification of a model from closed-loop operation without any external perturbations is of prime importance for such control aspects as controller tuning, fault detection and isolation (FDI), and performance monitoring. Previous work has shown that, if the controller is of higher order than the process or with significant nonlinearities, then it is possible to identify the process successfully (Ljung, 1999). Furthermore, it has been shown that, if a reference signal is sufficiently persistently exciting to identify the open-loop process, then it is possible to identify the process from closed-loop data (Ljung, 1999). However, these statements are qualitative, in that they do not specify the exact (minimal or otherwise) requirements for closed-loop process identification.

Recently, Gevers, Bazanella, and Ljubiša (2008) determined the conditions for closed-loop identification based on the sensitivity functions. In their paper, general autoregressive processes with exogenous input (ARX) with a transfer function given as

$$y_t = \frac{\sum_{i=1}^{n_b} \beta_i z^{-i}}{1+\sum_{i=1}^{n_a} \alpha_i z^{-i}} u_t + \frac{1}{1+\sum_{i=1}^{n_a} \alpha_i z^{-i}} e_t \tag{1}$$

where n_a is the order of the A-polynomial and n_b is the order of the B-polynomial and are regulated by a controller whose transfer function is given as

$$u_t = -\frac{\sum_{i=1}^{n_x} \xi_i z^{-i}}{1+\sum_{i=1}^{n_y} \psi_i z^{-i}} y_t \tag{2}$$

where n_x is the order of the numerator of the controller and n_y is the order of the denominator of the controller, were investigated in the presence or absence of a reference signal. It was determined that, in the absence of a reference signal, (Gevers, Bazanella, & Ljubiša, 2008)

$$\max\left(n_x - n_a, n_y - n_b\right) \geq 0 \tag{3}$$

must be satisfied in order for the system to be identifiable. This result only gives restrictions on the orders of the system, which is a qualitative description of the closed-loop identifiability conditions.

This paper has four objectives. The first objective is to quantify the parametric region of identifiability for a first-order ARX process regulated by a lead-lag controller without any external excitation, that is, determine the region of identifiability as a function of the controller and process parameters. The second objective is to use the analytical results to determine the restrictions or limitations on identification of the process when the true process itself is unknown or uncertain. The third objective is to verify the analytical results using Monte Carlo simulations. A related issue is to explain, using the analytical results, the observed behaviour. The fourth and final objective of this paper is to verify closed-loop identifiability based on the conditions proposed by Gevers, Bazanella, and Ljubiša (2008).

2. THEORY

2.1 Closed-Loop System to be Analysed

Given a closed-loop process described similar to that described in Figure 1, where

$$
\begin{aligned}
r_t &= 0 \\
G_c &= \frac{\delta + \varepsilon z^{-1}}{1 + \eta z^{-1}} y_t \\
G_p &= \frac{\beta z^{-1}}{1 + \alpha z^{-1}} \\
G_l &= \frac{1}{1 + \alpha z^{-1}}
\end{aligned}
\tag{4}
$$

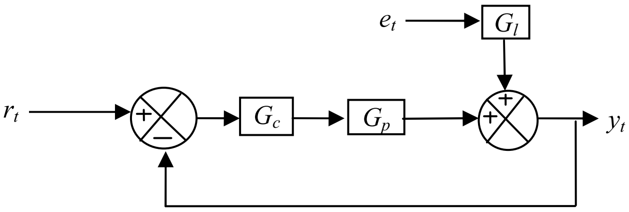

Figure 1: Generic Closed-loop Process

then the transfer function for the process can be written as (Ljung, 1999)

$$y_t = \frac{\beta z^{-1}}{1 + \alpha z^{-1}} u_t + \frac{1}{1 + \alpha z^{-1}} e_t \tag{5}$$

while the transfer function for the input can be written as

$$u_t = \frac{\delta + \varepsilon z^{-1}}{1 + \eta z^{-1}} y_t . \tag{6}$$

where y_t is the output from the process, e_t is white noise, and u_t is the input into the process. The reference signal, r_t, is assumed to be equal to zero. The parameters α, β, δ, ε, and η are free parameters whose values can range over the real number line $(\mathbb{R})$. For simulating an autoregressive model with exogenous input (ARX), α is constrained to be in the interval]-1, 1[, *i.e.* $-1 < \alpha < 1$. Equations (5) and (6) can be rewritten as a closed-loop system in terms of the noise term as

