Connection between Model Predictive Control and Anti-Windup Control Schemes

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Abstract

We show that the general Anti-Windup-Bumpless-Transfer (AWBT) controller structure naturally emerges from the structure of Model Predictive Control (MPC) with input constraints and plant model structure that is linear or nonlinear affine in the input variables. The key to establishing that relationship between AWBT control and MPC is a particular interpretation of the maximum principle.

Keywords

Model predictive control, Anti-windup-bumpless-transfer control, Maximum principle, Input saturation, Nonlinear control

Introduction

Controller design for linear or nonlinear processes with actuator saturation nonlinearities has long been studied within various contexts (Kothare, 1997). There are two distinct classes of control structures that handle input saturation nonlinearities: (a) On-line optimization based control structures, such as Model Predictive Control (MPC), and (b) anti-windup bumpless transfer (AWBT) controllers that have a closed form and do not perform on-line optimization. If properly designed, MPC can provide optimality, robustness, and other desirable properties. However, because of the time needed to perform the on-line optimization, MPC is usually implemented on relatively slow processes. On the other hand, AWBT controllers completely bypass on-line optimization; therefore they inherently have lower computational requirements and can be used on faster processes.

The AWBT controller design approach is based on the following two-step design paradigm: Firstly, a linear controller is designed ignoring input constraints. In the next step, an anti-windup scheme is added to compensate for the adverse effects of input constraints on closed-loop performance. Campo (1997) and Kothare et al. (1994) unified all heuristically developed AWBT control schemes into the structure shown in Figure 1, and developed a general framework for studying stability and robustness issues. The importance of that work lies in that model uncertainty can be taken into account systematically and theory exists to analyze, at least in principle, the closed-loop system for stability and robustness. However, that analysis is also based on the standard conic sector nonlinear stability theory. Therefore, the results could be potentially conservative. The design of AWBT controllers for SISO systems relies on a mix of intuitive and rigorous arguments, which become difficult to use in the MIMO case (Peng et al., 1998). As pointed by Doyle et al. (1987), for MIMO controllers, the saturation may cause a change in the plant input direction resulting in disastrous consequences. Through an example, Doyle et al. (1987) showed that all anti-windup schemes of the time failed to work on MIMO systems. Recently, Kothare and Morari (1997) described three performance requirements that should be incorporated in a multi-objective multivariable AWBT controller synthesis framework. Although promising lines for designing an AWBT controller using dynamic output feedback and one-step design were outlined, many of the details like “recovery of linear performance” need to be worked out.

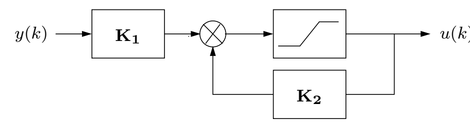

Figure 1: Classical AWBT controller structure.

The MPC design approach naturally and explicitly handles multivariable input and output constraints by directly incorporating them into the on-line optimization problem. The issues of stability and robustness of MPC are now a fairly well understood topic (Rawlings and Muske, 1993; Mayne et al., 2000; Nikolaou, 2000).

For linear plants the MPC problem can be reduced to a quadratic program (QP) which can be solved efficiently (Cutler and Ramaker, 1980; García and Morshedi, 1986). Alternatively, to reduce the computational load, MPC may use a cascaded on-line optimization approach in which a steady-state target is first calculated on-line via linear programming (cost minimization) and then an unconstrained least-squares problem steers the controlled system towards the steady-state optimum (Kassmann et al., 2000; Rao and Rawlings, 1999). It should be stressed that the solution of the least-squares problem (inputs to the controlled system) should satisfy constraints, even though the latter are not explicitly considered in the least-squares problem. Least-squares solution inputs that do not satisfy constraints are sim-

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ply clipped, clearly a non-optimal solution.

MPC for nonlinear plants naturally leads to nonlinear programs (NLP) which are in general non-convex and computationally demanding (Biegler and Rawlings, 1991; Mayne and Michalska, 1991, 1990; Kreshenbaum et al., 1994; Qin and Badgwell, 2000).

Both MPC and AWBT controllers each have their own advantages and disadvantages. However, to the best of our knowledge, no clear relationship between these two control schemes has been established.

