Integration of control theory and scheduling methods for supply chain management

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1 Introduction

The supply chain is a system comprising organizations, decision makers, and technology decision policies that is responsible for transforming raw materials into finished products that are delivered to end customers. As expanded upon later in the paper, the supply chain is traditionally characterized by counter-current flows of information and material. Material flows from the raw material suppliers through the production and distribution facilities to the end customers, while information, in the form of demands and orders, flows from the end customers upstream to the suppliers [5, 9].

The decisions for supply chain management can be broadly classified into three categories: strategic, tactical and operational. The strategic decisions are the long term planning decisions that may include, among others, where to locate production facilities and warehouses, and in which technologies to invest. On a medium time range, tactical decisions include selecting supply chain partners such as raw material suppliers, transportation companies, etc. The operational decisions are the short term decisions, which are related to optimally operating the supply chain. These decisions include planning and scheduling in the production facilities, and distribution decisions such as inventory management, ordering and shipping policies, etc. [79, 34].

Shapiro [82] lists the challenges in enlarging the scope of strategic planning in supply chains. Among the listed challenges are integrating manufacturing, purchase and sales decisions, multiperiod analysis and optimizing the overall supply chain profits. Stadtler [85] is an excellent overview paper about advanced planning in supply chains. The authors emphasize linking organizational units to improve competitiveness of the supply chain. However, from an operational viewpoint, they focus on advanced planning systems (APS) that uses information and communication technology to coordinate all the flows (material, information, financial) in the supply chain to best improve customer satisfaction.

The combined strategic and operational planning is a challenging optimization problem, but researchers have made efforts to solve it; see, for instance, [74, 91, 97]. The optimization problems formulated for combined strategic and operational planning typically involve selecting a supply chain network from a family of networks or a network superstructure. Recent developments in combined strategic and operational planning, including handling of uncertainties and multi-objective formulations, are described in the review paper [63].

At the operational level of the supply chain, the need for simultaneous decision making at the manufacturing and the distribution sites to operate a *coordinated* supply chain has been recognized. The focus of this paper is on methods to achieve such simultaneous decisions. This simultaneous decision making is also known as enterprise wide optimization [36].

Modern supply chains operate over multiple locations and products, and are highly interconnected. In a competitive economy, neglecting these interactions may result in lower profits. A central coordinator who controls the supply chain can account for these interactions and provide optimal operation. However, centralized coordination may not always be practical for a supply chain as (i) different nodes may belong to different firms, (ii) there may be a conflict of objectives among nodes (iii) information sharing may not be perfect and (iv) a centralized decision maker is the most vital cog in a supply chain, and its failure may be catastrophic for the supply chain. Therefore,

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distributed coordination structures for supply chain operation is needed.

We focus here on tailoring model predictive control (MPC) as a general purpose method for optimal supply chain operation. Model predictive control uses a dynamic model of the system to predict future outcomes and solves a constrained optimization problem over the predicted outcomes to find the best operational decisions. Therefore, it is well suited as a basis for supply chain operation because it makes full use of the dynamic model and knowledge of the interactions between the various nodes to predict and optimize an overall supply chain objective function.

We propose cooperative MPC as a tool for coordinating supply chains as it retains the same structure as traditional supply chains wherein each node makes its own local decisions, but instead of optimizing the local objective functions, the nodes optimize the overall supply chain objective function. It has been shown that cooperative MPC can perform as well as centralized MPC under certain assumptions. Closely related to cooperative MPC, is noncooperative MPC, in which the nodes have global knowledge, via information sharing with the other nodes, but they continue to do local optimizations. It has been shown that noncooperative MPC can destabilize a stable system [71, Chapter 6].

In Section 2, we provide a brief review of the different control theory based and distributed decision making approaches to supply chain optimization and operation. In section 3, we describe the dynamic modeling of supply chains. In Section 4, we show an example of a two-tank system which is closed-loop unstable with noncooperative MPC but closed-loop stable using the algorithms for cooperative MPC presented in Section 5. In section 6. we implement cooperative MPC on a single-product, two-echelon supply chain. Finally, we summarize our results and present future directions of research in Section 7.

2 Literature survey

A well defined supply chain optimization model requires a detailed dynamic description of the supply chain and an objective function that captures all the essential costs and trade-offs in the supply chain. Beamon [9] classifies supply chain modeling in four broad categories: Deterministic models where all the parameters are known, stochastic models with at least one unknown parameter (typically demands) that follows a known probability distribution, economic game theory based models, and simulation based models. As pointed out in [75], a majority of these models are steady-state models based on average performance, and hence are unsuitable for dynamic analysis. In the review of dynamic models for supply chains, Riddalls et al. [73] classify the models as continuous time models, discrete time models, discrete event simulations, and operations research (OR) based models.

The pioneering work of "industrial dynamics" awakened the control community's interest in supply chain optimization. The industrial dynamics models are the continuous (and discrete) time dynamic models mentioned in [2]. Industrial dynamics captures the dynamics of the supply chains using differential (or difference) equations, and therefore, control theory is a natural choice to study supply chain dynamics. In their simplest form, these models capture inventory dynamics based on the shipments and orders leaving the node

$$\mathrm{Iv}_i(k) = \mathrm{Iv}_i(k-1) - \sum_{j \in \mathrm{Dn}(i)} S_{ij}(k) + \sum_{j \in \mathrm{Up}(i)} S_{ji}(k - \tau_{ji})$$

in which $\mathrm{Iv}_i(k)$ is the inventory in node $i \in \mathcal{I}$ at discrete time k, $\mathrm{Dn}(i)$ is the set of nodes to which node i ships material and $\mathrm{Up}(i)$ is the set of nodes from which node i receives material. The shipment delay between nodes i and j is denoted τ_{ij}, and S_{ij} is the amount shipped by node i to node j.

In order to compare different methods of supply chain operation, supply chain performance has to be quantified. Beamon [8, 9] classify the performance measures in a supply chain as quantitative measures like cost minimization, profit maximization, customer response time minimization and, qualitative measures like customer satisfaction, flexibility etc. An important performance measure that supply chain operation strives to reduce is the *bullwhip effect*, which is defined as the amplification of demand fluctuations as one moves upstream in the supply chain. It has been observed that the orders placed by a node to its upstream nodes amplify (with respect to the customer demand) as one moves towards the supplier in a supply chain. This effect increases the cost of operating the supply chain. It has been estimated that a potential 30 billion dollar opportunity exists in streamlining the inefficiencies of the grocery supply chain, which has more than 100 days of inventory supply at various nodes in its supply chain [44, 43]. Among the reasons cited for the bullwhip effect is information distortion as one moves upstream in the supply chain. Information sharing has been shown to alleviate the bullwhip effect and is part of industrial practice such as vendor managed inventory (VMI), etc. The other reasons often cited for the bullwhip effect are: (i) the misunderstanding of feedback, which occurs because the nodes do not understand the dynamics of the supply chain, and

(ii) the use of local optimization without global vision, in which each node tries to maximize its local profit without accounting for the effects of its decisions on the other nodes in the supply chain [57]. Centralized operation of supply chains is best suited to mitigate bullwhip effect, as it has exact knowledge of the dynamics and complete information.

Classical control theory

The earliest applications of control theory to supply chains involved studying the transfer functions and developing single input single output (SISO) controllers for tracking inventory to its targets. Frequency domain analysis was used to analyze and evaluate alternative supply chain designs. In classical control approaches to controlling supply chain, the nodes were analyzed as linear systems using Laplace and Z-transforms. In the work of Towill [90], a block diagram based approach to modeling a node was proposed. The single product node consisted of two integrators to capture the dynamics of inventory and backorders, while the order rate was the manipulated variable. The disturbance to the system, market demand, was incorporated in a feed-forward manner in the model. Time delays were also incorporated in the model. A feedback control law was proposed for controlling the inventory deviations from a target inventory. By varying some of these parameters like delay, controller gain etc., a family of models for a single node called as the input-output based production control system (IOBPCS) can be studied [42]. The feedback law, in its simplest form, takes the form of an order-up-to policy, that is order up-to the inventory target, if the current inventory is below its target. This policy can be viewed as a saturated proportional controller, although other forms of the controller can also be studied. Upon having a control policy and after defining other system details like delays, forecast smoothing etc, the transfer function of the node can be derived and analyzed [24]. White [94], Wikner et al. [96] developed a PID controller without feed-forward forecasting for the node. A review of stability analysis for the IOBPCS family of models is presented in Disney et al. [26].

Classical control theory has also been studied for controlling the dynamics of the entire supply chain as well. Grubbström and Tang [37] provides a review of the input-output modeling of supply chains and its analysis using Laplace transforms. Input-output modeling is the matrix form description of the supply chain dynamics. Burns and Sivazlian [16], Wikner et al. [95] analyzed multiechelon supply chains using the block diagram based approach. They analyzed the effect of ordering policies, delays and information availability at the nodes to analyze the supply chain response and bullwhip effect. Burns and Sivazlian [16] used Z-transforms in their approach and found that information distortion led to bullwhip effect. Wikner et al. [95] found out that information sharing and echelon inventory policies (in which each echelon considers inventory in all the nodes downstream to it) can mitigate bullwhip effect. Perea López et al. [67], Perea López et al. [66] have developed a continuous time model to describe a supply chain. The model is similar to deterministic supply chain models but uses differential equations to track dynamics. They simulated the model using a heuristic shipping policy and studied the closed-loop supply chain under three different proportional controllers for placing orders. They developed controllers to track inventory, backorder or a combination of both. The objective of the paper was to demonstrate that the model was capable of capturing the dynamics. Hence, they did not suggest any tuning methods for the controllers. Lin et al. [51] presented an approach to analyze the closed-loop stability of a supply chain and an approach for controller synthesis using a transfer function approach. The controller policy and shipping policy were similar to the Perea López et al. [67] paper. They analyzed stability considering three extreme closed-loop scenarios: (i) high inventories and infinite replenishment from upstream nodes (infinite production),(ii) a low inventory and infinite replenishment from upstream nodes and (iii) limited production/supply. The effect of controller gains on the bullwhip effect was also analyzed. The authors proposed a controller tuning criterion based on frequency domain analysis of the Z-transfer functions. Venkateswaran and Son [93] also studied the supply chain response using Z-transform and derived stability conditions for the supply chain. Hoberg et al. [38] applied linear control theory on a two-echelon supply chain and concluded that order-up-to policy based on inventory on hand can lead to instabilities. They found that the use of an echelon policy provides the best performance. Dejonckheere et al. [25] studied information enrichment where in, each node receives the final customer demand as well as the orders placed by its downstream nodes, using a linear control theory based approach and concluded that information enrichment is beneficial to the supply chain. Papanagnou and Halikias [64] used a proportional controller to place orders and analyzed the bullwhip effect by estimating the state covariance matrix, for a supply chain responding to a random demand (modeled as a white noise) at the retailer node.

Sarimveis et al. [75], Ortega and Lin [60] provide extensive reviews of classical control approaches to supply chain design and operation.

Stochastic optimal control

Stochastic optimal control has been used to obtain ordering policies that minimize the expected costs of a node responding to random demands. We assume that the probability distribution of the demand is given. In its simplest form, the inventory control problem can be formulated as a dynamic optimization problem. The order-up-to policy is one such policy that is obtained by solving the dynamic optimization problem. The single node inventory control problem can be cast as a Markov decision problem. See Puterman [69] for details on setting up the problem and algorithms. The order-up-to policy is optimal for independent and identically distributed demands as shown in the seminal paper by Clark and Scarf [23]. By considering setup costs, it can be shown that the $(s, S), s < S$ policy, in which the node orders $S - \mathrm{Iv}$ whenever the inventory Iv falls below s, is the optimal policy for an infinite horizon problem; see for instance [92, 39, 30]. Optimality of similar policies have been shown for Markovian demands [84, 78], compound Poisson and diffusion demands [12], etc. These results, derived for a single inventory holding facility, have been extended to multiechelon systems, [29, 81, 27, 33, 20]and capacitated systems; [47, 31, 32], to better capture the dynamics of modern supply chains. Chen et al. [21, 22] quantify the bullwhip effect for order-up-to policy under exponential smoothing and moving average forecasts. We refer the readers to the books by Zipkin [98] and Axsäter [4] for more details.