$$y_t = \frac{\left(1 + \eta z^{-1}\right)}{1 + \left(\alpha + \eta - \beta\delta\right) z^{-1} + \left(\alpha\eta - \beta\varepsilon\right) z^{-2}} e_t \tag{7}$$

$$u_t = \frac{\delta + \varepsilon z^{-1}}{1 + \left(\alpha + \eta - \beta\delta\right) z^{-1} + \left(\alpha\eta - \beta\varepsilon\right) z^{-2}} e_t . \tag{8}$$

Based on Söderström and Stoica (1989), the consistency of the parameter estimates of an ARX model can be determined by solving the following system of equations:

$$\begin{bmatrix} E\left(y_t y_t\right) & -E\left(y_t u_t\right) \\ -E\left(y_t u_t\right) & E\left(u_t u_t\right) \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} -E\left(y_t y_{t-1}\right) \\ E\left(y_t u_{t-1}\right) \end{bmatrix} \tag{9}$$

where E is the expectation operator, $\hat{\alpha}$ is the estimated value of the parameter α, and $\hat{\beta}$ is the estimated value of the parameter β.

It should be noted that, according to the inequality in (3), the first order ARX system should be identifiable for any combination of controller and process parameters.

2.2 Symbolic Expectation Operator

In order to solve (9), symbolic calculation of the expectoration operator is required. Since there does not exist an appropriate method for symbolically determining the expectation operator, a new method was derived that allows easy computation of a symbolic expectation.

A general ARMA process can be written as

$$y_t = \sum_{j=0}^{n_y} \frac{\phi_j z^{-d}}{1 + \theta_j z^{-1}} e_t + \sum_{i=0}^{n_{MA}} \phi_{MA_i} e_{t-i} \tag{10}$$

where n_y represents the number of autoregressive components in the process and n_{MA} represents the number of moving average components. The parameter ϕ is a real number, while θ can be imaginary. If θ is imaginary, then there will be a corresponding imaginary term that is the complex conjugate of θ. Thus, it can be assumed that the imaginary poles come in pairs. Equation (10) can be obtained by performing partial fractioning of the general ARMA transfer function. Rewriting (10) as an infinite series in terms of the error gives

$$y_t = \sum_{i=0}^{\infty} \sum_{j=1}^{n_y} \phi_j \left(-\theta_j\right)^i e_{t-i-d} + \sum_{i=0}^{n_{MA}} \phi_{MA_i} e_{t-i} \tag{11}$$

Thus, the k-lag cross-correlation between two signals, u_t and y_t, that are expressed in a form similar to (10) can be written as

$$
\begin{aligned}
E\left(y_t u_{t-k}\right) = &\left(\sum_{i=0}^{\infty} \sum_{j=1}^{n_y} \phi_j \left(-\theta_j\right)^i e_{t-i-d_y} + \sum_{i=0}^{n_{MA_y}} \phi_{MA_i} e_{t-i} \right) * \\
&\left(\sum_{n=0}^{\infty} \sum_{l=1}^{n_u} \phi_l \left(-\theta_l\right)^n e_{t-n-d_u+k} + \sum_{m=0}^{n_{MA_u}} \phi_{MA_m} e_{t-m+k} \right)
\end{aligned}
\tag{12}
$$