In this work, we rigorously show that there is a direct relationship between MPC and AWBT control. In fact, we show that the heuristically proposed AWBT structure of Figure 1 naturally emerges from the structure of MPC with quadratic objective, input constraints, and (linear or nonlinear) plant model structure affine in the input variables. This realization is important for a number of reasons:

- It provides theoretical justification for the heuristically proposed AWBT structure of Figure 1.
- It allows direct substitution of MPC with input constraints by controllers with a closed-form structure, which allows computations to be performed significantly faster. Computational efficiency has been pursued by several other investigators using a variety of different approaches, such as approximation of the on-line optimization (Zheng, 1999), or a priori determination of active constraints (De Dona and Goodwin, 2000; Bemporad et al., 2000).
- It facilitates the design of both MPC and AWBT controllers, because the insight into a controller from either class can be augmented by using insight into an equivalent controller from the other class.
- It allows constrained least squares to be used with computational efficiency in MPC systems that follow a cascaded structure of linear programming followed by (unconstrained) least squares (Kassmann et al., 2000; Rao and Rawlings, 1999).

The proposed approach works for linear models as well as nonlinear models in which the input appears affinely. It also works equally well for SISO and MIMO systems.

From MPC to AWBT

MPC and On-line Optimization

Consider a discrete-time non-linear system in which the input $u(i)$ appears affinely in the right-hand side of the system difference equation, i.e.:

$$\begin{aligned} x(i{+}1) &= f[x(i)] + g[x(i)]u(i) + d(i), \quad x(0) = x_0 \\ y(i) &= h[x(i)] \end{aligned} \tag{1}$$

where $x(i) \in \Re^n$ is the state vector, $u(i) \in \Re^m$ is the control vector, $d(i) \in \Re^n$ is the disturbance vector, $f[x(i)] \in \Re^n$, and $g[x(i)] \in \Re^{n\times m}$. For the linear case the system is

$$\begin{aligned} x(i{+}1) &= \Phi x(i) + \Gamma u(i) + d(i), \quad x(0) = x_0 \\ y(i) &= Cx(i) \end{aligned} \tag{2}$$

where $\Phi \in \Re^{n\times n}$ and $\Gamma \in \Re^{n\times m}$.

The vector of manipulated variables is constrained as

$$u_{\min} \le u(i) \le u_{\max} \tag{3}$$

where $u_{\min}$ and $u_{\max}$ are real vectors.

To simplify the discussion, we assume that $x(i)$ is measured.

According to standard MPC practice, the optimization problem to be solved at time step k is

$$\min_{z,v} \sum_{i=0}^{N} \left[z_d^T(k{+}i) Q z_d(k{+}i) + \Delta v^T(k{+}i) R \Delta v(k{+}i) \right] \tag{4}$$

subject to

$$z_d(k{+}i{+}1) = z(k{+}i{+}1) - r(k{+}i{+}1) \qquad \text{(feedback error)}$$

$$z(k) = x(k) \qquad \text{(feedback measurement)}$$

$$\begin{cases} z(k{+}i{+}1) = f_m\left[z(k{+}i)\right] + g_m\left[z(k{+}i)\right] v(k{+}i) + d(k{+}i) \\ \text{or} \\ z(k{+}i{+}1) = \Phi_m z(k{+}i) + \Gamma_m v(k{+}i) + d(k{+}i) \end{cases} \qquad \text{(prediction)}$$

$$u_{min} \le v(k{+}i) \le u_{max} \qquad \text{(input constraints)}$$

where $i = 0, 1, \ldots, N{-}1$; Q and R are *diagonal* positive definite matrices; f_m and g_m are the nonlinear plant model vector functions; Φ_m and Γ_m are the linear plant model matrices; r is the desired state; z_d is the deviation from the desired state; x is the current state; $\Delta v(j) \equiv v(j) - v(j-1)$ is the change in control vector at time j; $d(k)$ is a load disturbance (bias) at time k estimated, for simplicity, as

$$d(k) = x(k) - (f_m[x(k{-}1)] + g_m[x(k{-}1)]u(k{-}1))$$

for the nonlinear case, or

$$d(k) = x(k) - (\Phi_m x(k{-}1) + \Gamma_m u(k{-}1))$$

for the linear case.