Distributed decision making in supply chains

Supply chain decisions have traditionally been made by managers at each node. From a decentralized operation perspective, supply chains can be analyzed using the tools of game theory. In decentralized decision making, the payoff (profits) for each node depends not only on its decisions, but also on the decisions made by the other nodes. Therefore, supply chain operations can be viewed as a strategic game between the various nodes. Game theory based analysis can be further classified into noncooperative and cooperative game theory.

In noncooperative game theory, each node simultaneously makes decisions and then the payoff is obtained. Such games are characterized by the Nash equilibrium that is the set of game outcomes for which no node has a unilateral incentive to move away from the outcome. At the Nash Equilibrium, no node can increase its payoff by changing its decision while the choices made by the other nodes remain the same. This result is attributed to Nash in his seminal paper [59]. Related to Nash equilibrium is the Stackelberg equilibrium attributed to the mathematician von Stackelberg. In a Stackelberg game the nodes make their decisions sequentially. We refer the reader to the excellent text by Başar and Olsder [7] for detailed analysis into game theory tools and methods. Leng and Parlar [45], Cachon and Netessine [18] provide excellent reviews of game theoretic methods applied to supply chains.

If the nodes make the supply chain optimal decision in a noncooperative game, then the supply chain is said to be coordinated [19]. One of the methods to coordinate supply chains is to modify the interactions between the nodes of the supply chain (for example, by adjusting contracts) so that each node, optimizing its local objective, makes the globally optimal decision. For example, a two node newsvendor type supply chain can be coordinated using buy-back contracts. A two node newsvendor supply chain consists of a retailer and a supplier. The retailer faces a random demand with a known probability distribution at each period. In order to respond to this demand, the retailer buys product from the supplier at the beginning of the period. The supplier is assumed to ship products instantaneously. In the buy-back contract, the supplier agrees to buy back unsold stock at the end of the season from the retailer. The buy-back contract transfers some of the risk of maintaining inventory to the supplier and divides the supply chain optimal profit (the centralized profit) among the partners. In contrast, the performance of the wholesale (price only) contract, in which the supplier supplies product at a wholesale price to the retailer, can be arbitrarily poor. Under wholesale contract, the retailer takes all the risk of excess inventory and orders safely[17, 19]. Perakis and Roels [65] quantified the inefficiencies in the supply chain (the ratio of the decentralized supply chain profits to that of the centralized supply chain profits) for the price only or wholesale contracts. Moses and Seshadri [56] showed that a two-echelon supply chain can be coordinated only if the manufacturer agrees to share a fraction of the holding costs of the retailer's safety stock. Golany and Rothblum [35] also studied linear reward/penalty as a contract modification to induce coordination in the supply chain. Li and Wang [49] provide a survey of the various coordinating mechanisms. Axsäter [3] studied the Stackelberg game in the supply chain. Axsäter [3] assumed that the manufacturer is the leader in the Stackelberg game. The manufacturer minimized the system-wide costs and declared its policies to the retailers. The retailers then optimized a modified cost function that considers the policies of the manufacturer. They implemented an iterative optimization algorithm such that the policies at every iterate was better than the initial policy. The authors also noted that the iterations may not converge to the centralized solution.

On the other hand, cooperative game theory is a branch of game theory that studies the benefits of coalitions. A coalition between nodes is formed when the nodes cooperate. These studies allocate payoffs to various coalitions and these payoffs are analyzed via different techniques like Shapley value [83] or nucleolus [76]. Raghunathan [70] studied incentives for nodes to form information sharing partnerships. Leng and Parlar [46] studied different coalitions in a three-echelon supply chain. For example, if the manufacturer and distribution center form a coalition, then it is assumed that the orders placed by the retailer are known to both the nodes. Under the grand coalition, the final customer demand is shared among all the three nodes. Leng and Parlar [46] defined the payoff of a coalition as the cost savings obtained when extra information due to the coalition is available to the nodes. Using the payoff of all the possible coalitions, they studied the stability of different coalitions. The authors noticed that the bullwhip effect is reduced when the manufacturer and distribution center formed a coalition. Bartholdi and Kemahlioğlu-Ziya [6] studied a two-echelon supply chain in which a manufacturer supplies to multiple retailers. They used the concepts of cooperative game theory to find profit allocation rules after cooperation. Since the value allocation was in the core of the cooperative game, it ensured that none of the participants in the coalition have incentive to leave. Nagarajan and Sošić [58] provide a comprehensive survey of cooperative game theory applications to supply chains.

MPC for supply chains

Perea López et al. [68] developed a detailed multi-product model including time delays and a mixed integer model for the manufacturing facility. They modeled the shipment rates with a "best I can do" policy that satisfies all the accumulated orders at a given time if stocks are available; otherwise it ships all of its available stock. This model was used for supply chain control using MPC maximize profit. They considered three cases in their implementation: a centralized case, and two other cases that they termed "decentralized" control. In one decentralized control scheme, they optimized the mixed integer production facility while operating the supply chain under a nominal control policy (like a proportional controller for the orders). In the other decentralized control scheme, they optimized only the orders in the supply chain subject to a nominal production schedule. The authors advocated the use of "centralized MPC". Mestan et al. [55] developed a supply chain model using a hybrid systems approach and implemented centralized, decentralized, and noncooperative MPC as described in [72]. They compared customer satisfaction and supply chain profit for the centralized and decentralized MPC. The objective functions were chosen such that the retailer objective of maximizing customer satisfaction was in conflict with the objective of other nodes. Decentralized MPC had the highest customer satisfaction metric but the supply chain operated at a loss. The bullwhip effect was high in the decentralized approach. In centralized MPC, the supply chain found the trade-off between maximizing customer satisfaction and minimizing overall supply chain costs. The centralized approach showed a small bullwhip effect because all the shipment and order rates were determined by a central policy. The authors also noted that the performance of noncooperative MPC was much better than the performance of decentralized MPC. Dunbar and Desa [28] solved a three-echelon, one-product supply chain using a noncooperative MPC. They developed a bidirectionally coupled model, by considering two types of delay: pipeline delay or the transportation delay and a first order material delay to quantify delays in clearing backlogs. The algorithm was found to be better than a nominal control policy. They also observed that the ordering policy was not very aggressive, indicating that the bullwhip effect may be mitigated by distributed MPC. Seferlis and Giannelos [77] presented a two-layer MPC strategy for multi-echelon supply chains. They used MPC to find shipments and orders placed to other nodes, subject to a total order constraint. The total orders placed was the manipulated variable of a PID controller to track inventory. The authors suggest that the performance can be improved by better tuning the PID controller and suggest a bi-level optimization problem in which the PID controller is replaced with an optimization-based controller. Kempf [40] and Braun et al. [15] developed a model predictive control framework for the supply chain in the semiconductor industry. They developed models that are specific to the semiconductor industry. Braun et al. [15] implemented decentralized MPC and studied the control performance under plant model mismatch. Kempf [40] described a two-loop optimization technique for the supply chain optimization problem. The coarse first loop optimizer is used to generate the inventory and order setpoints (reference trajectories), while the fine inner loop MPC is used to track these setpoints. Bose and Penky [14] also used an MPC framework. They focused on forecasting the demand signal and studied the sensitivity of the MPC framework to fluctuations in the demand signal. Maestre et al. [52] proposed a cooperative MPC algorithm for a two-layer supply chain. In their formulation, each node minimized its local objective function, not only over its own decision space, but also over the decision spaces of the other nodes. Based on the multiple optimal objective function values (one for each node), the algorithm determined a consensus input. The drawback of the approach is that it is not scalable for large supply chains with multiple nodes. Bemporad et al. [11] showed the applications of hybrid MPC [10] on a centralized supply chain management problem. Li and Marlin [48] implemented robust MPC using an economic objective function on a multiechelon supply chain.

While it has been established that supply chain operations can benefit from a rolling horizon optimization approach, the closed-loop properties of the supply chain under this operations policy have not been studied. As shown in Section 4, the rolling horizon approach, while being feasible and optimal at each sample time, can lead to arbitrarily poor closed-loop operation (instability). On the other hand, closed-loop properties of the rolling horizon implementation of optimal control policies have been studied intensely in the control community during the last twenty years, and a wealth of results and design guidelines are available. See [54, 71] for comprehensive reviews of these results. In this paper, we extend the recent closed-loop stability results on distributed, cooperative MPC [71, Ch. 6], [86], and establish that any dynamic system that can be stabilized by centralized MPC can be stabilized by cooperative MPC. We show in Section 4, that while noncooperative MPC can render a simple system of integrators closed-loop unstable, we can easily design cooperative MPC schemes for the same system that have guaranteed stability and robustness properties. In the following section, we show that the supply chain can be modeled as a system of integrators.

3 Dynamic modeling of the supply chain

A dynamic model is the heart of any feedback control algorithm. While developing a dynamic model of a supply chain, the components of a supply chain (like the production facility, distributor, retailer etc.) are called as nodes. The supply chain network is the vertices or arcs, which depict the connections between the various nodes. We assume that the network is fixed and given to us. We denote the set of nodes by $\mathcal{I}$. The nodes to which a particular node supplies material are called its downstream nodes, while the nodes from which a particular node obtains material are called its upstream nodes. For each node $i \in \mathcal{I}$, we define the set $\mathrm{Up}(i)$ as the set of all nodes $j \in \mathcal{I}$ that are connected by an arc with i and are upstream to node i. Similarly, we define the set of downstream nodes to i as $\mathrm{Dn}(i)$. For each arc in the the supply chain, material flows downstream and orders (or information) flows upstream. The supply chain in the form of nodes and arcs is shown in Figure 1.

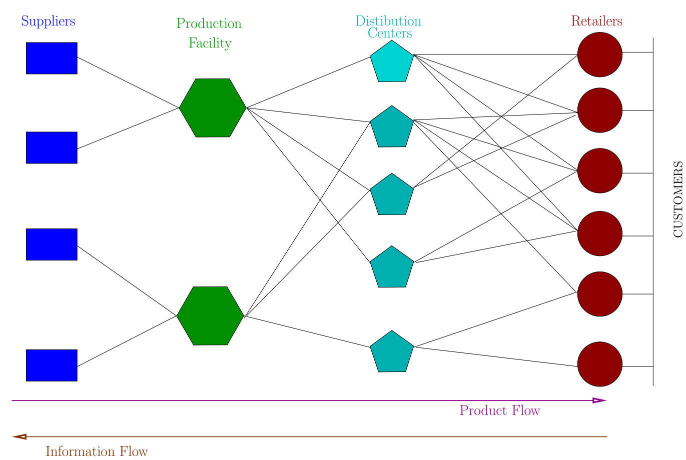

Figure 1: Supply chain as nodes and arcs.

From a classical chemical engineering perspective, each node can be modeled as two tanks, the inventory tank and the backorder tank. The flows out of the inventory tank are the shipments to the downstream nodes and the shipments from the upstream nodes make up the flow into the inventory tank. The flows out of the backorder tank are the shipments to the downstream nodes, which alternatively can be viewed as the orders that have been met; the flows into the backorder tank are the orders arriving at the node. For nodes that handle multiple products, we have as many inventory and backorder tanks as the number of product handled by the node. We develop the supply chain model for a single product, but the model can be easily generalized for multiple-products. Figure 2 depicts the 'tanks' model of a node in the supply chain handling a single product.

