ROBUST COUNTERPART OPTIMIZATION: UNCERTAINTY SETS, FORMULATIONS AND PROBABILISTIC GUARANTEES

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Abstract

Robust counterpart optimization techniques are studied in this paper. Different uncertainty sets, including those studied in literature (i.e., interval set; combined interval and ellipsoidal set; combined interval and polyhedral set) and new ones (i.e., adjustable box; pure ellipsoidal; pure polyhedral; combined interval, ellipsoidal, and polyhedral set) are studied in this work and their geometric relationship is discussed. Robust counterpart optimization formulations induced by those different uncertainty sets are derived. For those robust formulations, their corresponding probability bounds on constraint violation are derived based on the distributional information of the uncertainty (i.e., bounded or unbounded uncertainty, with or without known probability distribution function). The tightness of the different probability bounds and the conservatism of different robust counterpart optimization formulations are illustrated through a case study.

Keywords

Uncertainty Set, Robust Optimization, Probabilistic Guarantee.

Introduction

In many optimization applications, the problem data is assumed to be known with certainty. However, that is seldom the case in practice. Very often, the realistic data are subject to uncertainty due to their random nature, measurement errors or other reasons. Since the solution of an optimization problem often exhibits high sensitivity to the data perturbations as illustrated by Ben-Tal and Nemirovski (2000), ignoring the data uncertainty could lead to solutions which are suboptimal or even infeasible for practical applications.

Robust optimization belongs to an important methodology for dealing with optimization problems with data uncertainty. In this type of method, a deterministic data set is defined within the uncertain space, and the best solution which is feasible for any realization of the data uncertainty in the given set is computed through the solution of the robust counterpart optimization problem. One major motivation for studying robust optimization is that in many applications the data set is an appropriate notion of parameter uncertainty, e.g., for applications in which infeasibility cannot be accepted at all (e.g., design of engineering structures like bridges (Ben-Tal & Nemirovski, 1997)), and for those cases that the parameter uncertainty is not stochastic, or if no distributional information is available.

One of the earliest papers on robust counterpart optimization is related to the work of Soyster (1973), who considered simple perturbations in the data and aimed at finding a reformulation of the original linear programming problem such that the resulting solution would be feasible under all possible perturbations. This approach, however, is the most conservative one since it ensures feasibility against all potential realizations. Thus, it is highly desirable to provide a mechanism to allow tradeoff between robustness and performance. To address the issue of over-conservatism in worst-case models, Ben-Tal and Nemirovski (2000) and El-Ghaoui and co-workers (El-Ghaoui et al., 1998; ElGhaoui & Lebret, 1997) independently proposed the ellipsoidal set based robust counterpart formulation for dealing with parameter uncertainty within linear and quadratic programming problems. ElGhaoui and Lebret (1997) studied the robust solutions to the uncertain least-squares problems, and El-Ghaoui et al. (1998) studied uncertain semidefinite problems. Ben-Tal and Nemirovski (1999) showed that when the uncertainty sets for a linear constraint are ellipsoids, the robust formulation turns out to be a conic quadratic problem.

The robust optimization formulation introduced for linear programming problems with uncertain linear

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coefficients was extended by Lin et al. (2004) and Janak et al. (2007) to mixed integer linear optimization (MILP) problems under uncertainty. They developed the theory of the robust optimization framework for general mixed-integer linear programming problems and considered both bounded and several known probability distributions. The robust optimization framework is later extended by Verderame and Floudas (2009) who studied both continuous (general, bounded, uniform, normal) and discrete (general, binomial, Poisson) uncertainty distributions and applied the framework to operational planning problems. The work was further compared with the conditional-value risk based method in (Verderame & Floudas, 2010).

In this paper, we systematically study the robust counterpart optimization problem. The general idea of robust optimization is introduced first in next section. Several different uncertainty sets are presented, followed by the formulation of the robust counterpart model. Probability bounds on constraint violation are introduced based on the distribution of the uncertainty. Finally, a case study is given to illustrate the robust counterpart optimization and their probabilistic bounds.

Robust Counterpart Optimization

In set induced robust optimization, the uncertain data are assumed to be varying in a given uncertainty set, and the aim is to choose the best solution among those "immunized" against data uncertainty, that is, candidate solutions that remain feasible for all realizations of the data from the uncertainty set.