The controlled system is assumed to be controllable. Of the N control moves $\hat{v}(k), \ldots, \hat{v}(k{+}N{-}1)$ computed at time k, only the first one is implemented: $u(k) \equiv \hat{v}(k)$. At the next time instant $k{+}1$, when the new value for state $x(k{+}1)$ becomes available, the minimization of Equation 4 is performed with the new initial condition, to provide $u(k{+}1)$.

Necessary conditions satisfied by the solution of the optimization problem of Equation 4 can be obtained using the discrete maximum principle (Polak and Jordan, 1964; Halkin, 1966). Note that Boltyanskii (1978, pg. 54) has showed that not all formulations of the discrete

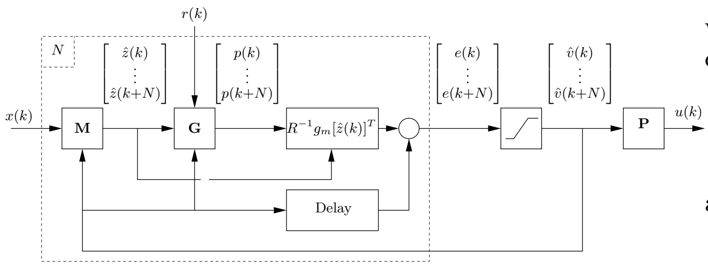

Figure 2: Block diagram for the system of Equations 5 through 10.

maximum principle that have appeared in literature are correct.

Next, we apply the discrete maximum principle to the MPC control problem with input bounds, in order to obtain an analytical solution. The maximum principle is a necessary condition. Under the additional assumption that the problem is convex, the maximum principle is also a sufficient condition for optimality (Theorem 47.7 in Boltyanskii, 1978).

The Discrete Maximum Principle and MPC

We apply the maximum principle to the optimization problem of Equation 4. There are two crucial points to stress:

- The matrix R in Equation 4 is *diagonal*, and
- The input constraints set is $U \equiv \{v | u_{\min} \leq v \leq u_{\max}\}$.

Under these conditions, we get the following necessary conditions for the optimal solution:

System Equations:

$$\begin{aligned}\hat{z}(k+i+1) - \hat{z}(k+i) &= f_m\left[\hat{z}(k+i)\right] \\ &\quad - \hat{z}(k+i) + g_m\left[\hat{z}(k+i)\right]\hat{v}(k+i) + \underbrace{d(k)}_{\substack{=x(k)-f_m(x(k-1))- \\ g_m(x(k-1))u(k-1)}} \quad (5) \\ &\equiv F_i\left(\hat{z}(k+i), \hat{v}(k+i)\right), \qquad i = 0, \ldots, N-1 \\ \hat{z}(k) &= x(k)\end{aligned}$$

Adjoint equations:

$$\begin{aligned}p(k+i) &= \\ &\underbrace{\left(\frac{\partial f_m[\hat{z}(k+i)]}{\partial z} + \sum_{j=1}^{m} \frac{\partial g_{m,j}[\hat{z}(k+i)]}{\partial z}\hat{v}_j(k+i)\right)^T}_{\eta[\hat{z}(k+1),\hat{v}(k+i)]} p(k+i+1) \\ &\quad - Qz_d(k+i), \quad i = 0, \ldots, N-1 \quad (6)\end{aligned}$$

where $g_{m,j}$ denotes the j^{th} column of the matrix g_m.

Transversality equation:

$$p(k+N) = 0 \tag{7}$$

Minimization of the Hamiltonian:

$$\hat{v}(k+i) = sat\left[\hat{v}(k+i-1) + R^{-1}g_m[\hat{z}(k+i)]^T p(k+i+1)\right] \tag{8}$$

where $\hat{v}(k-1) = u(k-1)$ and the saturation function is defined in a standard way, i.e., for $u \in \Re^m$,

$$sat(u) \equiv \left[sat(u_1) \ \cdots \ sat(u_m)\right]. \tag{9}$$

Based on the above, the input to the controlled process at time k is

$$u(k) = \hat{v}(k). \tag{10}$$

A Revealing Block Diagram

Equations 5 through 10 correspond to a static (algebraic) system, namely knowledge of $x(k)$ is, in principle, sufficient for computation of everything else. Figure 2, shows a block diagram interpretation of Equations 5 through 10.