Since, it can be assumed that the error terms are white noise, (12) can be simplified using the following property of white noise (Ljung, 1999)

$$E\left(e_t e_{t-k}\right) = 0 \quad \forall k \neq 0 \tag{13}$$

to give

$$
\begin{aligned}
E\left(y_t u_{t-k}\right) &= \sum_{j=1}^{n_y}\sum_{l=1}^{n_u} \frac{\theta_j \theta_l \left(-\phi_l\right)^{d_y - d_u + k}}{1-\left(\phi_j \phi_l\right)} \sigma_w^2 \\
&+ \sum_{i=0}^{n_{MA_y}}\sum_{j=1}^{n_u} \theta_{MA_{y_i}} \theta_j \left(-\phi_j\right)^{2i-d_u-k} \sigma_w^2 \\
&+ \sum_{m=0}^{n_{MA_u}}\sum_{j=1}^{n_y} \theta_{MA_{u_m}} \theta_j \left(-\phi_j\right)^{2m-d_y+k} \sigma_w^2 \\
&+ \sum_{i=0}^{n_{MA_y}} \theta_{MA_{y_m}} \theta_{MA_{u_{2i+k+1}}} \sigma_w^2
\end{aligned} \tag{14}
$$

It should be noted that (14) only applies if the following condition is satisfied

$$\left|\theta_j \theta_i\right| < 1 \tag{15}$$

Furthermore, the two middle sums in (14) are only valid for terms such that

$$
\begin{aligned}
(i - d_u - k) &\geq 0 \\
(m - d_y + k) &\geq 0
\end{aligned} \tag{16}
$$

Using MATLAB's symbolic mathematics toolbox, a symbolic expectation operator programme was written that will return the cross-covariances based on solving (14).

3. ANALYTICAL RESULTS

3.1 Analytical Conditions for Identifiability

Using (14), all of the covariances in (9) were calculated. Next, the matrix was inverted to obtain the parameter estimates, $\hat{\alpha}$ and $\hat{\beta}$. The equations for the parameter estimates were extremely long. Finally, the parameter estimates were equated to the true parameter values to obtain a system of equations with two equations. The solution to this system of equations gave the conditions for identifiability.

Two primary conditions for identifiability were obtained. The first condition for identifiability is:

$$\boxed{\frac{\alpha}{\beta} = \frac{\varepsilon}{\eta}} \tag{17}$$

The second condition can be derived by noting that, in order to obtain identification over the stable region, (7) must be stable. This essentially implies that the roots of the denominator must lie inside the unit circle (in terms of z). Combining (17) with this stability requirement for (7) gives the second condition for identifiability:

$$\boxed{\left|\alpha + \eta - \beta\delta\right| < 1} \tag{18}$$

Additional conditions for identifiability can be classified as trivial, since these deal with the case when (14) fails to be summable, or either of the parameters to be identified is zero. The results are summarised in Table 1. It should be noted that the same corresponding signs are taken based on the location, while if the parameter is specified as the variable itself, then any value of that parameter applies. If $\beta = 0$, then a pure AR model is being fitted. In this case, the system is only stable for $|\alpha| < 1$, which is confirmed by Table 1. Furthermore, it can be seen that α and η have conditions that are often complementary. The conditions stated in terms of ε can be rewritten in terms of the system parameters α and β. Finally, it can be noted that some of the conditions do not have any real significance. For example, if any of the parameter values are imaginary numbers, then the system is undefined, unless there is a corresponding complex conjugate number that would combine to give 2 real numbers.

Table 1: Summary of the Trivial Conditions. It should be noted that, if there is a choice of signs, then the same sign location should be selected for each of the choices.

Case	α	β	δ	ε	η
1	α	0	δ	ε	±1
2	α	β	δ	$\frac{\eta\alpha \pm \alpha + 1 \mp \beta\delta \pm \eta}{\beta}$	η
3	±1	0	δ	ε	η
4	α	β	δ	$\frac{-1+\eta\alpha}{\beta}$	η
5	η^{-1}	0	δ	ε	η
6	α	0	δ	ε	$\pm i$
7	$\pm i$	0	δ	ε	η
8	α	β	δ	$-0.5\frac{\left(\begin{array}{l}\eta^2 - 2\eta\beta\delta + \alpha^2 \\ -2\alpha\beta\delta + \beta^2\delta^2 + 2 \\ -\left(\eta + \alpha - \beta\delta \pm 2i\right)\eta \\ -\left(\eta + \alpha - \beta\delta \pm 2i\right)\alpha \\ +\left(\eta + \alpha - \beta\delta \pm 2i\right)\beta\delta\end{array}\right)}{\beta}$	η