In general, consider the following linear optimization problem with uncertainty in the left hand side (LHS) constraint coefficients, right hand side (RHS) and objective function coefficients:

$$\begin{aligned} \max \quad & \sum_j \tilde{c}_j x_j \\ \text{s.t.} \quad & \sum_j \tilde{a}_{ij} x_j \le \tilde{b}_i \quad \forall i \end{aligned} \tag{1}$$

where x_j can be either a continuous or an integer variable. Note that the objective and RHS uncertainty can be transformed into LHS uncertainty as follows:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - \sum_j \tilde{c}_j x_j \le 0 \\ & \tilde{b}_i x_0 + \sum_j \tilde{a}_{ij} x_j \le 0 \quad \forall i \\ & x_0 = -1 \end{aligned} \tag{2}$$

So, without loss of generality, we focus on the following general i-th constraint of a (mixed integer) linear optimization problem considering only LHS uncertainty:

$$\sum_j \tilde{a}_{ij} x_j \le b_i \tag{3}$$

and $\tilde{a}_{ij}$ are subject to uncertainty. Define the uncertainty as follows

$$\tilde{a}_{ij} = a_{ij} + \xi_{ij} \hat{a}_{ij} \quad \forall j \in J_i, \tag{4}$$

where a_{ij} represent the nominal value of the parameters, $\hat{a}_{ij}$ represent positive constant perturbations, ξ_{ij} represent independent random variables which are subject to uncertainty and J_i represents the index subset that contains the variables whose coefficients are subject to uncertainty.

Constraint (3) can be rewritten by grouping the deterministic part and the uncertain part for the LHS of (3) as follows:

$$\sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \le b_i \tag{5}$$

In the set induced robust optimization method, the aim is to find solutions that remain feasible for any ξ in the given uncertainty set U so as to immunize against infeasibility, that is,

$$\sum_j a_{ij} x_j + \max_{\xi \in U} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \right\} \le b_i \tag{6}$$

Uncertainty Sets

The formulation of robust counterpart optimization model is connected with the selection of the uncertainty set U. In the sequel, several different uncertainty sets are introduced. For the sake of simplicity, we eliminate the constraint index i in the random vector ξ.

Box Uncertainty Set

$$U_\infty = \left\{ \xi \;\middle|\; \|\xi\|_\infty \le \Psi \right\} = \left\{ \xi \;\middle|\; |\xi_j| \le \Psi, \forall j \in J_i \right\} \tag{7}$$

Ellipsoidal Uncertainty Set

$$U_2 = \{ \xi \mid \|\xi\|_2 \le \Omega \} = \left\{ \xi \;\middle|\; \sum_{j \in J_i} \xi_j^2 \le \Omega^2 \right\} \tag{8}$$

Polyhedral Uncertainty Set

$$U_1 = \left\{ \xi \;\middle|\; \|\xi\|_1 \le \Gamma \right\} = \left\{ \xi \;\middle|\; \sum_{j \in J_i} |\xi_j| \le \Gamma \right\} \tag{9}$$

"Box+Ellipsoidal" Uncertainty Set

$$U_{2\infty} = \left\{ \xi \;\middle|\; \sum_{j \in J_i} \xi_j^2 \le \Omega^2, \;\; |\xi_j| \le \Psi, \; \forall j \in J_i \right\} \tag{10}$$

"Box+Polyhedral" Uncertainty Set

$$U_{1\infty} = \left\{ \xi \;\middle|\; \sum_{j \in J_i} |\xi_j| \le \Gamma, \;\; |\xi_j| \le \Psi, \; \forall j \in J_i \right\} \tag{11}$$

"Box+Ellipsoidal+Polyhedral" Uncertainty Set

$$U_{12\infty} = \left\{ \xi \;\middle|\; \sum_{j \in J_i} |\xi_j| \le \Gamma, \sum_{j \in J_i} \xi_j^2 \le \Omega^2, |\xi_j| \le \Psi, \forall j \in J_i \right\} \tag{12}$$

where Ψ, Ω, Γ are the adjustable parameter controlling the size of the uncertainty sets. Note that the "box+ellipsoidal", "box+polyhedral" and "box+ellipsoidal+polyhedral" uncertainty sets are the intersection between ellipsoid and box, the intersection between the polyhedral and box, the intersection between the ellipsoidal, polyhedral and box set, respectively.