Each node i has two states: the inventory in the node, Iv_i, and the backorders in the node, BO_i; two input vectors: the shipments made to each downstream node $j \in \mathrm{Dn}(i)$, S_{ij}, and the orders placed to each upstream node $j \in \mathrm{Up}(i)$, O_{ij}. The shipments coming from the upstream nodes $S_{ji}, j \in \mathrm{Up}(i)$ and the orders arriving from

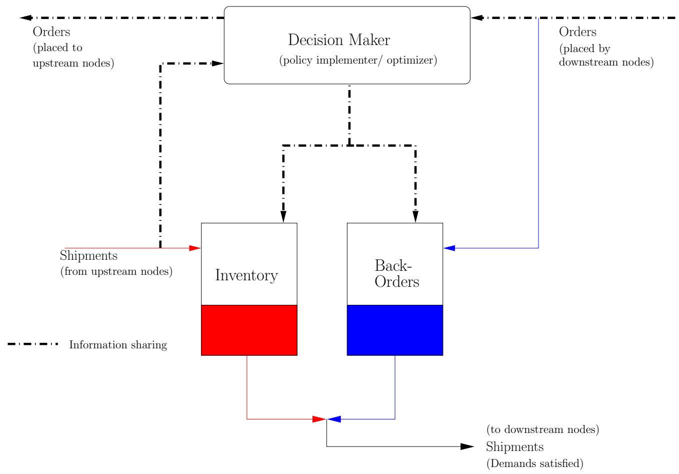

Figure 2: Tank analogy for modeling a node.

the downstream nodes $O_{ji}, j \in \mathrm{Dn}(i)$ are the disturbances arriving to the node. Denoting discrete sample time by integer k, the dynamic equations for node i can be written as

$$\mathrm{Iv}_i(k+1) = \mathrm{Iv}_i(k) + \sum_{j \in \mathrm{Up}(i)} S_{ji}(k - \tau_{ji}) - \sum_{j \in \mathrm{Dn}(i)} S_{ij}(k) \tag{1}$$

$$\mathrm{BO}_i(k+1) = \mathrm{BO}_i(k) + \sum_{j \in \mathrm{Dn}(i)} O_{ji}(k) - \sum_{j \in \mathrm{Dn}(i)} S_{ij}(k) \tag{2}$$

in which τ_{ji} is the transportation delay. We assume that there are no delays for order transfers between the nodes. Denoting $x_i(k) = \begin{bmatrix}\mathrm{Iv}_i(k) & \mathrm{BO}_i(k)\end{bmatrix}'$, $u_i(k) = \begin{bmatrix}S_{ij} & O_{ij'}\end{bmatrix}', j \in \mathrm{Dn}(i), j' \in \mathrm{Up}(i)$, the previous dynamic equations for the nodes can be written in the familiar state-space form for MPC applications

$$\begin{aligned} x_i(k+1) &= A_{ii} x_i(k) + B_{ii} u_i(k) \\ &\quad + \sum_{j \in \mathrm{Up}(i)} B_{ji}^{\tau_{ji}} u_j(k - \tau_{ji}) + \sum_{j \in \mathrm{Dn}(i)} B_{ji} u_j(k) \end{aligned}$$

The decision maker shown in Figure 2 can take several different forms:

- Each decision maker can implement a simple ordering policy that depends only on the incoming shipments and orders. Such an ordering policy could be a PI controller to control the inventory levels, or (s, S) policies that are obtained from stochastic inventory control optimization. Such decision makers are implementations of classical control theory approaches to supply chain control.
- Each decision maker can implement an MPC controller to regulate its local states by optimizing a local objective function (for example, the profit function for the node). The nodes can share information regarding upstream shipments, downstream orders, etc. This form of control is termed noncooperative MPC.
- Each decision maker can implement an MPC controller that considers the effect of the nodes' decision on the entire supply chain (for example, each node optimizes the supply chain profit function). The nodes still share information. This form of control is termed cooperative MPC.
- We can replace all the decision makers at the nodes with a single decision maker at the supply chain level. This single decision maker makes decisions for all the nodes. This form of control is termed centralized MPC.

The overall supply chain dynamic model is the individual node dynamic equations collected for all nodes $i \in \mathcal{I}$. The only required change in the node dynamic equation is for the retailer and the production facility nodes.

Retailer models

For the retailer nodes $i \in \mathcal{R}$, the dynamic equations are modified as,

$$\begin{aligned} \mathrm{Iv}_i(k+1) &= \mathrm{Iv}_i(k) + \sum_{j \in \mathrm{Up}(i)} S_{ji}(k - \tau_{ji}) - S_{ic}(k) \\ \mathrm{BO}_i(k+1) &= \mathrm{BO}_i(k) + \mathrm{Dm}_{ic}(k) - S_{ic}(k) \end{aligned}$$

in which S_{ic} is the shipment made by the retailer, and Dm_{ic} is the customer orders (demands). *The only disturbances in the overall supply chain model are the customer demands* $d = \begin{bmatrix}\mathrm{Dm}_{ic}\end{bmatrix}', i \in \mathcal{R}$, which drive all the flows (shipments and orders) in the supply chain.

Production facility models

The production facility needs to be modeled separately because material conversion takes place in this node. In multiple product supply chains, the same production facility handles multiple products. Thus a model for the production facility needs to incorporate a scheduling model to optimize the sequence of production. In this paper, we assume that the production facilities belong to the first echelon. We further assume an ideal supplier of raw materials to the production facilities, implying that we have infinite supply of raw materials without transportation delay.

Planning models. In this paper, we shall use an "approximate production model" to model the production facility. In the approximate production model, we replace the detailed scheduling model with convex constraints that represent the feasible region of production. This idea is similar to the process attainable region [89]-a convex region of production quantities for which there exists some feasible schedule. The process attainable region can be computed by using computational geometry tools [89, 53, 88] or parametric programming tools [50]. Let $\mathcal{M}$ be the set of production facility nodes. Then, for each $i \in \mathcal{M}$, the modified dynamic equations for the final products are

$$\begin{aligned}
&\mathrm{Iv}_i(k+1) = \mathrm{Iv}_i(k) + S_{ip}(k) - \sum_{j\in \mathrm{Dn}(i)} S_{ij}(k) \\
&\mathrm{BO}_i(k+1) = \mathrm{BO}_i(k) + \sum_{j\in \mathrm{Dn}(i)} O_{ji}(k) - \sum_{j\in \mathrm{Dn}(i)} S_{ij}(k) \\
&f(S_{ip}(k)) \leq 0
\end{aligned}$$

in which S_{ip} are the manipulated inputs denoting production during the period. Note that, for multiproduct production facilities, each of the inputs S_{lip} for products $l \in \mathcal{L}$ are coupled by the convex production feasibility constraint. The set $\mathcal{L}$ represents the set of products.

$$f(S_{1ip}(k), S_{2ip}(k), \dots, S_{lip}(k), \dots) \leq 0$$

Scheduling models. The state task network (STN) approach is probably the most popular method to model a production facility in which multiple products are produced using shared resources [41, 80]. In STN modeling, the final products, intermediates and raw materials are states that are processed using tasks like reactions, separation, etc. These tasks can be carried out in units capable of handling multiple tasks. A detailed schedule is the sequence of operation of the tasks in the units so that a production objective can be met at minimal cost without violating the scheduling constraints.

Detailed scheduling models are formulated as mixed integer linear programs (MILPs) or mixed integer nonlinear programs (MINLPs). If we chose to model the production facility using a detailed scheduling model, then the resulting supply chain MPC problems become mixed integer programs. Although, research progress has been made in the theory of MIQP and hybrid MPC (see [10]), in this paper, we do not consider detailed scheduling models in the formulation of the supply chain model.

Summary

In this section, we wish to bring to the readers' attention, three salient features of the supply chain dynamic model presented in this section.

Uncontrollable local models. Controllability implies that there exist inputs that can move the state of the system from any initial state to any final state in finite time. Examining (1) and (2) for the inventory and backorder balance for node i, we observe that while nodes $j \in \mathrm{Up}(i)$ respond to orders O_{ij} placed by node i, node i has no knowledge of the subsequent dynamics of its own orders. Therefore, we need to provide the node some model of how its orders affect the later shipments coming into the node. To do so in a noncooperative or decentralized control arrangement, we track another state (or output) Ip_i termed the inventory position. The dependence of orders on incoming shipments is modeled through the function $g(\cdot)$.

$$\mathrm{Ip}_i(k+1) = \mathrm{Ip}_i(k) + \sum_{j\in \mathrm{Up}(i)} g(O_{ij}(k-\tau_{ji})) - \sum_{j\in \mathrm{Dn}(i)} S_{ij}(k)$$

In the centralized control framework, the actual dynamics of the entire supply chain is available to the decision maker, and the relationship of the orders at node i to its subsequent incoming shipments is captured by the upstream nodes backorder balance equations and the supply chain performance metric. As we show later in the paper, the same argument also holds for cooperative MPC.

Unstable models. The supply chain is modeled as a system of integrators whose response to an input step change is a ramp. Such systems need to be stabilized in the closed loop, otherwise the states can keep growing (think of it as backorders keep rising as time increases). Therefore, we emphasize establishing closed-loop properties of the algorithms that we propose for supply chain optimization.

Stabilizable centralized model. We notice that all the nodes belonging to the manufacturing facilities are controllable because we manipulate the production rates. Therefore, the manufacturing nodes do not require an inventory position model. Since the manufacturing facility model has this property, the overall supply chain model is also controllable. The controllability of the centralized model is an important feature that we use to design closed-loop stable centralized and cooperative MPC frameworks for supply chain optimization.

4 Two-tank example

In this section, we introduce centralized, cooperative and noncooperative MPC using the two-tank system shown in Figure 3. We choose the two-tank system because its model is a system of integrators like the supply chain model.

As noted previously, in model predictive control, at each time a constrained optimization problem is solved with a future trajectory of control inputs as decision variables. The control objective is based on a forecast of the system behavior coming from the dynamic system model. Only the first input of the optimal input sequence is injected into the system. The system then evolves from its current state given the applied optimal input, and the procedure is repeated at the next sample time. Although this procedure is reasonable and has been used in many situations, there is no a priori guarantee that the trajectory of the closed-loop system remains close to the forecast made during each of the open-loop optimizations. Indeed, keeping the actual closed-loop trajectory close to the optimal open-loop forecast is the focus of stability theory. With careful design of the online optimization problems, we can ensure that the closed-loop system trajectory is also optimal given the control objective [71, Chapter 2].

In this section, we introduce the cooperative and noncooperative distributed MPC algorithms and show that we can stabilize the system using cooperative MPC (see Figure 4). The theory to ensure stability of cooperative MPC is presented in the next section.

Models and objective functions

The system, shown in Figure 3, consists of two tanks with levels x_1 and x_2. The two tanks are considered as two separate subsystems for implementing distributed MPC. Subsystem-1 controls the level x_1 and has the inputs u_{11}, u_{12}, u_{13} at its disposal. The input u_{12} drains water from the first tank into the second tank. Input u_{13} directly drains water from the first tank, but it is assumed that manipulating input u_{13} is more expensive than manipulating input u_{12}. Subsystem-2 controls the level x_2 and has the inputs u_{21} and u_{22} at its disposal. The input u_{21} recycles a fraction of the water back into the first tank according to the recycle ratio r. Similar to subsystem-1, input u_{22}, which directly drains water out from subsystem-2, is assumed to be more expensive to operate compared to input u_{21}. To add some complexity, we assume that the recycle flow from subsystem-2 to subsystem-1 introduces a further disturbance in the system,which is perfectly modeled. This disturbance introduces water into the first tank at a rate proportional to the flow out of the second tank through u_{21}. Such interactions can arise when there is tight heat and mass integration in chemical plants.

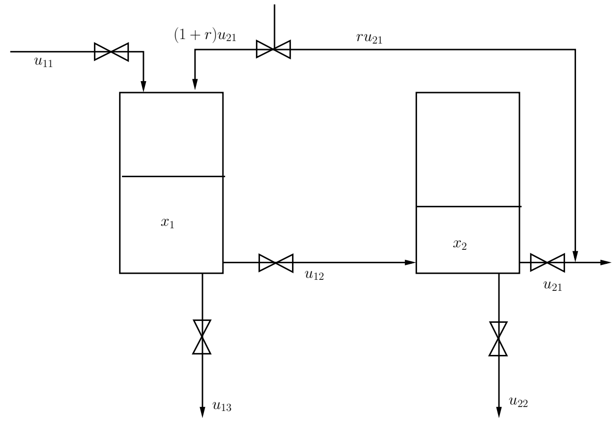

Figure 3: The two-tank system.