The block **M** is the following set of algebraic equations:

$$\begin{aligned}\hat{z}(k+1) &= f_m[x(k)] + g_m[x(k)]v(k) + d(k) \\ \hat{z}(k+2) &= f_m[\hat{z}(k+1)] + g_m[\hat{z}(k+1)]\hat{v}(k+1) + d(k) \\ &\vdots \\ \hat{z}(k+N) &= f_m[\hat{z}(k+N-1)] \\ &\quad + g_m[\hat{z}(k+N-1)]\hat{v}(k+N-1) + d(k)\end{aligned}$$

where

$$d(k) = x(k) = f_m(x(k-1)) - g_m(x(k-1))u(k-1). \tag{11}$$

The block **G** is the following set of algebraic equations:

$$\begin{aligned}p(k+N) &= 0 \\ p(k+N-1) &= \eta\left[\hat{z}(k+N-1), \hat{v}(k+N-1)\right] p(k+N) \\ &\quad - Qz_d(k+N-1) \\ &\vdots \\ p(k) &= \eta\left[\hat{z}(k), \hat{v}(k)\right] p(k+1) - Qz_d(k)\end{aligned}$$

The block **P** is the projection matrix

$$\mathbf{P} = \begin{bmatrix} I_m & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{N \times N}.$$

Remarks.

- An essential trait of the structure in Figure 2 is that there is a feedback loop with the saturation function in the forward path and algebraic equations in both the forward and feedback paths. This structure bears strong resemblance to the general AWBT structure of Figure 1 (Kothare, 1997). However, unlike the classical AWBT structure, which is based on experience, the structure of Figure 2 emerges naturally as a result of formulating the controller design problem through MPC. In addition, Figure 2, provides guidelines on how to design an AWBT controller starting from MPC.

- Figure 2 implies that

$$e = N(Se, x, r), \quad \hat{v} = Se, \quad u = \mathbf{P}\hat{v} \tag{12}$$

where S denotes the saturation function (Equation 9). On the other hand, if the saturation block were placed right after the output of the feedback loop, before $\mathbf{P}$, in Figure 2 (i.e., if a saturation block were appended to a controller designed without taking input constraints into account), then we would have

$$e = N(e, x, r), \quad \hat{v} = Se, \quad u = \mathbf{P}\hat{v} \tag{13}$$

The above equations make it clear that the second alternative, Equation 13 is different from the first one, Equation 12 and show what is the missing element in controller design that does not take saturation explicitly into account during the design. The above comment will be more concrete in the linear case, discussed below.

On-line Implementation

The controller structure of Figure 2, albeit optimal in the MPC sense, is not suitable for direct on-line implementation, the reason being that the set of algebraic equations 12 must be solved at each time step. A time-recursive set of equations would be required, so that $\hat{v}(k)$ could be computed from data up to and including time k. To circumvent that difficulty, we use the following heuristic:

Let the optimal input sequence computed at time $k-1$ be $\{\hat{w}(k-1), \hat{w}(k), \ldots, \hat{w}(k+N-2)\}$. Then at time k we use

$$\{\hat{v}(k), \hat{v}(k+1), \ldots, \hat{v}(k+N-2), \hat{v}(k+N-1)\} = \{\hat{w}(k), \hat{w}(k+1), \ldots, \hat{w}(k+N-2), \hat{w}(k+N-2)\} \tag{14}$$

in Equations 5 and 8. This heuristic introduces a memory (delay) in the feedback path of Figure 2, thus making the structure suitable for on-line implementation. Note that the choice $\hat{v}(k+N-1) = \hat{w}(k+N-2)$ implicitly assumes that the optimal input sequence over the finite optimization horizon reaches a virtually flat profile towards the end of the horizon.