As $\Psi = 1$, the above set (10) defines the intersection between interval and ellipsoid, which is referred as "interval+ellipsoidal" uncertainty set in this paper. This type of uncertainty set is important for bounded uncertainty since it makes no sense to construct an uncertainty set exceeding the bounded uncertain space. For this kind of uncertainty set, when $\Omega = 1$, the ellipsoid is exactly inscribed by the box; when $\Omega = \sqrt{|J_i|}$, the ellipsoid is circumscribed by the box (i.e., the intersection between the box and ellipsoid is exactly the box). Figure 1 illustrates the geometry of this uncertainty set for the case that the dimension of the uncertain parameter space is 2 (i.e., $|J_i| = 2$).

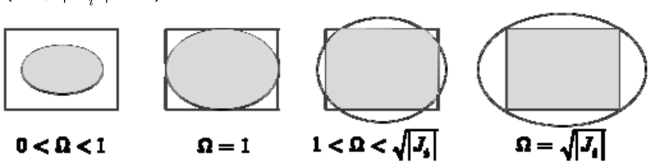

Figure 1. The "interval+ellipsoidal" uncertainty set

As $\Psi = 1$, the above set (11) defines the intersection between the interval and polyhedral set, which is referred as "interval+polyhedral" uncertainty set. For this uncertainty set, when $\Gamma = 1$, the polyhedron is exactly inscribed by the box and the intersection between the polyhedron and the box is exactly the polyhedron; when $\Gamma = |J_i|$, the intersection between the polyhedron and the box is exactly the box, as shown in Figure 2.

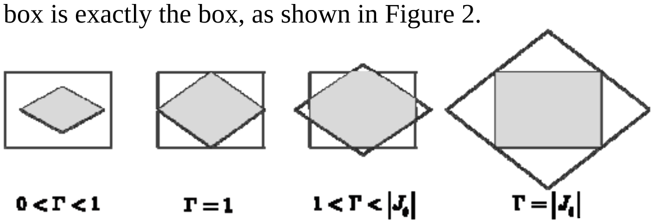

Figure 2. The "interval + ellipsoidal" uncertainty set

Robust Counterpart Optimization Formulations

For constraint (5), its robust counterpart optimization formulation (6) is derived for different uncertainty sets introduced above as follows.

Property 1 If the set U is the box uncertainty set (7), then the corresponding robust counterpart constraint (6) is equivalent to the following constraints:

$$\begin{cases} \sum_j a_{ij} x_j + \Psi \sum_{j \in J_i} \hat{a}_{ij} u_j \le b_i \\ -u_j \le x_j \le u_j \end{cases} \quad (13)$$

Property 2 If the set U is the ellipsoidal uncertainty set (8), then the corresponding robust counterpart constraint (6) is equivalent to the following constraint:

$$\sum_j a_{ij} x_j + \left[\Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2} \right] \le b_i \quad (14)$$

Property 3 If the set U is the polyhedral uncertainty set (9), then the corresponding robust counterpart constraint (6) is equivalent to the following constraints

$$\begin{cases} \sum_j a_{ij} x_j + \Gamma p_i \le b_i \\ p_i \ge \hat{a}_i u_j, \forall j \in J_i \\ -u_j \le x_j \le u_j, \forall j \in J_i \end{cases} \quad (15)$$

Property 4 If the set U is the "box+ellipsoidal" uncertainty set (10), then the corresponding robust counterpart constraint (6) is equivalent to the following constraints:

$$\begin{cases} \sum_j a_{ij} x_j + \Psi \sum_{j \in J_i} \hat{a}_{ij} u_{ij} + \Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_{ij}^2} \le b_i \\ -u_{ij} \le x_j - z_{ij} \le u_{ij} \end{cases} \quad (16)$$

Notice that when $\Psi = 1$ (i.e., the set U is defined as "interval+ellipsoidal" uncertainty set), the corresponding "interval+ellipsoidal" based robust counterpart optimization formulation reduces to the robust counterpart formulation proposed by Ben-Tal and Nemirovski (2000) (i.e., a special case of the combined adjustable box and adjustable ellipsoidal based robust counterpart).