3.2 Discussion of the Analytical Conditions for Identifiability

The above conditions for identifiability raise some interesting issues. Firstly, the condition given by (17) suggests that the coefficient of the second order term in the denominator of the closed-loop transfer function (see (7) and (8)) must equal zero, that is,

$$\alpha\eta - \beta\varepsilon = 0 \tag{19}$$

This would suggest that, if the order (in terms of z^{-1}) of the denominator of the transfer function for the closed loop system is greater than the corresponding order of the numerator, then there are issues with identifiability. Thus, it would seem that the orders of the numerator and denominator must be at least equal. It is possible, but unproven, whether the numerator's order (in terms of z^{-1}) can be greater than that of the denominator. In other words, it is not known whether the number of parameters in the numerator can be greater than the number in the denominator.

Secondly, an internal model controller where the controller parameters are defined as

$$\begin{aligned}\delta &= -\beta^{-1} \\ \varepsilon &= -\alpha\beta^{-1} \\ \eta &= -1\end{aligned} \tag{20}$$

should be able to identify the true process parameters for all parameter values, since it can be easily verified that the conditions given by (17) and (18) are satisfied for all possible values of α and β. This suggests that, despite the fact that the internal model controller is an aggressive controller, it can always identify the correct model using routine operating data. However, there is an interesting problem with using an internal model controller and routine operating data. In order to tune the internal model controller, the true model parameters are required. Thus, in practice, there could potentially exist an iterative procedure that alternates between model identification and controller tuning.

Thirdly, the requirement of knowing the true parameter values and the consequent inability to identify the model raise the issue of whether, if starting from arbitrarily close enough initial estimates of the parameters, an iterative approach will identify the true parameter values. This iterative approach would involve initially guessing some parameter values and designing a controller, which would then be improved upon using the newly identified parameter estimates. This process would be repeated until the parameter estimates stopped changing. The question is then whether or not the parameters obtained converge to the true parameter values. This observation could lead to interesting research in iterative, closed-loop identification.

Fourthly, the inequality presented in (3) (Gevers, Bazanella, & Ljubiša, 2008), is always satisfied in this example. However, the analytical results suggest that this condition is insufficient to describe the identifiability of the process. It would seem that when the inequality reaches its lower bound, zero, the true parameter values need to be known (in order to avoid the issues mentioned in point three above), and the controller must satisfy some rather restrictive conditions in order to guarantee identifiability. Thus, solely considering the orders of the polynomials in the numerator and denominator is insufficient to guarantee identifiability.

4. SIMULATION RESULTS

4.1 Simulation Set-Up

In order to verify the analytical results that were obtained above, a Monte Carlo simulation was performed. The following parameters were used:

- 10,000 data points were simulated for each process.
- 1,000 simulations of the process were run. Changing the number of simulations run did not change the observed results.
- Parameter estimates were averaged to obtain the mean values for the parameters. The 95% confidence intervals for the parameters were calculated using the following formula

$$t_{\frac{0.05}{2},1000}\sqrt{\frac{\operatorname{Var}\left(\bar{\hat{\alpha}}\right)}{N}} \tag{21}$$

where N is the number of simulations performed, $\bar{\hat{\alpha}}$ is the mean value of the parameter estimate, t is the value of Student's t-test, and $\operatorname{Var}\left(\bar{\hat{\alpha}}\right)$ is the variance of the parameter estimates. Since the sample size selected for the Monte Carlo simulation is sufficiently large, the estimate's bias will be used to verify consistency. This can be verified by computing the 95% confidence intervals for the estimated parameter values.

- The following values were used for the process parameters $\beta = 1$ and $\alpha \in]-1, 1]$.

Three different types of controller were designed:

I. An internal model controller, which satisfies all of the constraints given above. The values for the controller are given by (20). It is expected that this controller should estimate the correct parameter over all of the given range.