The subsystem-1 model for the two-tank system is

$$x_1^+ = \underbrace{1}_{A_1} x_1 + \underbrace{\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}}_{B_{11}} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix}}_{u_1} + \underbrace{\begin{bmatrix} 1+r & 0 \end{bmatrix}}_{B_{12}} \underbrace{\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}}_{u_2}$$

The subsystem-2 model for the two-tank system is

$$x_2^+ = \underbrace{1}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{B_{21}} \underbrace{\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix}}_{u_1} + \underbrace{\begin{bmatrix} -1 & -1 \end{bmatrix}}_{B_{22}} \underbrace{\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}}_{u_2}$$

The overall (centralized) model of the two-tank system is the minimum realization of

$$x^+ = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}_{B} u$$

in which $x = (x_1, x_2)$ and $u = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22})$. Each input is constrained to lie between $[0, \bar{u}]$ in which 0 corresponds to the valve completely closed and $\bar{u}$ corresponds to the valve completely open.

We define stage costs $\ell_1(\cdot)$ and $\ell_2(\cdot)$:

$$\ell_1(x_1, u_1) = x_1^2 + u_{11}^2 + u_{12}^2 + 100u_{13}^2$$
$$\ell_2(x_2, u_2) = x_2^2 + u_{21}^2 + 100u_{22}^2$$

To demonstrate cooperative MPC, we follow the steps outlined in Section 5 and design the terminal penalty $V_f(\cdot)$ as the solution to the Riccati equation using the stage costs above. The terminal penalty is:

$$V_f(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1.76 & 0.44 \\ 0.44 & 1.51 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The feedback controller corresponding to the terminal penalty is $\kappa(x) = Kx$.

For noncooperative MPC, we use the following terminal penalties:

$$V_f^1(x) = 1.76x_1^2 \qquad V_f^2(x) = 1.51x_2^2$$

We define the cost function for each subsystem as

$$V_N^{1,\beta}(x_1, \mathbf{u}_1, \mathbf{u}_2) = \sum_{i=0}^{N-1} \ell_1(x_1(i), u_1(i)) + \beta V_f^1(x_1(N))$$

$$V_N^{2,\beta}(x_2, \mathbf{u}_1, \mathbf{u}_2) = \sum_{i=0}^{N-1} \ell_2(x_2(i), u_2(i)) + \beta V_f^2(x_2(N))$$

The overall cost function is:

$$V_N^{\beta}(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell_1(x_1(i), u_1(i)) + \sum_{i=0}^{N-1} \ell_2(x_2(i), u_2(i)) + \beta V_f(x(N))$$

The centralized MPC problem can now be defined as

$$\begin{aligned} \mathbb{P}(x): \quad & \min_{\mathbf{u}} V_N^{\beta}(x, \mathbf{u}) \\ \text{s.t. } & x^+ = Ax + Bu \\ & \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U}_1^N \times \mathbb{U}_2^N \end{aligned} \tag{3}$$

in which $\mathbf{u}_i = (u_i(0), u_i(1), \ldots, u_i(N-1))$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $x = (x_1, x_2)$, and N is the length of the horizon. The input constraint set for subsystem-1 is $\mathbb{U} = \{u \mid u_i \in [0, \bar{u}], i \in \{11, 12, 13\}\}$. Similarly, $\mathbb{U}_2$ is the input constraint set for subsystem-2, $\mathbb{U}_2 = \{u \mid u_i \in [0, \bar{u}], i \in \{21, 22\}\}$.

Distributed MPC algorithms

In this section, we present the distributed MPC algorithms [71, Chapter 6]. It is interesting to note that the two-tank system presented in this section does not satisfy the assumptions presented in Chapter 6 of [71] for stabilizing, cooperative MPC. We shall use the new theory developed in Section 5 to establish stability.

Parallel optimization algorithm. Consider the optimization problems in variables y_1, y_2:

$$\mathbb{J}_1(v_2): \qquad \min_{y_1} V_1(y_1, v_2), \text{ s.t } y_1 \in \Omega_1$$

and

$$\mathbb{J}_2(v_1): \qquad \min_{y_2} V_2(v_1, y_2), \text{ s.t } y_2 \in \Omega_2$$

We now present the parallel optimization algorithm, which is a Gauss-Jacobi type of algorithm discussed in Bertsekas and Tsitsiklis [13], Chapter 3. We assume that the constraint sets Ω_1, Ω_2 and the objective function $V_1(\cdot), V_2(\cdot)$ are convex.

Algorithm 1 (Parallel optimization)**.**

Data: Initial values $y_1^{(0)}, y_2^{(0)}$, $p_m \geq 0$, $\omega \in (0,1)$,
Problems $\mathbb{J}_1(v_2), \mathbb{J}_2(v_1)$
Result: Iterates y_1, y_2
Set $p = 0$
while $p < p_m$ **do**
Subsystems share $y_i^{(p)}, i \in \{1, 2\}$ with each-other.
Solve $\mathbb{J}_1(y_2^{(p)})$ to obtain $y_1^o(y_2^{(p)})$.
Solve $\mathbb{J}_2(y_1^{(p)})$ to obtain $y_2^o(y_1^{(p)})$.
Set $(y_1^{(p+1)}, y_2^{(p+1)}) =$
$\omega(y_1^o(y_2^{(p)}), y_2^{(p)}) + (1-\omega)(y_1^{(p)}, y_2^o(y_1^{(p)}))$.
Set $p = p + 1$.
end
Set $y_i = y_i^{(p_m)}, i \in \{1, 2\}$.

Noncooperative MPC. The local subsystem MPC optimization problems for noncooperative MPC is defined as (for $i \in \{1, 2\}$):

$$\begin{aligned} \mathbb{P}_i^{nc}(x_i, \mathbf{v}_j): \quad & \min_{\mathbf{u}} V_N^{i,\beta}(x_i, \mathbf{u}_1, \mathbf{u}_2) \\ \text{s.t. } & x_i^+ = A_i x_i + \sum_{l \in \{1,2\}} B_{il} u_l \\ & \mathbf{u}_i \in \mathbb{U}_i, \quad \mathbf{u}_j = \mathbf{v}_j, j \neq i \end{aligned} \tag{4}$$

Algorithm 2 is the noncooperative MPC algorithm.

Algorithm 2 (Noncooperative MPC)**.**

Data: Starting state $x_1(0), x_2(0)$, initial guess $\tilde{\mathbf{u}}_1(0), \tilde{\mathbf{u}}_2(0)$, $p_m \geq 0$, $\omega \in (0,1)$
Result: Closed loop $x(k), u(k)$, $k \in \mathbb{I}_{\geq 0}$
Set $k = 0$
while $k \geq 0$ **do**
- Implement Algorithm 1 for problems $\mathbb{P}_1^{nc}(x_1(k), \tilde{\mathbf{u}}_2(k)))$ and $\mathbb{P}_2^{nc}(x_2(k), \tilde{\mathbf{u}}_1(k))$ with initial values $(\tilde{\mathbf{u}}_1(k), \tilde{\mathbf{u}}_2(k), p_m, \omega)$.
- Obtain input $u_i(k) = \mathbf{u}_i(0; x_i(k)), i \in \{1,2\}$.
- Set $x_i(k+1) = A_i x_i(k) + \sum_{l\in\{1,2\}} B_{il} u_l(k), i \in \{1,2\}$.
- Obtain the warm start as: $\tilde{\mathbf{u}}_i(k+1) = (\mathbf{u}_i(1; x_i(k)), \mathbf{u}_i(2; x_i(k)), \ldots, \mathbf{u}_i(N-1; x_i(k)), 0), i \in \{1,2\}$.
- Set $k = k+1$.

end

Cooperative MPC. The local subsystem MPC optimization problems for cooperative MPC are defined as (for $i \in \{1,2\}$):

$$\begin{aligned} \mathbb{P}_i^c(x, \mathbf{v}_j): \quad & \min_{\mathbf{u}} V_N^\beta(x, \mathbf{u}_1, \mathbf{u}_2) \\ & \text{s.t. } x^+ = Ax + Bu \\ u = (u_1, u_2), \quad & \mathbf{u}_i \in \mathbb{U}_i, \quad \mathbf{u}_j = \mathbf{v}_j, j \neq i \end{aligned} \tag{5}$$

Algorithm 3 is the cooperative MPC algorithm.

Algorithm 3 (Cooperative MPC)**.**

Data: Starting state $x(0)$, initial guess $\tilde{\mathbf{u}}_1(0), \tilde{\mathbf{u}}_2(0)$, such that $V_N(x, \mathbf{u}) \leq \bar{V}$, $p_m \geq 0$, $\omega \in (0,1)$
Result: Closed loop $x(k), u(k)$, $k \in \mathbb{I}_{\geq 0}$
Set $k = 0$
while $k \geq 0$ **do**
- Implement Algorithm 1 for problems $\mathbb{P}_1^c(x(k), \tilde{\mathbf{u}}_2(k))$ and $\mathbb{P}_2^c(x(k), \tilde{\mathbf{u}}_1(k))$ with initial values $(\tilde{\mathbf{u}}_1(k), \tilde{\mathbf{u}}_2(k), p_m, \omega)$.
- Obtain input sequence $\mathbf{u}_i$ and input $u_i(k) = \mathbf{u}_i(0; x(k)), i \in \{1,2\}$.
- Set $x(k+1) = Ax(k) + Bu(k), u(k) = (u_1(k), u_2(k))$
- Obtain the warm start as: $\tilde{\mathbf{u}}_i(k+1) = (\mathbf{u}_i(1; x(k)), \mathbf{u}_i(2; x(k)), \ldots, K_i \phi(N, x, (\mathbf{u}_1, \mathbf{u}_2))), i = \{1,2\}$
- Set $k = k+1$.

end

In Algorithms 2 and 3, $\mathbf{u}_i$ is the output of the optimizations obtained by implementing Algorithm 1. The first input in the sequence $\mathbf{u}_i$ is denoted by $\mathbf{u}_i(0; x)$, in which x denotes the state for which we have calculated the input. We denote the state at the l^{th} time instance evolving from x at time 0 under the input sequence $\mathbf{u}$ with $\phi(l; x, \mathbf{u})$. We introduce parameters $\bar{V} > 0$, and the linear gain $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ in Algorithm 3. These two parameters along with the parameter β multiplying the terminal cost functions are used to design closed-loop stable cooperative MPC in Section 5.

Notice that we use the same parallel optimization algorithm in both noncooperative and cooperative MPC. The attractive closed-loop properties of cooperative MPC is by the design of the subsystem optimization problems. In noncooperative MPC, each subsystem minimizes its local control objective function $V_N^{i,\beta}(x_i, \mathbf{u}_i)$. This control objective function depends on the shared input sequence of the other subsystem (via the system dynamics). In cooperative MPC, each subsystem still solves for its local decision variables (that is, the size of the optimization problem solved by each subsystem in cooperative as well as noncooperative MPC remains the same), but the subsystems minimize the overall objective function $V_N^\beta(x, \mathbf{u})$.

Example

We now revisit the two-tank example and implement cooperative, noncooperative and centralized MPC. The system starts at steady state with tank levels $(7,7)$ and all valves closed. At time $t = 0$, we change the setpoint of the two tanks to level $(3,3)$ and all valves closed.

The responses are shown in Figure 4. In noncooperative MPC, each subsystem uses the cheap input u_{12}, u_{21} to change the tank levels; unaware that this choice of inputs leads to instability by introducing more water into the system. The subsystems manipulate the cheap inputs because the influence of their inputs on the other subsystem is not captured in the noncooperative MPC optimization problem. At each iteration, the two subsystems, optimizing independently, harm each other because they do not want to operate the expensive valves u_{13} and u_{22}. In cooperative MPC, subsystem-2 realizes that operating valve u_{21} is not desirable, because it optimizes the overall objective function. The subsystems now judiciously use the expensive valves to maintain stability.