Property 5 If the set U is the "box+polyhedral" uncertainty set (11), then the corresponding robust counterpart constraint (6) is equivalent to the following constraints:

$$\begin{cases} \sum_j a_{ij} x_j + \Psi \sum_{j \in J_i} w_{ij} + \Gamma z_i \le b_i \\ z_i + w_{ij} \ge \hat{a}_{ij} u_j, \quad \forall j \in J_i \\ -u_j \le x_j \le u_j, \quad \forall j \in J_i \\ z_i \ge 0, w_{ij} \ge 0 \end{cases} \quad (17)$$

Notice that when $\Psi = 1$ (i.e., the set U is defined as the "interval+polyhedral" uncertainty set), the corresponding robust counterpart optimization formulation reduces to the robust counterpart proposed by Bertsimas and Sim (2004).

Property 6 If the set U is the "interval+ellipsoidal+polyhedral" uncertainty set (12), then the corresponding robust counterpart constraint (6) is equivalent to the following constraints:

$$\begin{cases} \sum_j a_{ij} x_j + \left[\sum_{j \in J_i} |p_{ij}| + \Omega \sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma z_i \right] \le b_i \\ z_i \ge |\hat{a}_{ij} x_j - p_{ij} - w_{ij}| \quad \forall j \in J_i \end{cases} \quad (18)$$

For detailed proofs for the above properties, the reader is directed to the paper (Li et al., 2011).

Probabilistic Guarantee

In the uncertainty set induced robust counterpart optimization framework, the uncertainty set is defined by the decision maker. If the uncertainty set covers the whole uncertain space containing all the possible realizations of uncertain parameters, then it is sure that the robust solution (if it exists) is feasible for any realizations of uncertainty (i.e. the probabilistic guarantee on constraint satisfaction is 1). However, in reality, the uncertainty set is not necessarily defined to cover the whole uncertain space because the decision maker might allow for a certain degree of constraint violation. For instance, it is impossible to define a finite set to cover unbounded uncertainty space.

In those cases where the uncertainty set does not cover the whole uncertainty space, the following question naturally arises: Before we solve a robust optimization problem, what size of the uncertainty set is necessary to ensure that the degree of constraint violation does not exceed a certain level? Upon solution of the robust optimization problem, what is the degree of constraint violation? The answers to those questions are related to the probabilistic guarantee on the constraint satisfaction, or the upper bound on the probability of constraint violation.

In general, two different types of methodologies can be used in evaluating the probabilistic guarantees. The first type of methods derives the probability using the uncertainty set information before we solve the problem (or in other words, from the robust counterpart constraint since the robust counterpart is derived from the uncertainty set) and the bound is called a priori probability bound. The second method derives the probability directly from the robust counterpart optimization solution, which can also be viewed as checking the probability of constraint violation, and the bound is also called a posteriori bound. For both methodologies, different probability bounds can be derived with different levels of uncertainty information.

For constraint (3), we define the probability of constraint violation as

$$P^{vio} = \Pr\left\{\sum_j a_{ij}x_j + \sum_{j\in J_i}\xi_j \hat{a}_{ij}x_j > b_i\right\} \quad (19)$$

In this paper, we introduce the following probability bounds for robust counterpart constraint (13)-(17) without giving the proof, which can be referred from (Li & Floudas, 2011). Specifically, we study the case $\Psi = 1$ for robust counterpart optimization formulation (16) and (17).

Lemma 1 For every robust counterpart constraint (13)-(17), if it is satisfied, then the following relationship, which represents an upper bound on the probability that the original constraint is violated, holds:

$$P^{vio} \le \Pr\left\{\sum_{j\in J_i}\xi_j\delta_j > \Delta\right\} \quad (20)$$

where the parameter Δ and δ are defined as follows:

1) For box based robust counterpart (13)

$$\Delta = \Psi,\ \left|\delta_j\right| \le 1,\quad \forall j \in J_i,\ \sum_{j\in J_i}\delta_j \le 1 \quad (21)$$

2) For ellipsoid based robust counterpart (14)

$$\Delta = \Omega,\ \left|\delta_j\right| \le 1,\quad \forall j \in J_i,\ \sum_{j\in J_i}\delta_j^2 = 1 \quad (22)$$

3) For polyhedron based robust counterpart (15),

$$\Delta = \Gamma,\ \left|\delta_j\right| \le 1,\quad \forall j \in J_i \quad (23)$$

4) For "interval+ellipsoidal" based robust counterpart (16) with $\Psi = 1$,

$$\Delta = \Omega,\ \left|\delta_j\right| \le 1,\quad \forall j \in J_i,\ \sum_{j\in J_i}\delta_j^2 = 1 \quad (24)$$