II. A modified internal model controller, which only satisfies the condition given by (17). Its parameter values are given as

$$\begin{aligned}\delta &= -\beta^{-1} \\ \varepsilon &= -0.9\alpha\beta^{-1} \\ \eta &= -0.9\end{aligned} \tag{22}$$

It is expected that this controller should obtain the correct parameter estimates only over the region for which the condition given by (18) is satisfied, that is,

$$-1.0 < \alpha < 0.9 \tag{23}$$

III. A controller, whose parameters are given as

$$\begin{aligned}\delta &= -\beta^{-1} \\ \varepsilon &= -0.9\alpha\beta^{-1} + 0.25 \\ \eta &= -0.9\end{aligned} \tag{24}$$

This controller does not satisfy either of the conditions and should not provide unbiased estimates throughout the region. There may also be regions for which the process is closed-loop stable, but closed-loop unidentifiable.

4.2 Simulation Results and Discussion

Figure 2 presents the parameter estimates for α and β as a function of the true value of α for the controller given in Case I. It can be seen that the controller identifies the correct parameter values for all stable process systems. It is clear that the confidence intervals for the parameter estimates are small and include the true values. Hence, there is no evidence suggests that the parameter estimates are biased.

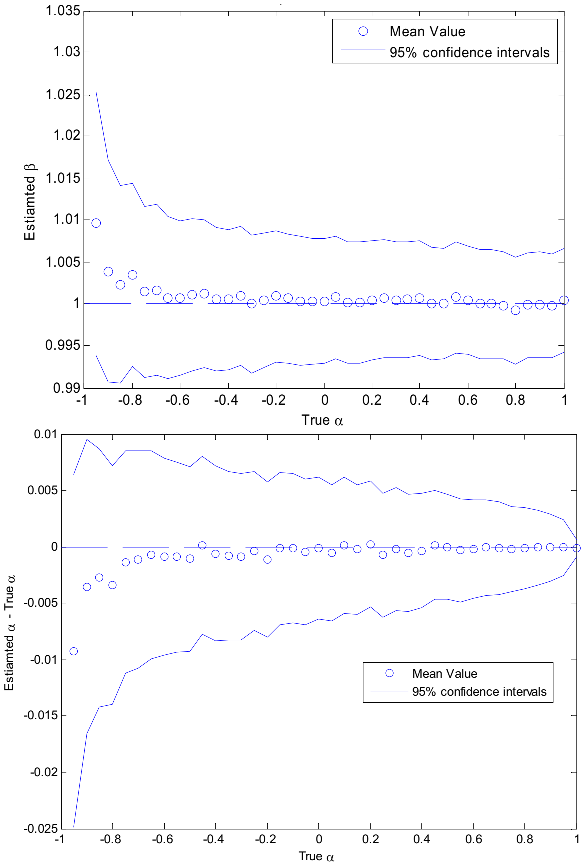

Figure 2: (above) Estimated β as a function of the true α; (below) Estimated α – True α as a function of the true α for Case I.

Figure 3 presents the parameter estimates for α and β as a function of the true value of α for the controller given in Case II. It can be seen that the controller identifies the correct parameter values only in the region given by (23). It can be noted that, unlike for Case I, where the system was identified at $\alpha = 1$, in this case, the system is not identifiable. This is expected given the condition stated in (18). Similarly to Case I, it can be concluded that the parameter estimates are unbiased where the system is stable.

Finally, Figure 4 presents the parameter estimates for α and β as a function of the true values of α for the controller given in Case III. Although for most of the region, where the condition stated in (18) is satisfied, the estimated parameter values appear to match the true values, there appears to be a rather large confidence interval for the parameter estimates around -0.7. In order to investigate this region, a more detailed simulation of the region for which $\alpha \in [-0.8, -0.7]$ was performed. Figure 5 presents the parameter estimates for α and β as a function of the true values of α for the smaller region identified above and the same controller as for Case III. Clearly, Figure 5 shows that there appears to be a discontinuity in the identified parameter values around the point -0.72. In fact, the behaviour of the curve around this point strongly resembles that of a hyperbolic curve.