Analytical Solution for a Linear System with Quadratic Objective

Consider the system of Equation 1 and corresponding model used in Equation 4. In this case, the vector of predicted optimal states satisfies the following equation:

$$\begin{bmatrix} \hat{z}(k) \\ \vdots \\ \hat{z}(k+N) \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ \Phi_m \\ \vdots \\ \Phi_m^N \end{bmatrix}}_{\mathbf{L_2}} \otimes x(k) + \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & \ddots & & \vdots \\ \Phi_m & \ddots & & \ddots \\ \vdots & \ddots & \ddots & \ddots \vdots \\ \Phi_m^{N-1} & \cdots \Phi_m^T & I & 0 \end{bmatrix}}_{\mathbf{L_2}} \otimes \Gamma_m \begin{bmatrix} \hat{v}(k) \\ \vdots \\ \hat{v}(k+N) \end{bmatrix} + \begin{bmatrix} d(k) \\ \vdots \\ d(k) \end{bmatrix}$$

where $d(k)$ is estimated as in Equation 11 and $\otimes$ denotes the Kronecker product. The adjoint equations, Equation 6, imply that the costate vectors , satisfy the following equations:

$$\begin{bmatrix} p(k+N) \\ \vdots \\ p(k) \end{bmatrix} = -\underbrace{\begin{bmatrix} 0 & \cdots\cdots & 0 \\ I & \ddots & \vdots \\ \Phi_m^T & \ddots & \ddots \vdots \\ \vdots & \ddots & \ddots 0 \\ \Phi_m^{T^{N-1}} & \cdots & \Phi_m^T I \end{bmatrix}}_{\mathbf{L_3}} \otimes Q \begin{bmatrix} \hat{z}(k+N-1) \\ \vdots \\ \hat{z}(k) \end{bmatrix} \tag{15}$$

The minimization of the Hamiltonian, Equation 8, yields

$$\hat{v}(k+i) = sat[\hat{v}(k+i-1) + R^{-1}\Gamma_m^T p(k+i+1)], \quad i = 0, \ldots, N-1. \tag{16}$$

Again, the approximation of Equation 14 can be used.

Remarks.

- It is straight forward to modify the previous discussion for the objective function in Equation 4 containing a term quadratic in v instead of Δv. The well-known advantage of using the Δv is that step disturbance or setpoint changes result in zero offset.
- Figure 2 indicates that

$$e = L_x x + L_r r + L_v \hat{v} = L_x x + L_r r + L_v Se \tag{17}$$

$$\Rightarrow (I - L_v S)e = L_x x + L_r r \tag{18}$$

Because the quadratic minimization of Equation 4 is convex, there must exist a unique optimal solution, which implies that the above equation must have a unique solution for e, i.e., $e = (I - L_v S)^{-1}(L_x x + L_r r)$, from which we get

$$u = \mathbf{P}\hat{v} = \mathbf{P}Se = \mathbf{P}S(I - L_v S)^{-1}(L_x x + L_r r) \tag{19}$$

It is interesting to note again that if the controller was designed without taking input saturation into account, and a saturation block were appended to it, then the corresponding mapping between (x, r) and u would be

$$u = \mathbf{P}\hat{v} = \mathbf{P}Se = \mathbf{P}S(I - L_v)^{-1}(L_x x + L_r r) \quad (20)$$

The above equation trivially shows that this design approach is not optimal.

Conclusions

In this work we established a direct relationship between multivariable AWBT control and MPC with quadratic objective, input constraints and plant model structure affine in the input variables. The key to establishing that relationship was application of the discrete maximum principle to the on-line optimization problem solved by MPC.

The results of this work are important for both theoretical and practical reasons.

From a theoretical viewpoint, these results provide fundamental justification for the empirical realization that virtually all heuristically developed AWBT control structures (Figure 1) follow a similar pattern involving a nonlinear (saturation) block and linear transfer functions (Figure 2). The structure of Figure 2 is actually valid for nonlinear systems as well.

From a practical viewpoint, the substitution of model predictive controllers with input constraints by controllers with a closed-form structure allows computations to be performed significantly faster. This is particularly important for MPC systems that follow a cascaded structure of linear programming followed by (unconstrained) least squares (Kassmann et al., 2000; Rao and Rawlings, 1999), because it allows *constrained* least squares to be used with *computational efficiency* in place of unconstrained least squares.

Of both theoretical and practical importance is the fact that insight into a controller from either the MPC or AWBT class can be augmented by using insight from an equivalent or related controller of the other class.

A number of simulation examples can be downloaded from `http://athens.chee.uh.edu`.

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