5) For "interval+polyhedral" based robust counterpart (17) with $\Psi = 1$,

$$\Delta = \Gamma,\ 0 \le \delta_j \le 1,\quad \forall j \in J_i \quad (25)$$

Theorem 1 If $\{\xi_j\}_{j\in J_i}$ are independent and subject to a bounded and symmetric probability distribution supported on [-1,1]. Then:

1) for the box, ellipsoidal, and "interval+ellipsoidal" uncertainty sets induced robust counterparts, we have the following bound B1:

$$P^{vio} \le \exp\left(-\Delta^2/2\right) \quad (26)$$

2) for the box, ellipsoidal, polyhedral, "interval+ellipsoidal" and "interval+polyhedral" uncertainty sets induced robust counterparts, we have the following bound B2:

$$P^{vio} \le \exp\left(-\Delta^2/2\left|J_i\right|\right). \quad (27)$$

where Δ represents the adjustable parameter for the different uncertainty sets as detailed in Lemma 1.

A tighter bound $B(\left|J_i\right|,\Delta)$ proposed by Bertsimas and Sim (2004) for the "interval+polyhedral" based robust counterpart is also valid for all the five robust counterparts since it is a valid upper bound for $\Pr\left\{\sum_{j\in J_i}\xi_j\delta_j \ge \Delta\right\}$ and the condition in (25) is satisfied by the conditions in (21)-(24), thus we have the following bound B3 for the five robust counterparts:

$$P^{vio} \le B(\left|J_i\right|,\Delta) \quad (28)$$

Theorem 2 If $\{\xi_j\}_{j\in J_i}$ are independent and subject to symmetric probability distribution, then we have the following bound B4:

$$P^{vio} \leq \exp\left(\min_{\theta>0} - \theta\Delta + \sum_{j\in J_i} \ln E[e^{\theta\xi_j}] \right) \quad (29)$$

where Δ represents the adjustable parameter for the different uncertainty sets as detailed in Lemma 1.

The above bounds are *a priori* probabilistic guarantees derived from uncertainty set information. We can also evaluate the *a posteriori* probabilistic guarantees based on the robust counterpart optimization solutions.

Theorem 3 If the uncertain parameters $\{\xi_j\}_{j\in J_i}$ are independent and subject to a bounded probability distribution supported on [-1,1], and the robust counterpart solution is x^* with $d_i(x^*) > 0$, where $d_i(x) \triangleq b_i - \sum_j a_{ij}x_j - E\{\sum_{j\in J_i} \xi_j \hat{a}_{ij} x_j\}$, then we have the following bound B5:

$$P^{vio}(x^*) \leq \exp\left(-\frac{d_i(x^*)^2}{\sum_{i\in J_i} 2\hat{a}_{ij}^2 x_j^{*2}} \right) \quad (30)$$

Theorem 4 If the uncertain parameters $\{\xi_j\}_{j\in J_i}$ are independent and the robust counterpart solution is x^*, then for any $\theta > 0$, we have the following bound B6:

$$P^{vio}(x^*) \leq \exp\{\min_{\theta>0} \theta(\sum_j a_{ij}x_j^* - b_i) + \sum_{j\in J_i} \ln E[e^{\theta\xi_j \hat{a}_{ij} x_j^*}]\} \quad (31)$$

Case Study

In this section, we compare the solution of the different robust counterpart optimization models and also the tightness of the different probability bounds through the following case study.

Example 1 Consider the following optimization problem

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ s.t. \quad & \tilde{a}_1 x_1 + \tilde{a}_2 x_2 \leq 140 \\ & 6x_1 + 8x_2 \leq 72 \\ & x_1, x_2 \geq 0 \end{aligned}$$

In the above problem, $\tilde{a}_1, \tilde{a}_2$ are uncertain coefficients and they are defined by $\tilde{a}_j = a_j + \hat{a}_j \xi_j$, $j = 1,2$, where $[a_1 \ \ a_2] = [10 \ \ 20]$, $\hat{a}_j = 0.1a_j$, and ξ_1, ξ_2 are independent uncertain parameters.