In order to determine the value of the discontinuity, the roots of the numerator and denominator of the transfer functions defined by (7) and (8) can be equated

$$
\begin{aligned}
-\frac{\delta}{\varepsilon} &= \frac{-(\alpha+\eta-\beta\delta) \pm \sqrt{(\alpha+\eta-\beta\delta)^2 - 4(\alpha\eta-\beta\varepsilon)}}{2(\alpha\eta-\beta\varepsilon)} \\
-\frac{1}{\eta} &= \frac{-(\alpha+\eta-\beta\delta) \pm \sqrt{(\alpha+\eta-\beta\delta)^2 - 4(\alpha\eta-\beta\varepsilon)}}{2(\alpha\eta-\beta\varepsilon)}
\end{aligned} \tag{25}
$$

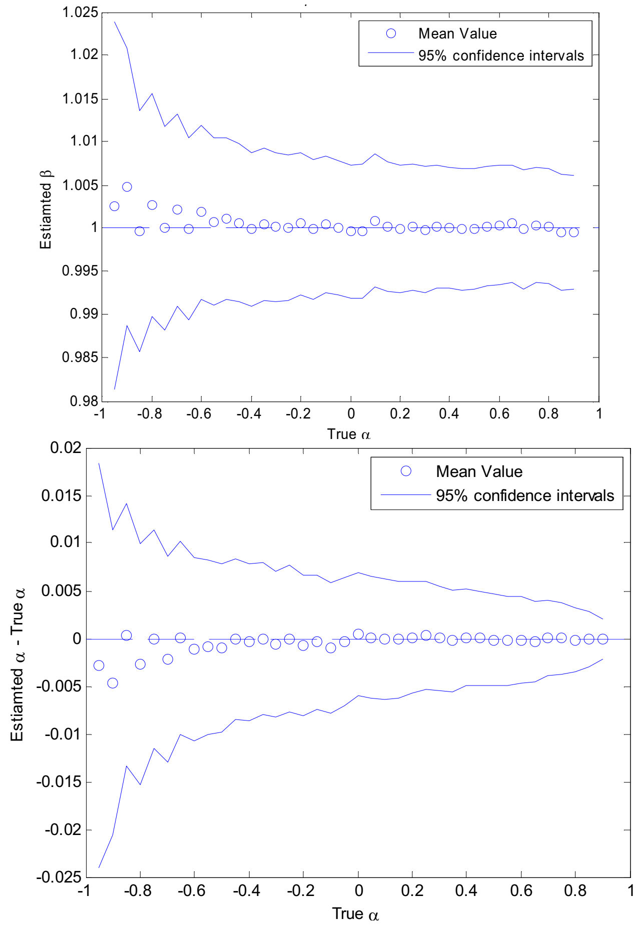

Figure 3: (above) Estimated β as a function of the true α; (below) Estimated α – True α as a function of the true α for Case II.

Simultaneously solving (25) based on the values of the controller parameters given in (24) and assuming that $\beta = 1$ shows that the discontinuity has the exact value

$$\alpha = -\frac{13}{18} \tag{26}$$

It can be noted that the negative sign was taken in (25) in order to obtain a solution. The positive solution did not provide any points of interest. Furthermore, the remaining root of the denominator lies inside the unit circle. Thus, it can be concluded that, although the closed-loop system is stable, it cannot be identified. In fact, if a simulation is performed at this exact point, the parameter estimates are effectively infinity.