5 Cooperative MPC

Stewart et al. [86] present a cooperative MPC algorithm that is stabilizing for only a subset of models that can be stabilized by centralized MPC. Unfortunately, the supply

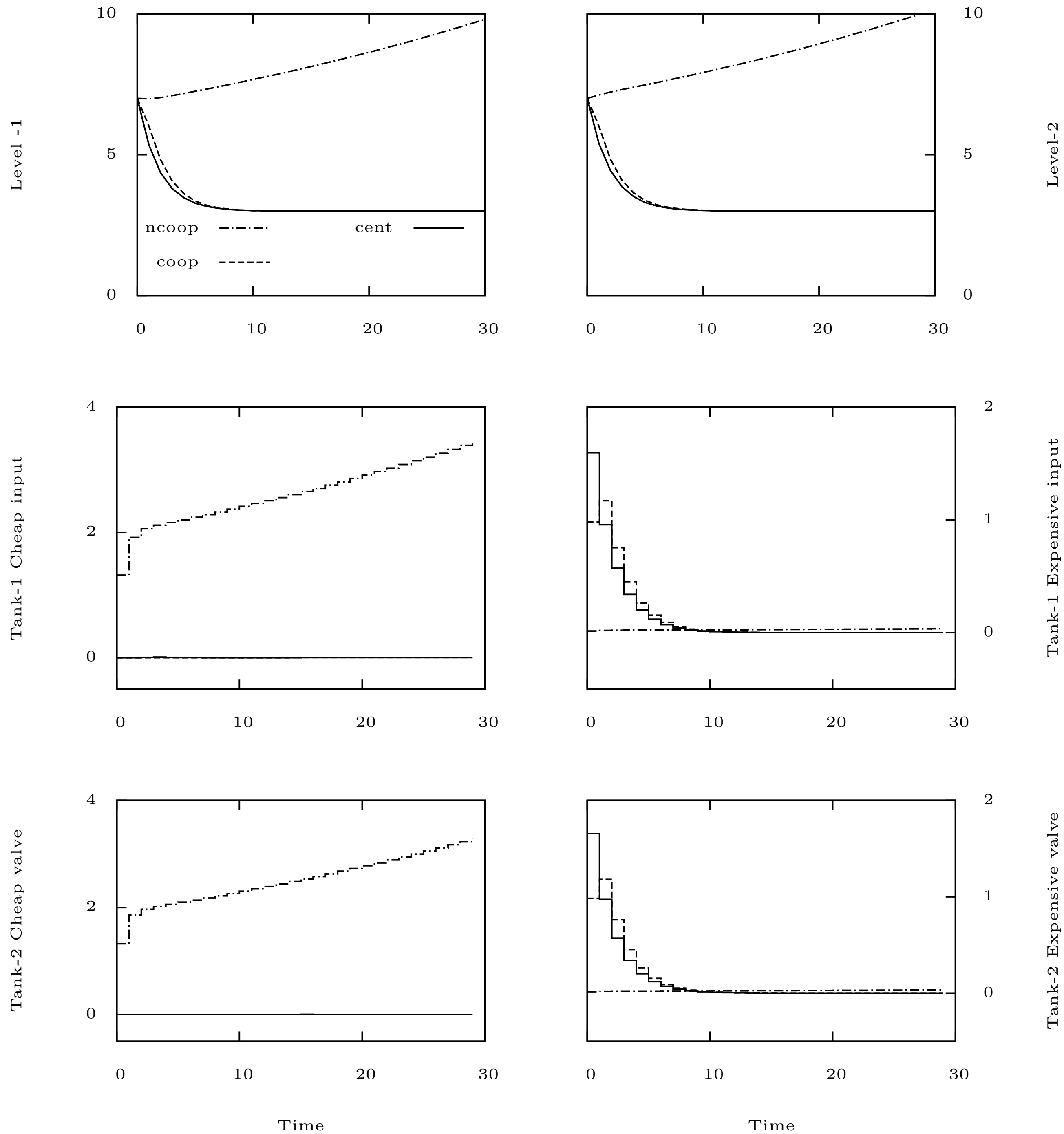

Figure 4: State and input profiles for two-tank system under distributed MPC (ncoop: noncooperative, coop: cooperative, cent: centralized).

chain model cannot be stabilized using the methods provided in [86]. In this section, we propose a new formulation of cooperative MPC that can stabilize any centralized system.

We first introduce some preliminaries and notation, followed by a short description of suboptimal MPC without a terminal region constraint[62, 87]. We then show that cooperative MPC presented in Algorithm 3 is an implementation of centralized, suboptimal MPC, under Assumptions 1– 6.

Optimal MPC

We consider the system:

$$x^+ = f(x,u) \tag{6}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and input while x^+ is the successor state. The system is constrained by the input constraints $u \in \mathbb{U} \subset \mathbb{R}^m$. For a given finite horizon N, we define the input sequence as $\mathbf{u} = (u(0), u(1), \ldots, u(N-1)) \in \mathbb{U}^N$. The state at time $i \geq 0$ for a system starting at state x at time $i = 0$, under control $\mathbf{u}$ is given by $\phi(i; x, \mathbf{u})$.

The MPC cost function is defined as:

$$V_N^\beta(x,\mathbf{u}) := \sum_{i=0}^{N-1} \ell(\phi(i;x,\mathbf{u}), u(i)) + \beta V_f\left(\phi(N;x,\mathbf{u})\right)$$

in which $\ell(x,u)$ is the stage cost and $\beta V_f(x)$ is the terminal cost with $\beta \geq 1$. The set of feasible state-input sequence pairs is given by:

$$\mathbb{Z}_N^\beta := \{(x,\mathbf{u}) \mid \mathbf{u} \in \mathbb{U}^N, V_N^\beta(x,\mathbf{u}) \leq \bar{V}, \quad V_N^\beta(x,\mathbf{u}) \leq \beta V_f(x), \text{ if } x \in \mathcal{B}_r\} \tag{7}$$

in which $\bar{V} > 0$ is an arbitrary large number and $r > 0$ is an arbitrarily small number. The set of feasible initial states $\mathcal{X}_N^\beta$ is:

$$\mathcal{X}_N^\beta := \left\{x \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ s.t. } (x,\mathbf{u}) \in \mathbb{Z}_N\right\}$$

For a given $x \in \mathcal{X}_N^\beta$, the set of feasible inputs is:

$$\mathcal{U}_N^\beta(x) := \left\{\mathbf{u} \mid (x,\mathbf{u}) \in \mathbb{Z}_N^\beta\right\}$$

We now define the MPC problem as:

$$\mathbb{P}_N^\beta(x): \quad \min_{\mathbf{u}} V_N^\beta(x,\mathbf{u}) \quad \text{s.t. } \mathbf{u} \in \mathcal{U}_N^\beta(x) \tag{8}$$

We make the following assumptions:

Assumption 1. *The functions $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous and $f(0,0) = 0$, $\ell(0,0) = 0$, and $V_f(0) = 0$.*

Assumption 2. *The set $\mathbb{X}_f$ is closed and contain the origin in its interior. The set $\mathbb{U}$ is compact and contains the origin.*

Assumption 3 (Basic stability assumption)**.** *For each $x \in \mathbb{X}_f$, there exists $u \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_f$ and*

$$\beta V_f(f(x,u)) \leq \beta V_f(x) - \ell(x,u), \quad \beta \geq 1$$

Assumption 3 implies that we can define a terminal region control law $u = \kappa_f(x)$ such that for all $x \in \mathbb{X}_f$, $\kappa_f(x) \in \mathbb{U}$, $f(x, \kappa_f(x)) \in \mathbb{X}_f$, and $\beta V_f(f(x,\kappa_f(x))) \leq \beta V_f(x) - \ell(x, \kappa_f(x))$.

Assumption 4. *There exist positive constants a, a_1', a_2', a_f and r, such that the cost function $V_N(x,\mathbf{u})$ satisfies:*

$$\begin{aligned} \ell(x,u) &\geq a_1'|(x,u)|^a & (x,u) &\in \mathbb{X} \times \mathbb{U} \\ V_N(x,\mathbf{u}) &\leq a_2'|(x,\mathbf{u})|^a & (x,u) &\in \mathcal{B}_r \\ V_f(x) &\leq a_f|x|^a & x &\in \mathbb{X} \end{aligned}$$

in which $\mathcal{B}_r$ is the ball of radius r.

Assumption 5. *The set $\mathbb{X}_f$ is a sublevel set of the terminal cost.*

$$\mathbb{X}_f := \{x \mid V_f(x) \leq a\}, \quad a > 0$$

Proposition 1. *Let Assumption 1 – 5 hold. Let the cost function be given by $V_N^\beta(x,\mathbf{u})$. For $\bar{V} > 0$, define $\bar{\beta} :=$ $\max(1, \bar{V}/a)$. Then, for any $\beta \geq \bar{\beta}$ and $(x,\mathbf{u}) \in \mathbb{Z}_N^\beta$, we have that $\phi(N;x,\mathbf{u}) \in \mathbb{X}_f$.*

Proof. For sake of contradiction, assume that $(x,\mathbf{u}) \in \mathbb{Z}_N^\beta$, $\beta \geq \bar{\beta}$, but $\phi(N;x,\mathbf{u}) \notin \mathbb{X}_f$, that is $V_f\left(\phi(N;x,\mathbf{u})\right) > a$. Since $(x,\mathbf{u}) \in \mathbb{Z}_N^\beta$, we know that $V_N^\beta(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(\phi(i;x,\mathbf{u}), u(i)) + \beta V_f\left(\phi(N;x,\mathbf{u})\right) \leq \bar{V}$. From Assumption 4, we know that $\ell(\cdot) \geq 0$, which implies that $\frac{\bar{V}}{a} V_f\left(\phi(N;x,\mathbf{u})\right) \leq \bar{V}$, which implies that $V_f\left(\phi(N;x,\mathbf{u})\right) \leq a$, which is a contradiction. Therefore, for $\beta \geq \bar{\beta}$, if $(x,\mathbf{u}) \in \mathbb{Z}_N^\beta$, then $\phi(N;x,\mathbf{u}) \in \mathbb{X}_f$. □

Let $V_N^0(x)$ and $\mathbf{u}^0(x)$ denote the optimal value and the optimal solution of $\mathbb{P}_N^\beta(x)$ in which $\beta \geq \bar{\beta}$ given by Proposition 1. We denote the MPC control law as $\kappa_N^0(x) := u^0(0;x)$, i.e., the control law is the first move of the optimal sequence. Under Assumptions 1–5, the optimal value function $V_N^0(\cdot)$ can be shown to be a Lyapunov function for the closed-loop system $x^+ = f(x, \kappa_N^0(x))$ and the origin is therefore asymptotically stable with $\mathcal{X}_N^\beta$ as its region of attraction [71, Theorem 2.24].

Suboptimal MPC

In suboptimal MPC, we do not solve $\mathbb{P}_N^\beta(x)$ to optimality. Instead, we inject a suboptimal input to the system. As the name suggests, the suboptimal input may not be the solution to the MPC optimization problem. We now define the warm start and successor input set that describes the closed-loop evolution of suboptimal MPC.

We denote the control action, , the first input in the input sequence $\mathbf{u}$ for state x as $\kappa(x) = \mathbf{u}(0; x)$.

Definition 1 (Warm Start)**.** *Let $(x, \mathbf{u})$ be a state-input vector pair such that $(x, \mathbf{u}) \in \mathbb{Z}_N^\beta$. Then the warm start for the successor initial state $x^+ = f(x, \kappa(x))$ is defined as:*

$$\tilde{\mathbf{u}} = (\mathbf{u}(1; x), \mathbf{u}(2; x), \ldots, \mathbf{u}(N; x), u_+)$$

in which $u_+ = \kappa_f(\phi(N; x, \mathbf{u}))$.

Definition 2 (Successor input set)**.** *Consider $(x, \mathbf{u}) \in \mathbb{Z}_N^\beta$. For the successor state $x^+ = f(x, \kappa(x))$, we define the set $G(x, \mathbf{u})$*

$$G(x, \mathbf{u}) = \left\{ \mathbf{u}^+ \mid \mathbf{u}^+ \in \mathcal{U}_N(x^+), V_N^\beta(x^+, \mathbf{u}^+) \leq V_N^\beta(x, \tilde{\mathbf{u}}) \right\}$$

in which $\tilde{\mathbf{u}}$ is the warm start given by Definition 1.