Five different robust counterpart formulations are solved for different Δ values which satisfy the relationship: $\Gamma = \Psi|J_i|$, $\Omega = \Psi\sqrt{|J_i|}$, and the results are shown in Figure 3. Comparing the three dashed lines which represent the results of the box, ellipsoidal and polyhedral set based solutions, for every Ψ value, the box set based solution is always better (larger for maximization problem in this case) than the ellipsoidal set based solution and the polyhedral set based solution is the worst. Comparing the red and blue solid line representing the "interval+ellipsoidal" and "interval+polyhedral" set based solutions, the red line is always above the blue line until the two lines overlap, this shows that the "interval+ellipsoidal" set induced model is less conservative than the "interval+polyhedral" set induced model from the worst case scenario point of view.

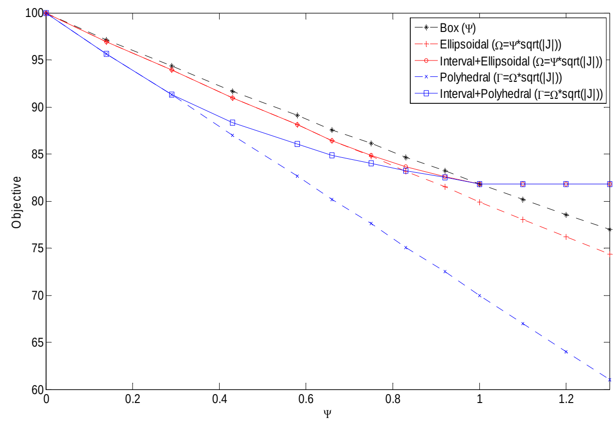

Figure 3. Robust solution comparison

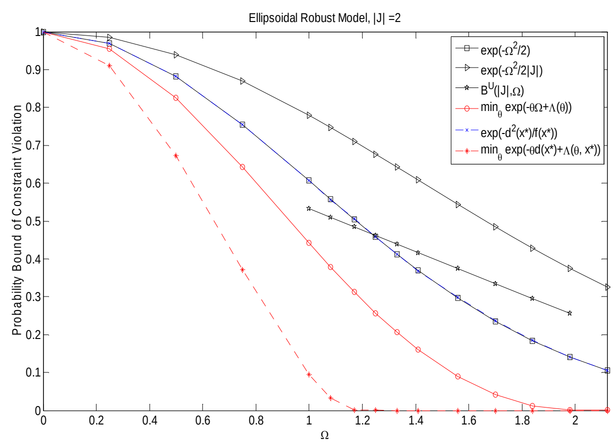

Figure 4. Probability bounds' tightness comparison

Different probability bounds are plotted in Figure 4. Note that B5 and B6 and obtained by first solving the robust optimization problem and then evaluating the probability bounds. Comparing the probability bounds B1, B2, B3, which do not use the probability distribution function or the robust solution, it is seen that for the ellipsoidal and "interval+ellipsoidal" model, bound B1 is the tightest bound. For polyhedral and "interval+polyhedral" model, bound B3 is the tightest bound. Comparing the probability bound B4 and the probability bound B6, although both of them use the probability distribution information of the uncertainty, bound B6 is tighter than B4 since B6 further uses the specific robust counterpart solution to evaluate the probability of constraint violation. Comparing the probability bound B5 and B6, B6 is tighter than B5 since it takes into consideration the detailed probability distribution information of the uncertainty. Furthermore, bound B6 is applicable to not only bounded uncertainties but also unbounded general uncertainties. On the other hand, bound B5 is only valid for bounded uncertainty.

In the process of assigning a parameter value for a specific uncertainty set, the above different probability bounds provide us the capability to select the tightest possible bound expression so as to define the size of the uncertainty set better and to avoid overly conservative solutions.

Summary

Set induced robust counterpart optimization techniques are studied in this paper. Several important uncertainty sets are studied, including those studied in the literature and also several new ones proposed in this work. New uncertainty sets such as the adjustable box, ellipsoidal, polyhedral and "interval + ellipsoidal + polyhedral" set are introduced and their relationship with some well known uncertainty sets presented in the literature is discussed. The relationships between those different uncertainty sets are discussed, and useful insights are gained for their corresponding robust counterpart models. We also derive probability bounds for constraint violation of the robust solution for five different set induced robust counterpart formulations. Probabilistic guarantees are derived for both bounded and unbounded uncertainty, with and without detailed probability distribution information. Numerical study is performed to illustrate the tightness of different probability bounds and the conservatism of different robust formulations. The insights gained provide the basis for the application of the robust counterpart optimization in practical problems.

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