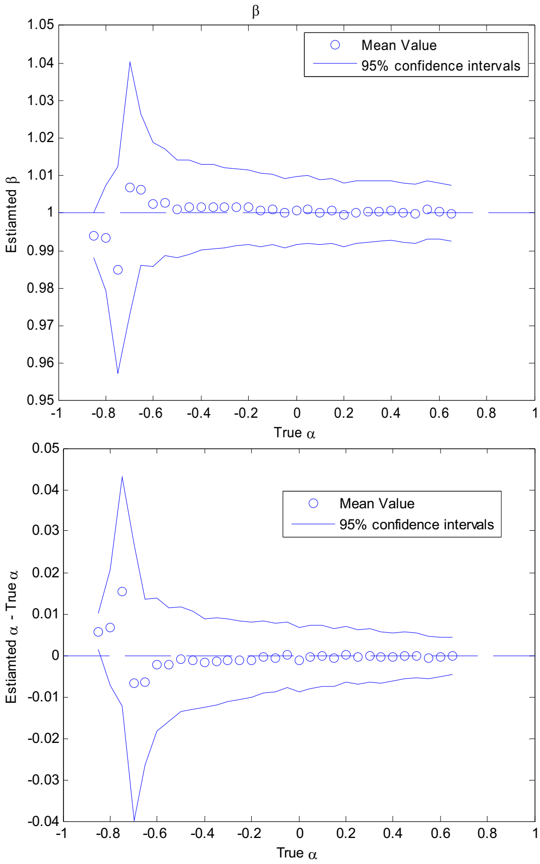

Figure 4: (above) Estimated β as a function of the true α; (below) Estimated α – True α as a function of the true α for Case III.

In control theory, when pole-zero cancellations occur, then it is said that a given mode is uncontrollable or unidentifiable (Chen, 1999). The above result would suggest that, for controllers that fail to satisfy the aforementioned conditions ((17) and (18)), simultaneous pole-zero cancellations occur that result in an inability to identify the correct parameter values.

Finally, it can be noted that, in Figure 5, the confidence intervals do not necessarily cover the true parameter value. Thus, it can be concluded that the parameter estimates are biased. This is especially pronounced around the discontinuity. This behaviour is to be expected given the analytical results.

5. CONCLUSIONS

In this paper, the conditions for identifiability for a first-order ARX process regulated by a lead-lag controller were analytically derived. It was furthermore shown that controllers that do not satisfy these conditions fail to be identifiable for all parameter values located in the stable region of the controller despite satisfying the conditions for identifiability proposed by Gevers, Bazanella, and Ljubiša (2008). The results obtained in this paper can be extended to include different types of model structures and controller.

Future work will focus on extending the results to other model structures, other types of controllers, and to higher order processes. The main difficulties with extending the results are obtaining sufficient computational power to resolve the messy algebraic equations and resolving issues related to the solution of non-ARX models.

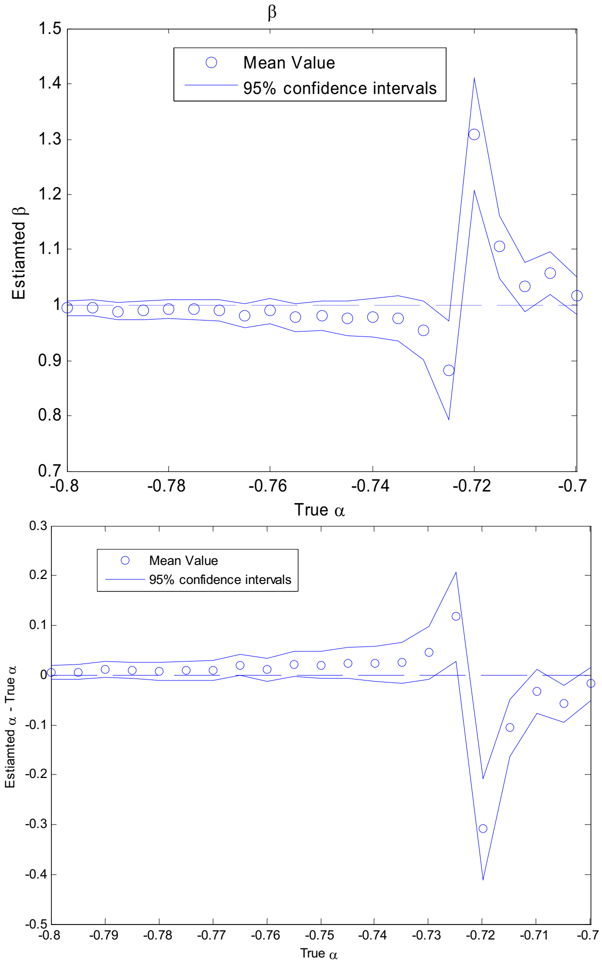

Figure 5: (above) Estimated β as a function of the true α; (below) Estimated α – True α as a function of the true α for Case III for the region [-0.8, -0.7].

ACKNOWLEDGEMENT

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