We now present the closed-loop stability theorem for suboptimal MPC.

Theorem 1. *Let Assumptions 1–5 hold. Choose $\bar{V} > 0$ and $\beta = \bar{\beta}$. For state $x \in \mathcal{X}_N^\beta$, choose input sequence $\mathbf{u} \in \mathcal{U}_N^\beta(x)$. Then, the origin of the closed-loop system*

$$x^+ = f(x, \kappa(x))$$
$$\mathbf{u}^+ \in G(x, \mathbf{u})$$

is asymptotically stable on (arbitrarily large) compact subsets of $\mathcal{X}_N^\beta$.

The proof of Theorem 1 is presented in [61].

Definition 3. *The optimization algorithm $\mathbb{O}$ applied to $\mathbb{P}_N^\beta(x)$ has the following properties:*

1. *It is an iterative algorithm starting from a feasible point $(x, \tilde{\mathbf{u}})$.*
2. *Every iteration $\mathbf{u}$ decreases the objective function. This property ensures $V_N^\beta(x, \mathbf{u}) \leq V_N^\beta(x, \tilde{\mathbf{u}})$.*
3. *Every iteration is feasible. This property ensures $\mathbf{u} \in \mathcal{U}_N^\beta(x)$.*

We now present the suboptimal MPC algorithm which uses an given optimization routine $\mathbb{O}$ satisfying properties given in Definition 3. Any optimization algorithm $\mathbb{O}$ to solve $\mathbb{P}_N^\beta(x)$ that satisfies the requirements mentioned in Definition 3 has the property that any iterate generated by $\mathbb{O}$ is a valid input to the system.

Algorithm 4 (Suboptimal MPC algorithm)**.**

Data: System $x^+ = f(x, u)$ and constraints $\mathbb{X}_f, \mathbb{U}$ and constants $a > 0$, $\bar{V} > 0$ and $\beta = \max(1, \bar{V}/a)$ satisfying Assumptions 1–5. Initial state $x(0)$, feasible input $\tilde{\mathbf{u}}(0) \in \mathcal{U}_N^\beta(x(0))$, optimizer $\mathbb{O}$ satisfying Definition 3 and $p > 0$.

Result: Asymptotically stable closed loop.

Set $k \leftarrow 0$; **while** $k > 0$ **do**

Do p iterations using optimizer $\mathbb{O}$ on the suboptimal MPC problem $\mathbb{P}_N^\beta(x(k))$ starting from $(x(k), \tilde{\mathbf{u}}(k))$ to obtain $\mathbf{u}(k)$ as the "suboptimal" input sequence.

Implement the first input, $\kappa(x(k))$ to move to $x(k+1) = f(x(k), \kappa(x(k)))$

Set "warm start" $\tilde{\mathbf{u}}(k+1) = (\mathbf{u}(1; x(k)), \mathbf{u}(2; x(k)), \ldots, \kappa_f(x(k+N)))$. Set $k \leftarrow k+1$

end

In Algorithm 4, $x(k+N) = \phi(k+N; x(k); \mathbf{u}(k))$.

Since optimizer $\mathbb{O}$ satisfies the requirements in Definition 3, we know that $\mathbf{u}(0) \in \mathcal{U}_N^\beta(x(0))$ and therefore, $\phi(N; x(0), \mathbf{u}(0)) \in \mathbb{X}_f$ as $\beta = \bar{\beta}$. Since we choose terminal function $V_f(\cdot)$ and terminal controller $\kappa_f(\cdot)$ satisfying Assumption 3 and, we know that warm start $\tilde{\mathbf{u}}(1) \in \mathcal{U}_N^\beta(x(1))$. Since every iterate generated by the optimization algorithm $\mathbb{O}$ is feasible and decreases the objective function value, we conclude that $\mathbf{u}(1) \in G(x(0), \mathbf{u}(0))$. By induction, we conclude that $\mathbf{u}(k+1) \in G(x(k), \mathbf{u}(k))$. We now satisfy all the requirements of Theorem 1 to establish that the origin of the closed loop system generated by Algorithm 4 is asymptotically stable.

Cooperative MPC

Consider the system

$$x^+ = Ax + B_1 u_1 + B_2 u_2 \tag{9}$$

subject to the constraint $u_1 \in \mathbb{U}_1, u_2 \in \mathbb{U}_2$. We make the following assumption on the sets $\mathbb{U}_1$ and $\mathbb{U}_2$.

Assumption 6. *The sets $\mathbb{U}_1$ and $\mathbb{U}_2$ are compact, convex and contain the origin in their interiors.*

Assumption 6 implies that the input constraint sets for u_1 and u_2 are uncoupled. Note that the constraint set $\mathbb{U}$ for the combined input $u = (u_1, u_2) \in \mathbb{U}$ with $\mathbb{U} := \mathbb{U}_1 \times \mathbb{U}_2$ is also convex, compact and contains the origin in its interior as required by Assumption 2.

We choose positive definite matrices Q, R_1, R_2, P and terminal region $\mathbb{X}_f$ so that the stage cost $\ell(x,u) = x'Qx + u_1'R_1u_1 + u_2'R_2u_2$, and the terminal cost $V_f(x) = x'Px$ satisfy Assumptions 1–5.

We modify the MPC problem $\mathbb{P}$ in(3) and the corresponding subsystem problem $\mathbb{P}_i^c$ in (5) by adding the constraint that $\mathbf{u}_i \leq d_i|x|$, if $x \in \mathcal{B}_r, i = \{1,2\}$, in which d_1, d_2, r are chosen according to Proposition 3.

Proposition 2. *For $Q, R_1, R_2, P > 0$, there exists positive constants a_1', a_2', a_f, a_3', a such that:*

$$(x, u_1, u_2) \geq a_1'|(x, u_1, u_2)|^a, (x, u_1, u_2) \in \mathbb{X} \times \mathbb{U}_1 \times \mathbb{U}_2$$
$$V_N^\beta(x, \mathbf{u}_1, \mathbf{u}_2) \leq a_2'|(x, \mathbf{u}_1, \mathbf{u}_2)|^a, (x, \mathbf{u}_1, \mathbf{u}_2) \in \mathbb{X} \times \mathbb{U}_1^N \times \mathbb{U}_2^N$$

and

$$a_3'|x|^a \leq V_f(x) \leq a_f|x|^a, x \in \mathbb{X}$$

Proof. Since $Q, R_1, R_2 > 0$, the eigenvalues of Q, R_1, R_2 are real and positive. Let $\underline{\lambda}(Q) = \min(\text{eig}(Q)) > 0$. Then, we have that $x'Qx \geq \underline{\lambda}(Q)|x|^2$. Therefore, $\ell(x, u_1, u_2) \geq (1/2)(\underline{\lambda}(Q)|x|^2 + \underline{\lambda}(R_1)|u_1|^2 + \underline{\lambda}(R_2)|u_2|^2)$. Choose $a_1' = (1/2)\min(\underline{\lambda}(Q), \underline{\lambda}(R_1), \underline{\lambda}(R_2))$ and $a = 2$. Denoting $u = (u_1, u_2)$, $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and $R = \text{diag}(R_1, R_2)$, the MPC cost function can be written as

$$V_N^\beta(x, \mathbf{u}) = \frac{1}{2}\begin{bmatrix} x \\ \mathbf{u} \end{bmatrix}' \mathcal{H} \begin{bmatrix} x \\ \mathbf{u} \end{bmatrix} \qquad \mathcal{H} = \begin{bmatrix} \mathcal{A}'\mathcal{Q}\mathcal{A} & \mathcal{A}'\mathcal{Q}\mathcal{B} \\ \mathcal{B}'\mathcal{Q}\mathcal{A} & \mathcal{B}\mathcal{Q}\mathcal{B} + \mathcal{R} \end{bmatrix} \tag{10}$$

in which

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ A \\ \vdots \\ A^{N-1} \end{bmatrix}}_{\mathcal{A}} x + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{\mathcal{B}} \mathbf{u}$$

and $\mathcal{Q} = \text{diag}(\underbrace{Q, Q, \dots, Q}_{N-1 \text{ times}}, \beta P)$ and $\mathcal{R} = \text{diag}(\underbrace{R, R, \dots, R}_{N \text{ times}})$. Note that $\mathcal{H}$ is a positive definite matrix. Hence, we obtain $V_N(x, \mathbf{u}) \leq \overline{\lambda}(\mathcal{H})|(x, \mathbf{u})|^2$ in which $\overline{\lambda}(\mathcal{H}) = \max(\text{eig}(\mathcal{H}))$. Choose $a_2' = \overline{\lambda}(\mathcal{H})$. Since $P > 0$, we can write $(1/2)\underline{\lambda}(P)|x|^2 \leq V_f(x) \leq \overline{\lambda}(P)|x|^2$, We choose $a_3' = (1/2)\underline{\lambda}(P)$ and $a_f = \overline{\lambda}(P)$. □

Proposition 3. *For positive definite Q, R_1, R_2, P, choose a_2', a_3', a according to Proposition 2. Then for any $r > 0$ and $\beta \geq 1$, there exist positive constants d_1, d_2 such that*

$$\mathbf{u}_1 \leq d_1|x|, \mathbf{u}_2 \leq d_2|x| \qquad x \in \mathcal{B}_r$$

implies that:

$$V_N^\beta(x, \mathbf{u}_1, \mathbf{u}_2) \leq \beta V_f(x) \qquad x \in \mathcal{B}_r$$

Proof. In the region $x \in \mathcal{B}_r$, notice that $\mathbf{u}_1 \leq d_1|x|$ and $\mathbf{u}_2 \leq d_2|x|$, implies $|(x, \mathbf{u}_1, \mathbf{u}_2)| \leq |x| + |\mathbf{u}_1| + |\mathbf{u}_2| \leq (1 + d_1 + d_2)|x|$. Therefore from Proposition 2, $V_N^\beta(x, \mathbf{u}_1, \mathbf{u}_2) \leq a_2'|(x, \mathbf{u}_1, \mathbf{u}_2)|^a \leq a_2'(1 + d_1 + d_2)^a|x|^a$. Choose $d_1, d_2 > 0$ such that

$$a_2'(1 + d_1 + d_2)^a \leq \beta a_3' \tag{11}$$

From Proposition 2, we know $a_3'|x|^a \leq V_f(x)$. Hence, for choices of d_1, d_2 satisfying (11), we conclude that for $x \in \mathcal{B}_r$,

$$V_N^\beta(x, \mathbf{u}_1, \mathbf{u}_2) \leq a_2'(1 + d_1 + d_2)|x|^a \leq \beta a_3'|x|^a \leq \beta V_f(x)$$

□

Proposition 4. *Choose set $\mathbb{X}_f$, positive definite $Q, R_1, R_2, P > 0$ and $a > 0$ such that Assumptions 3 – 5 hold. Let $(x, \mathbf{u})$ satisfy $V_N^\beta(x, \mathbf{u}) \leq \bar{V}$ in which $\beta \geq \max(1, \bar{V}/a), \bar{V} > 0$. Then, for the warm start $\tilde{\mathbf{u}}$ given in Definition 1 and the successor state $x^+ = Ax + B\kappa(x)$, we have that*

$$V_N(x^+, \tilde{\mathbf{u}}) \leq \bar{V}$$

Proof. Observe that

$$\begin{aligned} V_N(x^+, \tilde{\mathbf{u}}) = V_N(x, \mathbf{u}) - \ell(x, u(0; x)) - \beta V_f(x(N) \\ + \beta V_f(\phi(1; x(N), \kappa(x(N)))) + \ell(x(N), \kappa(x(N)))) \end{aligned}$$

Since $\beta \geq \bar{\beta}$, from Proposition 1 we know that $x(N) = \phi(N; x, \mathbf{u}) \in \mathbb{X}_f$. Therefore, from Assumptions 3 and 4, we conclude

$$V_N(x^+, \tilde{\mathbf{u}}) \leq V_N(x, \mathbf{u})$$

and the result is established. □

Proposition 5. *Let $V(\cdot,\cdot)$ be a convex function and sets Ω_1 and Ω_2 be convex, compact sets containing the origin in their interiors. Consider the optimization problem $\mathbb{J}$*

$$\mathbb{J}: \qquad \min_{y_1, y_2} V(y_1, y_2), \ \ s.t\ y_1 \in \Omega_1, y_2 \in \Omega_2$$

and its corresponding "cooperative" subsystem problems:

$$\mathbb{J}_1^c(v_2): \qquad \min_{y_1} V(y_1, v_2), \ \ s.t\ y_1 \in \Omega_1$$

and

$$\mathbb{J}_2^c(v_1): \qquad \min_{y_2} V(v_1, y_2), \;\; s.t\; y_2 \in \Omega_2$$

Algorithm 1 to problems $\mathbb{J}_1^c$ and $\mathbb{J}_2^c$ starting from a feasible initial point $(y_1^{(0)}, y_2^{(0)})$ generates iterates $y_i^{(p)}, p > 0$ satisfying $V(y_1^{(p)}, y_2^{(p)}) \leq V(y_1^{(0)}, y_2^{(0)})$, $y_i^{(p)} \in \Omega_i$.

Proof. Notice that problems $\mathbb{J}_1^c(y_2^{(0)})$ and $\mathbb{J}_2^c(y_1^{(0)})$ are feasible. Therefore, solutions $y_1^o(y_2^{(0)})$ and $y_2^o(y_1^{(0)})$ exist and satisfy $V(y_1^o(y_2^{(0)}), y_2^{(0)}) \leq V(y_1^{(0)}, y_2^{(0)})$ and $V(y_1^{(0)}, y_2^o(y_1^{(0)})) \leq V(y_1^{(0)}, y_2^{(0)})$

By convexity of $V(\cdot,\cdot)$,

$$\begin{aligned} V(y_1^{(1)}, y_2^{(1)}) &= V(\omega(y_1^o(y_2^{(0)}), y_2^{(0)}) + (1-\omega)(y_1^{(0)}, y_2^o(y_1^{(0)}))) \\ &\leq \omega V(y_1^o(y_2^{(0)}), y_2^{(0)}) + (1-\omega) V(y_1^{(0)}, y_2^o(y_1^{(0)})) \\ &\leq V(y_1^{(0)}, y_2^{(0)}) \end{aligned}$$

Since $y_1^{(0)}, y_1^o(y_2^{(0)}) \in \Omega_1$, by convexity of Ω_1,

$$y_1^{(1)} = \omega y_1^o(y_2^{(0)}) + (1-\omega) y_1^{(0)} \in \Omega_1$$

Similarly, $y_2^{(1)} \in \Omega_2$.

By induction, we can extend the result to any $p \geq 1$. $\square$

Theorem 2. *Consider the linear system* (9)*, and MPC cost function $V_N^{\beta}(x, \mathbf{u}_1, \mathbf{u}_2)$ with $\ell(x,u) = x'Qx + u_1'R_1u_1 + u_2'R_2u_2, V_f(x) = x'Px$, with $Q, R_1, R_2, P > 0$. Let Assumptions 1–6 hold. Choose $\bar{V} > 0$ and $\beta = \bar{\beta}$. For an initial state $x(0) \in \mathcal{X}_N^{\beta}$, choose input sequence $(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) \in \mathcal{U}_N^{\beta}(x)$ and implement cooperative MPC algorithm 3 for the centralized MPC problem* (3) *using the subsystem MPC problems $\mathbb{P}_i^C$* (5)*. The origin of the resulting closed-loop system is asymptotically stable on (arbitrarily large) compact subsets of $\mathcal{X}_N^{\beta}$.*

Proof. From 10, we know that V_N^{β} is a convex function since $\mathcal{H}$ is positive definite. Therefore, problem $\mathbb{P}(x)$ is a convex optimization problem with uncoupled constraints for $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence from Proposition 5, we know that Algorithm 1 satisfies all the requirements of Definition 3. Since $(\tilde{\mathbf{u}}_1(0), \tilde{\mathbf{u}}_2(0)) \in \mathcal{U}_N^{\beta}(x(0))$, we use Propositions 5 and 3 to establish that $(\mathbf{u}_1(0), \mathbf{u}_2(0)) \in \mathcal{U}_N^{\beta}(x(0))$. From Proposition 4, we conclude that the warm start $(\tilde{\mathbf{u}}_1(1), \tilde{\mathbf{u}}_2(1))$ implies $V_N^{\beta}(x(1), \tilde{\mathbf{u}}_1(1), \tilde{\mathbf{u}}_2(1)) \leq \bar{V}$. Hence $(\tilde{\mathbf{u}}_1(1), \tilde{\mathbf{u}}_2(1)) \in \mathcal{U}_N^{\beta}(x(1))$. Therefore, Proposition 5 implies

$$(\mathbf{u}_1(1), \mathbf{u}_2(2)) \in G(x(0), \mathbf{u}_1(0), \mathbf{u}_2(0))$$

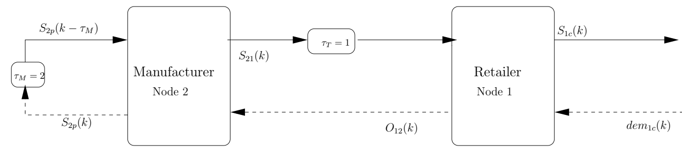

Figure 5: Two-stage supply chain.

By induction, we can extend the result to any $k \geq 0$, that is

$$(\mathbf{u}_1(k+1), \mathbf{u}_2(k+1)) \in G(x(k), \mathbf{u}_1(k), \mathbf{u}_2(k))$$

and we have established closed loop asymptotic stability using Theorem 1. $\square$

For a stabilizable pair (A, B), in which $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, stage cost $\ell(x,u) = x'Qx + u'Ru, Q, R > 0$ and controller gain K such that $A_K = (A + BK)$ is stable, we know that a solution $P > 0$ to the Lyapunov equation

$$A_K' P A_K + (Q + K'RK) = P$$

exists. Such a (P, K) is a convenient choice for terminal penalty and terminal controller in MPC. Therefore, we can implement cooperative MPC for any system that can be stabilized by centralized MPC.

6 Supply chain example

We simulate the supply chain shown in Figure 5 in this section. The plant has production delay of 2 time units and a transportation delay of 1 time unit. We assume that the manufacturer can start a batch of the product at every time instant. We label the retailer node 1, with the states Iv_1 and BO_1, the inventory and backorder at the retailer. The retailer inputs u_1 consist of orders placed and the shipments made by the retailer, S_{1c} and O_{12}. We label the manufacturer node 2, with states x_2 consisting of inventory Iv_2 and backorder BO_2. The manufacturer inputs are the shipments made to the retailer S_{21} and the production S_{2p}. The demand $d(k) = \mathrm{Dm}_{1c}$.

Models. We write a time invariant model for the supply chain that is also the process model (because we assume that a batch may start at every time) by writing the inventory and backorder balance equation. These models are

$$x_1(k+1) = A_1 x_1(k) + B_{11} u_1(k) + B_{12}^{(1)} u_2(k-1) + B_d d(k)$$
$$x_2(k+1) = A_2 x_2(k) + B_{22} u_2(k) + B_{22}^{(2)} u_1(k-2) + B_{21} u_1(k)$$

Stage cost. Each node (subsystem) has a local stage cost, given by

$$\ell_1(x_1, u_1) = |x_1|^2_{Q_1} + |u_1|^2_{R_1}, \quad \ell_2(x_2, u_2) = |x_2|^2_{Q_2} + |u_2|^2_{R_2}$$

The overall stage cost is $\ell(x, u) = \ell_1(x_1, u_1) + \ell_2(x_2, u_2)$.

Terminal cost. For centralized and cooperative MPC, following the theory outlined in Section 5, we chose the $P > 0, a > 0$ such that there exists a stabilizing control law $\kappa_f(x)$ in the terminal region given by:

$$\mathbb{X}_f = \{x \mid x'Px \leq a\}$$

We also choose a $\bar{V} > 0$ and fix $\beta = \max(1, \bar{V}/a)$. The positive definite matrix P is of the form $\begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}$. We choose the local terminal cost functions and the centralized terminal cost function as

$$V_f^1(x_1) = |x_1|^2_{P_{11}} \qquad V_f^2(x_2) = |x_2|^2_{P_{22}} \qquad V_f(x) = |x|^2_P$$

We now define the MPC cost functions. The subsystem cost functions are for $i \in \{1, 2\}$

$$V_N^{i,\beta}(x_i(0), \mathbf{u}_i) = \sum_{j=0}^{N-1} \ell_i(x_i(j), u_i(j)) + \beta V_f^i(x_i(N))$$

while the overall cost function is:

$$V_N^{\beta}(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + \beta V_f(x(N))$$

Note that since we defined the terminal costs differently for the subsystems, the overall cost function is not the sum of the subsystem cost functions. Associated with each input, we also have the input constraint set $\mathbb{U}_1$ and $\mathbb{U}_2$, which contain the minimum and maximum shipments and orders that can flow through the supply chain.

MPC implementation

Ordering policies. As mentioned in Section 3, the local retailer model does not have knowledge of how the orders placed by the retailer affects the supply chain. Therefore, in the implementation of noncooperative and decentralized MPC, we need to incorporate an ordering policy for the retailer. Since the manufacturer reacts to the orders placed by the retailer, the closed-loop performance of the supply chain is intimately connected to the ordering policy. We study two ordering policies in this paper:

1. Order-up-to policy: The order-up-to policy can be viewed as a saturated proportional controller.

$$O_{12}(k) = \begin{cases} \mathrm{Iv}_t - \mathrm{Iv}_1(k) & \text{if } \mathrm{Iv}_1 \leq \mathrm{Iv}_t \\ 0 & \text{otherwise} \end{cases}$$

   in which Iv_t is the inventory target.

2. Inventory position control: In inventory position control, the retailer, instead of controlling the inventory, controls the inventory position, which is a controlled output defined as:

$$\mathrm{Ip}(k) = \mathrm{Iv}_1(k) - S_{1c}(k) + O_{12}(k)$$

   Inventory position control introduces a new controlled output that is a function of the state and input. We penalize the deviations of Ip from the inventory target Iv_t in the optimizations.

Distributed MPC. In decentralized and noncooperative MPC with order-up-to policy, we modify the retailer subproblem, subsystem-1 in (4), by adding a constraint that enforces the order-up-to policy. Similarly, for decentralized and noncooperative MPC with inventory position control, we modify the retailer objective function in the subsystem-1 problem in (4) by modifying the stage cost to penalize inventory position Ip.

Decentralized and noncooperative MPC are implemented using Algorithm 2. In decentralized MPC, the subsystems do not share information.

We design the subproblems for cooperative MPC (5) using the methodology outlined in Section 5 for closed-loop stability. Cooperative MPC is implemented using Algorithm 3. In centralized MPC, we solve the overall problem $\mathbb{P}(x)$ given in (3).

Results and Discussion

We present the results of the different MPC implementations for a nominal demand of $d = 8$. In each of the simulations, the retailer starts with inventory $I_1 = 45$ and the manufacturer starts with inventory $I_2 = 30$. The control objective is to keep the inventories in the nodes as close to the starting inventory as possible while maintaining minimum backorder.

Figure 6 compares the results of centralized, cooperative, noncooperative and decentralized MPC in which we used the order-up-to ordering policy. Figure 7 compares the results of same controllers, but using the inventory control policy.

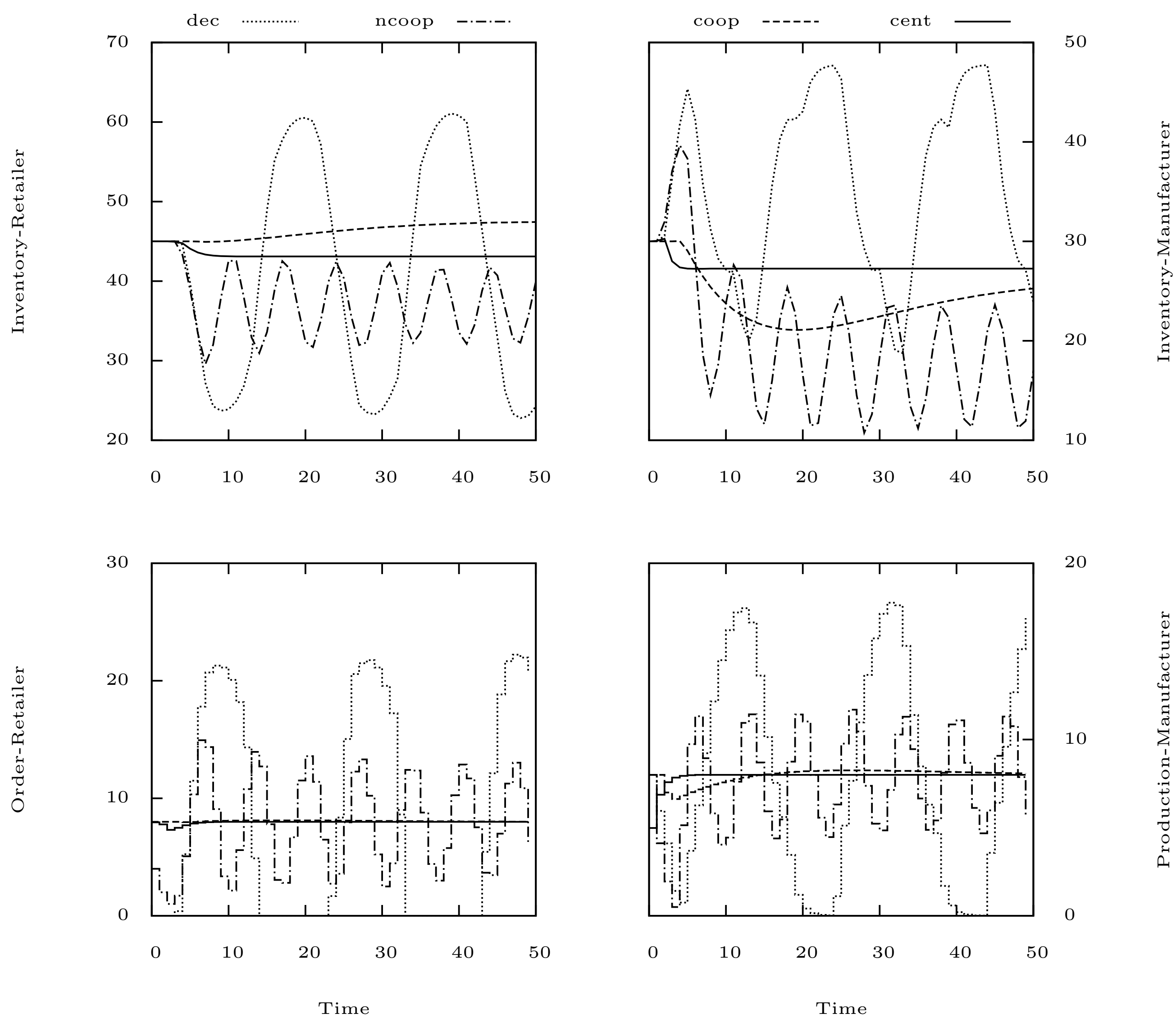

Figure 6: Inventories and orders placed in the supply chain: Order-up-to policy (dec: decentralized, ncoop: noncooperative, coop: cooperative, cent: centralized).

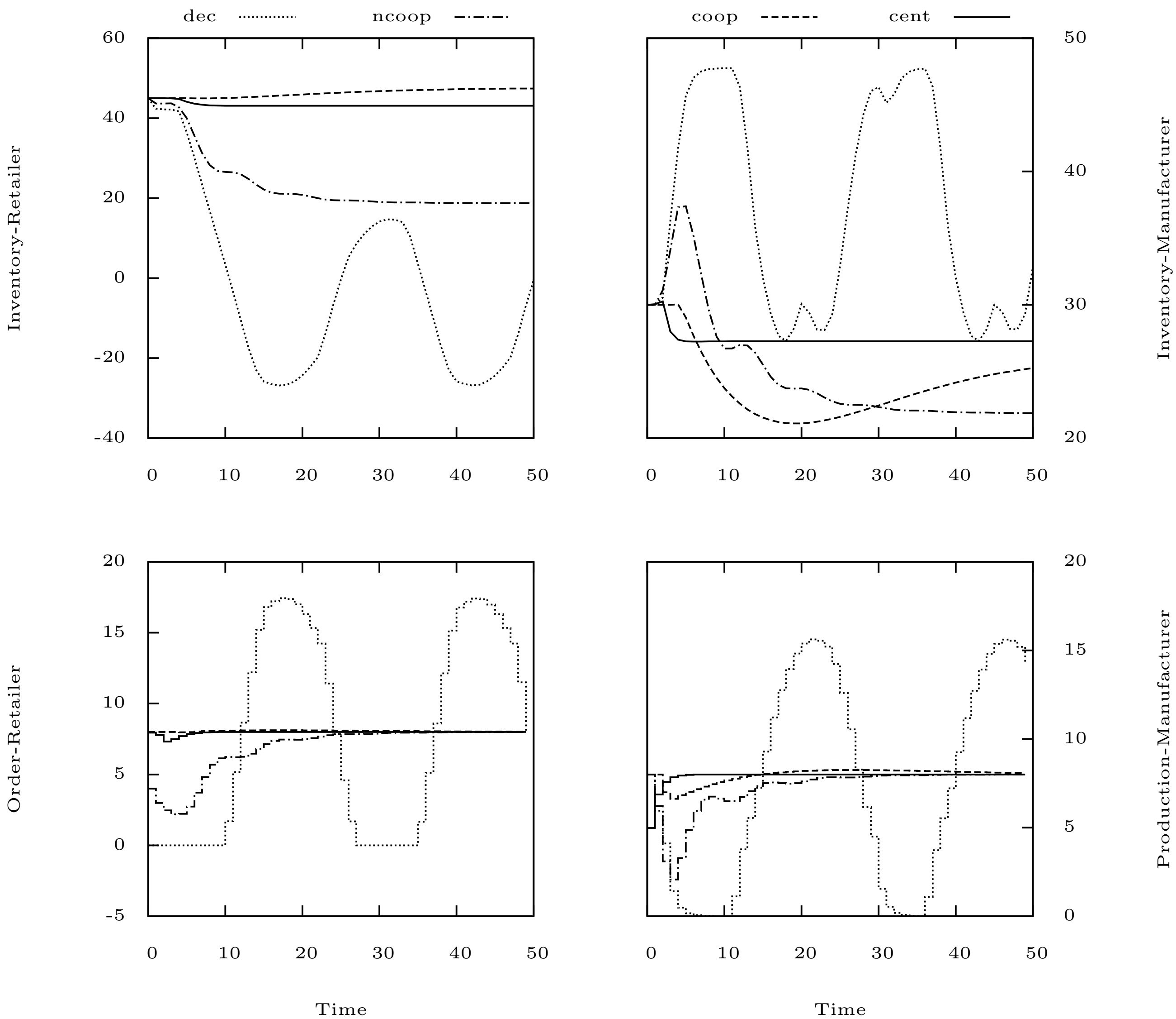

Figure 7: Inventories and orders placed in the supply chain: Inventory position control (dec: decentralized, ncoop: noncooperative, coop: cooperative, cent: centralized).

Value of information. We observe that, for both order-up-to and inventory position control, decentralized MPC produces large variations in the inventory and orders. These variations indicate a large bullwhip effect, and happen because the nodes have incomplete current information and no knowledge of the dynamics of the other nodes. At each time step, the retailer assumes some flow of materials from the manufacturer to make inventory predictions. Based on these predictions, the retailer places orders with the manufacturer. Similarly, the manufacturer knows only the current order quantity and makes some assumptions about the future orders from the retailer and makes production decisions. When the actual orders and shipments arrive at the nodes, their decisions are suboptimal.

In noncooperative MPC with the order-up-to policy, we see that the magnitude of inventory, order quantity and production quantity has drastically reduced. Since, each node now has more information about the ordering and production plans of the other node, both are able to make better forecasts and therefore, better decisions. In fact, with this extra information, noncooperative MPC based on inventory position control is able to reach a steady state.

Impact of Ordering policy. In noncooperative MPC with inventory position control, we observe that there are no inventory variations and the system reaches a steady state. All flows through the system settle at the nominal demand, which is the input steady state. The inventories, however, show offset from the target. In order-up-to policy, irrespective of the cost of placing large orders, the retailer is constrained to make orders if the inventory at any period falls below the target. In inventory position control, the orders placed are penalized, and therefore the retailer tends to order less, because the optimizer tries to balance ordering costs and inventory deviation costs.

Plant-Model mismatch. If we compare results for cooperative and centralized MPC with noncooperative MPC, we see that, cooperative and centralized MPC reach steady state more quickly. They achieve steady state because there is no information distortion in the system. Each node in cooperative control, optimizes not only the system-wide objective, it also accounts for the dynamics of the entire supply chain. In noncooperative MPC with inventory position control, since the retailer does not know the actual supply chain dynamics, it settles at a steady state that depends on the inventory position model. Therefore, we see the value of optimizing the actual dynamics instead of introducing a mismatch between the models used by the controller and the actual dynamics by using inventory position models.

Guaranteed stability. The third important result of the analysis is that cooperative and centralized MPC have been designed to guarantee closed-loop stability. Although, we see that noncooperative MPC using inventory position control has not made the supply chain unstable, we have no stability guarantees. On the other hand, using the theory developed in Section 5, we can guarantee closed-loop stability for cooperative MPC.

7 Conclusions

Although rolling horizon optimization for supply chains has been proposed many times, the closed loop performance of such implementations has not been previously discussed. Supply chain problems are natural problems for distributed implementations because various nodes may be owned by different companies or may be spread geographically across various locations. To our knowledge, although different distributed approaches to supply chain decision making have been proposed, no distributed MPC approach with guaranteed closed-loop nominal stability properties has been previously proposed. In this paper, we developed new theory for closed-loop stable cooperative MPC. With this new theory, we can ensure that cooperative MPC can stabilize any system which can be stabilized by centralized MPC. We used this new theory to implement closed-loop, stabilizing cooperative MPC for supply chains.

Future avenues for research in this area include the following:

- Integration of scheduling with control. We considered here an approximate model of the production facility. In the future, we would like to solve hybrid MPC problems with an exact production scheduling model included as one of the nodes. We are presently studying terminal constraints in rolling-horizon scheduling models. We plan to obtain these constraints by solving scheduling problems with cyclic or periodic constraints. Establishing closed-loop stability of schedules implemented in a rolling horizon framework, its integration with the supply chain and cooperative MPC are avenues for future research work.

- Economic distributed MPC. The control objective in this paper was to minimize deviations from a target inventory level. In practice, supply chain managers seek to optimize economic benefits or performance. We would like to implement economic MPC [1] for supply chains. Developing suboptimal and cooperative economic MPC controllers with guaranteed closed-loop properties is an motivating and open research problem.

- Robust distributed MPC. Another extension of MPC for supply chains would be to implement robust MPC. Theoretically, to implement robust distributed MPC, we need to find a cooperative “restart” point if the warm start becomes infeasible. To be robust to customer demands, we need to incorporate prior knowledge of the demands like seasonal variability, etc. into our problem formulation to ensure stability.

- Cooperative game theory. Much of our work on cooperative MPC has been developed for the process industries, in which we wish to coordinate different MPC's for the different units inside a single plant. Therefore, we avoid analyzing the stability of cooperative control from a cooperative game theory perspective. In a modern supply chain, however, often the different nodes are owned by different companies, and we need to study the incentives for cooperation among the nodes. Studying the stability of cooperative MPC from a cooperative game theory perspective is another avenue of future research.

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